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A Geometric Investigation of Multiplicative One-Parameter Motions in Lorentzian Geometry

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ABSTRACT

The trigonometric framework of the multiplicative Lorentzian plane, together with the notions of multiplicative rotations and motions, has been investigated in earlier studies. In this paper, parameterized motions defined on the multiplicative Lorentzian plane are examined. By employing multiplicative calculus, the fundamental properties of these motions are analyzed, and the velocity components, velocity law, and relationships among velocities are derived. Furthermore, acceleration quantities and their corresponding relations are obtained. To provide a geometric description of motion, moving coordinate systems are adapted to the multiplicative Lorentzian setting. In this context, the differential equations of the multiplicative Lorentzian moving frame are established, and the associated multiplicative Pfaffian forms are introduced. Moreover, a third multiplicative Lorentzian plane is defined, and the relative motions among the three planes are investigated. The results contribute to the development of multiplicative Lorentzian kinematics and provide a basis for future studies.

Keywords: Multiplicative One- Parameter Lorentzian Motion, Pole points, Multiplicative Lorentz Inner Product, Multiplicative Velocities, Multiplicative accelerations.

Multiplikatif Lorentz Geometride 1-Parametreli Hareketlerin Geometrik Bir Yaklaşımı

ÖZ

Multiplikatif Lorentz düzleminin trigonometrik yapıları önceki çalışmalarda incelenmiş, multiplikatif Lorentz düzlem üzerindeki dönme ve hareket kavramları tanımlanmıştır. Bu çalışmada, multiplikatif Lorentz düzleminde tanımlı bir parametreli hareketler ele alınmaktadır. Multiplikatif kalkülüs yöntemleri kullanılarak hareketlerin temel özellikleri incelenmiş; hız bileşenleri, hız yasası ve hızlar arasındaki ilişkiler elde edilmiştir. Ayrıca, harekete ait hız bileşenleri, hız yasası ve hızlar arasındaki ilişkiler elde edilmiştir Buna ek olarak hareketin ivmeleri ve bunlar arasındaki bağıntılar türetilmiştir. Hareketlerin geometrik tanımlanmasında önemli olan hareketli koordinat sistemleri multiplikatif Lorentz ortamına uyarlanmıştır. Bu kapsamda, multiplikatif Lorentz hareketli çatısının diferansiyel denklemleri elde edilmiş ve karşılık gelen çarpımsal Pfaff biçimleri tanıtılmıştır. Ayrıca üçüncü bir multiplikatif Lorentz düzlemi tanımlanarak üç düzlem arasındaki bağıl hareketler incelenmiştir. Elde edilen sonuçlar, multiplikatif Lorentz kinematiğinin geliştirilmesine ve bu alandaki ileri çalışmalar için bir temel oluşturulmasına katkı sağlamaktadır.

Anahtar Kelimeler: Multiplikatif 1- parametreli Lorentz Hareketi, Pol Noktaları, Multiplikatif Lorentz İç Çarpımı, Multiplikatif Hızlar, Multiplikatif İvmeler.

Introduction

Multiplicative calculus, first introduced by Michael Grossman and Robert Katz, is widely known in the literature as Non-Newtonian calculus. In contrast to classical Newtonian calculus, this theory is founded on two fundamental operators: the multiplicative derivative and the multiplicative integral [1]. Detailed explanations of the various forms of Non-Newtonian calculus and their applications can be found in the pioneering works of Grossman and Katz as well as in later contributions by Stanley, Campbell, Grossman, and Jane Grossman [2–7].

A more systematic and rigorous analytical framework for multiplicative calculus was later established by A. E. Bashirov, E. M. Kurpınar, and A. Özyapıcı, who developed the theoretical foundations of the subject. In addition, several researchers have extended multiplicative calculus to complex-valued functions, thereby enlarging the scope of the theory and enabling new analytical applications [8–13].

In this setting, Çakmak, Başar, and Uzer investigated matrix transformations acting on sequence spaces within multiplicative structures [14]. Furthermore, K. Boruah and B. Hazarika introduced the concept of geometric real numbers using a geometric coordinate representation and later defined geometric trigonometric ratios, establishing connections between these ratios and classical trigonometric functions [15–18].

Other algebraic aspects of multiplicative systems were studied by Gurefe, S. G. Georgiev, and K. Zennir, who introduced multiplicative vector spaces, multiplicative inner product spaces, and multiplicative matrices together with their main properties [19-21]. In a number of studies, the notions of geometric three–dimensional space and multiplicative quaternions were introduced by S. Aslan, M. Bekar, and Y. Yaylı [22]. Moreover, S. Nurkan, K. İ. Gürgil, and M. K. Karacan investigated vectorial structures in geometric calculus [23]. H. Es examined multiplicative motions under different metric structures and obtained several notable theoretical results in this field [24–27]. More recently, A. Has and B. Yılmaz established fundamental principles of multiplicative analytic geometry [28–29].

Taken together, these studies indicate that multiplicative calculus and its associated geometric and algebraic structures have experienced significant development in recent years. The growing body of research highlights the increasing role of this framework in both theoretical mathematics and applied investigations.

In this work, we focus on multiplicative Lorentzian one-parameter motions within the multiplicative Lorentzian plane. Using the tools of multiplicative calculus, we systematically derive the fundamental properties of such motions, including velocity and acceleration components and the relationships among them. Additionally, we formulate several definitions and theorems that provide a rigorous mathematical framework for these motions. The primary objective of this study is to establish a comprehensive kinematic analysis of multiplicative motions in the Lorentzian plane and to extend classical motion theory into the multiplicative setting. The results presented here not only deepen the theoretical understanding of multiplicative kinematics but also lay the groundwork for potential applications in geometric and algebraic structures.

MATERIALS and METHODS

For the basic definitions used in this paper, the reader may refer to the references listed in the bibliography, in particular [19], [21], [23] and [24].

Definition 1. In classical calculus, the derivative of a function f at a point x is defined through the limit

$$f^{\bullet}(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} .$$

If, instead of using the difference $f(x+h)-f(x)$ we consider the quotient $f(x+h)/f(x)$ and replace division by h with exponentiation by $\frac{1}{h}$, we obtain an alternative notion of differentiation. Accordingly, the multiplicative derivative of f at x is defined by

$$f^{\circ}(x) = \lim_{h\to 0}\left(\frac{f(x+h)}{f(x)}\right)^{\frac{1}{h}}.$$

According to [8], if f is a positive function defined on an open set $A \subset \mathbb{R}$ and its classical derivative exists, then the multiplicative derivative also exists. Moreover, these derivatives are related by

$$f^{\circ}(x) = e^{(\ln(f(x))^{\bullet}}.$$

In addition, the relationship between these derivatives for the n-th order is given as follows:

$$f^{\circ(n)}(x) = e^{(\ln f(x))^{(n)}},\ n = 0,1,\ldots$$

[8-10, 19,21,23-27].

Definition 2. Geometric and classical trigonometric functions are connected through exponential mappings, where

$$\operatorname{sing}\theta = \sin_*\theta = e^{\sin\theta},$$
$$\operatorname{cosg}\theta = \cos_*\theta = e^{\cos\theta},$$
$$\operatorname{tang}\theta = \tan_*\theta = e^{\tan\theta} = \sin_*\theta /_* \cos_*\theta$$

[16,21,23-27].

Geometric Real Numbers

The set of geometric real numbers can be expressed as

$$\mathbb{R}_* = \{\exp(\alpha) = e^{\alpha} : \alpha \in \mathbb{R}\}.$$

In the set $\mathbb{R}_*$, the binary operations of addition and multiplication are introduced in the following manner.
Addition

$$e^{\alpha} +_* e^{\beta} = e^{\alpha+\beta}$$

Multiplication

$$e^{\alpha} \cdot_* e^{\beta} = e^{\alpha\beta}$$

For every $e^{\alpha}, e^{\beta} \in \mathbb{R}_*$. The structure $\mathbb{R}_*$, equipped with the operations of geometric addition $+_*$ and geometric multiplication $._*$, forms a field. Within this system, the geometric zero is given by $e^0 = 1$, while the geometric identity element is $e^1 = e$.
Positive and negative geometric real numbers are defined in the following way

$$\mathbb{R}_*^+ = \{\mathbb{p} \in \mathbb{R}_* : \mathbb{p} > 1\}$$

and

$$\mathbb{R}_*^- = \{\mathbb{p} \in \mathbb{R}_* : 0 < \mathbb{p} < 1\}.$$

Geometric 2-Space

$$\mathbb{R}_*^2 = \{\hat{\alpha} = (e^{\alpha_1}, e^{\alpha_2}) : e^{\alpha_1}, e^{\alpha_2} \in \mathbb{R}_*\}.$$

The operations of addition and scalar multiplication in $\mathbb{R}_*^2$ are defined, respectively, as follows.

$$\begin{aligned} \hat{\alpha} +_* \hat{\beta} &= (e^{\alpha_1}, e^{\alpha_2}) +_* (e^{\beta_1}, e^{\beta_2}) \\ &= (e^{\alpha_1} +_* e^{\beta_1}, e^{\alpha_2} +_* e^{\beta_2}) \\ &= (e^{\alpha_1+\beta_1}, e^{\alpha_2+\beta_2}) \end{aligned}$$

And

$$\begin{aligned} e^{\mu} \cdot_* \hat{\alpha} &= e^{\mu} \cdot_* (e^{\alpha_1}, e^{\alpha_2}) \\ &= (e^{\mu} \cdot_* e^{\alpha_1}, e^{\mu} \cdot_* e^{\alpha_2}) \\ &= (e^{\mu\alpha_1}, e^{\mu\alpha_2}) \end{aligned}$$

Where $e^{\mu} \in \mathbb{R}_*$ and $\hat{\alpha}, \hat{\beta} \in \mathbb{R}_*^2$.

Example 1. Let $k = e^2 \in \mathbb{R}_*$ and $\hat{\alpha} = (e^3, e^4)$, $\hat{\beta} = (e^5, e^6)$ be given by

$$\begin{aligned} \hat{\alpha} +_* \hat{\beta} &= (e^3, e^4) +_* (e^5, e^6) \\ &= (e^3 +_* e^5, e^4 +_* e^6) \\ &= (e^{3+5}, e^{4+6}) \\ &= (e^8, e^{10}), \\ k \cdot_* \hat{\alpha} &= e^2 \cdot_* (e^3, e^4) \\ &= (e^2 \cdot_* e^3, e^2 \cdot_* e^4) \\ &= (e^{2.3}, e^{2.4}) \\ &= (e^6, e^8). \end{aligned}$$

In $\mathbb{R}_*^2$, the standard basis is given by the set

$$\left\{ \overrightarrow{\hat{1}_1} = (e, 1), \overrightarrow{\hat{1}_2} = (1, e) \right\}.$$

Any element $\hat{\alpha} \in \mathbb{R}_*^2$ is referred to as a vector in this space, and more specifically, it will be called a geometric vector. The origin of $\mathbb{R}_*^2$ is defined as

$$\hat{0} = (e^0, e^0) = (1,1).$$

In this paper, the multiplicative Lorentz inner product between $\hat{\alpha}$ and $\hat{\beta}$ will be represented by $\langle \hat{\alpha}, \hat{\beta} \rangle_*$.

The multiplicative Lorentzian inner (or scalar) product on $\mathbb{R}_*^2$ is defined by

$$\begin{aligned} \langle \hat{\alpha}, \hat{\beta} \rangle_* &= \langle (e^{\alpha_1}, e^{\alpha_2}), (e^{\beta_1}, e^{\beta_2}) \rangle_* \\ &= (e^{\alpha_1} \cdot_* e^{\beta_1}) -_* (e^{\alpha_2} \cdot_* e^{\beta_2}) \\ &= e^{\alpha_1\beta_1 - \alpha_2\beta_2} \end{aligned}$$

The geometric Lorentzian norm in $\mathbb{R}_*^2$ is defined as

$$\|\hat{\alpha}\|_* = \sqrt[*]{\langle \hat{\alpha}, \hat{\alpha} \rangle_*} = e^{\sqrt{|(\alpha_1)^2 - (\alpha_2)^2|}}.$$

The geometric vector $\hat{\alpha}$ is called a space-like unit vector if $\|\hat{\alpha}\|_* = e$.

Definition 3. $\hat{\alpha}, \hat{\beta}$ be two vectors in the multiplicative Lorentzian plane. If

$$\langle \hat{\alpha}, \hat{\beta} \rangle_* = 1,$$

then the vectors $\hat{\alpha}, \hat{\beta}$ are said to be orthogonal in the multiplicative Lorentzian sense**.**

Example 2. The vectors $\hat{\alpha} = (e^1, e^2)$ and $\hat{\beta} = (e^2, e^1)$ are not orthogonal with respect to the multiplicative Euclidean metric; however, they are orthogonal in the sense of the multiplicative Lorentzian metric.

Computing the multiplicative inner product of the vectors $\hat{\alpha}$ and $\hat{\beta}$ yields

$$\langle \hat{\alpha}, \hat{\beta} \rangle = \langle (e^1, e^2), (e^2, e^1) \rangle = e^{1.2+2.1} = e^4.$$

Let us now evaluate the multiplicative Lorentzian inner product of the vectors $\hat{\alpha}$ and $\hat{\beta}$. A direct computation yields

$$\langle \hat{\alpha}, \hat{\beta} \rangle_* = \langle (e^1, e^2), (e^2, e^1) \rangle_* = e^{1.2-2.1} = e^0 = 1.$$

Definition 4. Let $\hat{\alpha}$ be a vector in the multiplicative Lorentzian plane. The vector $\hat{\alpha}$ is classified as follows.

If $\langle \hat{\alpha}, \hat{\alpha} \rangle_* < 1$, then $\hat{\alpha}$ is called a multiplicative time-like vector**.**

If $\langle \hat{\alpha}, \hat{\alpha} \rangle_* > 1$, then $\hat{\alpha}$ is called a multiplicative space-like vector**.**

If $\langle \hat{\alpha}, \hat{\alpha} \rangle_* = 1$, $\hat{\alpha}$ is called a multiplicative null vector [30].

It is assumed that $\overrightarrow{\hat{1}_1} = (e, 1)$ is a multiplicative spacelike vector, whereas $\overrightarrow{\hat{1}_2} = (1, e)$ is a multiplicative time-like vector.

Theorem 1. Two vectors that are both multiplicative time-like (or both space-like) cannot be orthogonal to each other.

Corollary 1. For two multiplicative vectors to be orthogonal, one of them must be time-like and the other must be space-like.

Kinematic Analysis of one-Parameter Motions in the Multiplicative Lorentzian Plane

Let $\{\hat{O};\vec{\hat{I}}_1,\ \vec{\hat{I}}_2\}$ and $\{\hat{O}';\vec{\hat{I}}'_1\,\vec{\hat{I}}'_2\}$ denote the moving and fixed coordinate systems of L_* and L_*', respectively. Thus, the vector $\overrightarrow{\hat{O}\hat{O}'}$ can be expressed as

$$\overrightarrow{\hat{O}\hat{O}'}=\vec{\hat{a}}=e^{a_1}\cdot_*\vec{\hat{I}}_1+_*e^{a_2}\cdot_*\vec{\hat{I}}_2,e^{a_1},e^{a_2}\in\mathbb{R}_* \qquad (2)$$

Where, $\vec{\hat{a}}=(e^{a_1},e^{a_2})\in\mathbb{R}_*^2$,
When $t=t_0$, we assume that the origins $\hat{O}$ and $\hat{O}'$ overlap, and therefore

$$\left.\begin{aligned}\vec{\hat{I}}_1&=\cosh_*\theta\cdot_*\vec{\hat{I}}'_1+_*\sinh_*\theta\cdot_*\vec{\hat{I}}'_2\\ \vec{\hat{I}}_2&=\sinh_*\theta\cdot_*\vec{\hat{I}}'_1+_*\cosh_*\theta\cdot_*\vec{\hat{I}}'_2\end{aligned}\right\}$$

or

$$\begin{aligned}\vec{\hat{I}}_1&=e^{\cosh\theta}\cdot_*\vec{\hat{I}}'_1+_*e^{\sinh\theta}\cdot_*\vec{\hat{I}}'_2\\ \vec{\hat{I}}_2&=e^{\sinh\theta}\cdot_*\vec{\hat{I}}'_1+_*e^{\cosh\theta}\cdot_*\vec{\hat{I}}'_2\end{aligned} \qquad (3)$$

is obtained. In this context, θ represents the rotational angle.

We consider the functions a_1, a_2 and θ as real-valued functions depending on a real parameter t. In most applications, the parameter t will be interpreted as time. These functions are assumed to be defined on the domain $t_0\le t\le t_1$. Under these assumptions, the motion of the moving coordinate system $\{\hat{O};\vec{\hat{I}}_1,\vec{\hat{I}}_2\}$ with respect to the fixed coordinate system $\{\hat{O}';\vec{\hat{I}}'_1\,\vec{\hat{I}}'_2\}$ is called a multiplicative Lorentzian one-parameter motion, and it will be denoted by

$$B_1={}^{L_*}/_{L'_*}\ .$$

Hence, it follows that $\theta=\theta(t)$, $a_1=a_1(t)$, $a_2=a_2(t)$ and $t\in\mathbb{R}$. In both coordinate systems, the position vectors of the point x are given by

$$\begin{aligned}\vec{\hat{x}}&=\overrightarrow{\hat{O}\hat{X}}=e^{x_1}\cdot_*\vec{\hat{I}}_1+_*e^{x_2}\cdot_*\vec{\hat{I}}_2\\ \vec{\hat{x}}'&=\overrightarrow{\hat{O}'\hat{X}}=e^{x'_1}\cdot_*\vec{\hat{I}}'_1+_*e^{x'_2}\cdot_*\vec{\hat{I}}'_2\\ \overrightarrow{\hat{O}'\hat{X}}&=\overrightarrow{\hat{O}'\hat{O}}+\overrightarrow{\hat{O}\hat{X}}=-_*\overrightarrow{\hat{O}\hat{O}'}+\overrightarrow{\hat{O}\hat{X}}\end{aligned}$$

or

$$\begin{aligned}\vec{\hat{x}}'&=-_*\vec{\hat{a}}+_*\vec{\hat{x}}\\ &=(e^{-1}\cdot_*e^{a_1}+_*e^{x_1})\cdot_*\vec{\hat{I}}_1+_*(e^{-1}\cdot_*e^{a_2}+_*e^{x_2})\cdot_*\vec{\hat{I}}_2\end{aligned} \qquad (4)$$

The derivative formulations associated with the $B_1={}^{L_*}/_{L'_*}$ motion are obtained as follows

Based on equations (2) and (3), the differentiation formulas related to the $B_1={}^{L_*}/_{L'_*}$ are derived.

Let $\vec{\hat{I}}'_1=(e^1,e^0)$, $\vec{\hat{I}}'_2=(e^0,e^1)$.

$$\begin{aligned}\vec{\hat{I}}_1&=e^{\cosh\theta}\cdot_*\vec{\hat{I}}'_1+_*e^{\sinh\theta}\cdot_*\vec{\hat{I}}'_2\\ &=e^{\cosh\theta}\cdot_*(e^1,e^0)+_*e^{\sinh\theta}\cdot_*(e^0,e^1)\\ &=\left(e^{\cosh\theta}\cdot_*e^1,e^{\cosh\theta}\cdot_*e^0\right)+_*\left(e^{\sinh\theta}\cdot_*e^0,e^{\sinh\theta}\cdot_*e^1\right)\\ &=(e^{\cosh\theta},e^0)+_*(e^0,e^{\sinh\theta})\\ \vec{\hat{I}}_1&=(e^{\cosh\theta},e^{\sinh\theta})\\ \left(\vec{\hat{I}}_1\right)^{\circ}&=(e^{\theta^{\bullet}\sinh\theta},e^{\theta^{\bullet}\cosh\theta})\\ &=e^{\theta^{\bullet}}\cdot_*(e^{\sinh\theta},e^{\cosh\theta})\end{aligned}$$

is obtained. From this relation, the expression

$$\left(\vec{\hat{I}}_1\right)^{\circ}=e^{\theta^{\bullet}}\cdot_*(\sinh_*\theta\cdot_*\vec{\hat{I}}'_1+_*\cosh_*\theta\cdot_*\vec{\hat{I}}'_2)\ .$$

can be derived. Similarly, by computing $\left(\vec{\hat{I}}_2\right)^{\circ}$, the expression

$$\left(\vec{\hat{I}}_2\right)^{\circ}=e^{\theta^{\bullet}}\cdot_*(\cosh_*\theta\cdot_*\vec{\hat{I}}'_1+_*\sinh_*\theta\cdot_*\vec{\hat{I}}'_2)$$

is obtained. Moreover, the multiplicative derivative of $\vec{\hat{a}}$ is obtained in the following

$$\begin{aligned}\left(\vec{\hat{a}}\right)^{\circ}&=(e^{a_1})^{\circ}\cdot_*\vec{\hat{I}}_1+_*e^{a_1}\cdot_*\left(\vec{\hat{I}}_1\right)^{\circ}\\ &\quad+_*(e^{a_2})^{\circ}\cdot_*\vec{\hat{I}}_2+_*e^{a_2}\cdot_*\left(\vec{\hat{I}}_2\right)^{\circ}\end{aligned}$$

$$\left.\begin{aligned}\left(\vec{\hat{I}}_1\right)^{\circ}&=e^{\theta^{\bullet}}(\sinh_*\theta\cdot_*\vec{\hat{I}}'_1+_*\cosh_*\theta\cdot_*\vec{\hat{I}}'_2)\\ \left(\vec{\hat{I}}_2\right)^{\circ}&=e^{\theta^{\bullet}}\cdot_*(\cosh_*\theta\cdot_*\vec{\hat{I}}'_1+_*\sinh_*\theta\cdot_*\vec{\hat{I}}'_2)\\ \left(\vec{\hat{a}}\right)^{\circ}&=(e^{a_1})^{\circ}\cdot_*\vec{\hat{I}}_1+_*e^{a_1}\cdot_*\left(\vec{\hat{I}}_1\right)^{\circ}\\ &\quad+_*(e^{a_2})^{\circ}\cdot_*\vec{\hat{I}}_2+_*e^{a_2}\cdot_*\left(\vec{\hat{I}}_2\right)^{\circ}\end{aligned}\right\} \qquad (5)$$

is obtained.

$$\left.\begin{aligned}\left(\vec{\hat{I}}_1\right)^{\circ}&=e^{\theta^{\bullet}}\cdot_*\vec{\hat{I}}_2\\ \left(\vec{\hat{I}}_2\right)^{\circ}&=e^{\theta^{\bullet}}\cdot_*\vec{\hat{I}}_1\end{aligned}\right\} \qquad (6)$$

or

$$\begin{aligned}\left(\vec{\hat{a}}\right)^{\circ}&=(e^{a_1^{\bullet}}+_*e^{a_2}\cdot_*e^{\theta^{\bullet}})\cdot_*\vec{\hat{I}}_1\\ &\quad+_*\left(e^{a_2^{\bullet}}+_*e^{a_1}\cdot_*e^{\theta^{\bullet}}\right)\cdot_*\vec{\hat{I}}_2\end{aligned} \qquad (7)$$

Velocities Along with Their Linear Combinations

While the plane L_* performs a parameterized $B_1={}^{L_*}/_{L'_*}$ motion with respect to the multiplicative Lorentzian plane L'_*, a point $\hat{X}$ changes its position in the multiplicative Lorentzian moving plane L_* depending on the parameter t. In this case, two motions associated with the point $\widehat{X}$ arise, one with respect to the plane L_* and the other with respect to the multiplicative Lorentzian plane L'_*. In this section, the relationship between the velocities corresponding to these motions is examined.

Definition 3.1. The vector representing the motion of the point $\widehat{X}$ with respect to the multiplicative Lorentzian plane L_* along its path is termed the multiplicative Lorentzian relative velocity and is denoted by $\overrightarrow{\widehat{V}_r}$.

$$\vec{\widehat{x}} = e^{x_1} \cdot_* \overrightarrow{\widehat{l}_1} +_* e^{x_2} \cdot_* \overrightarrow{\widehat{l}_2} \tag{8}$$

$$\overrightarrow{\widehat{V}_r} = (\vec{\widehat{x}})^{\circ} = (e^{x_1})^{\circ} \cdot_* \overrightarrow{\widehat{l}_1} +_* (e^{x_2})^{\circ} \cdot_* \overrightarrow{\widehat{l}_2} \tag{9}$$

Definition 3.2. The velocity of the point $\widehat{X}$ measured relative to the multiplicative Lorentzian fixed plane L'_* is referred to as the absolute velocity of $\widehat{X}$ and is represented by $\overrightarrow{\widehat{V}_a}$.

$$\overrightarrow{\widehat{x}'} = -_* \vec{\widehat{a}} +_* \vec{\widehat{x}}. \tag{10}$$

Applying differentiation to equation (9) leads to the result

$$\overrightarrow{\widehat{V}_a} = \left(\overrightarrow{\widehat{x}'}\right)^{\circ} = -_* \left(\vec{\widehat{a}}\right)^{\circ} +_* \left(\vec{\widehat{x}}\right)^{\circ}$$

$$\begin{aligned}\overrightarrow{\widehat{V}_a} &= \left[-_*(e^{a_1^{\bullet}} +_* e^{a_2} \cdot_* e^{\theta^{\bullet}})\right] \cdot_* \overrightarrow{\widehat{l}_1} \\ &\quad +_* \left[-_*(e^{a_2^{\bullet}} +_* e^{a_1} \cdot_* e^{\theta^{\bullet}})\right] \cdot_* \overrightarrow{\widehat{l}_2} \\ &\quad +_* \left(\overrightarrow{\widehat{l}_1}\right)^{\circ} e^{x_1} +_* \overrightarrow{\widehat{l}_1} \cdot_* (e^{x_1})^{\circ} +_* \left(\overrightarrow{\widehat{l}_2}\right)^{\circ} e^{x_2} +_* \overrightarrow{\widehat{l}_2} \cdot_* (e^{x_2})^{\circ} \\ &= \left\{-_* e^{a_1^{\bullet}} -_* A \cdot_* e^{\theta^{\bullet}}\right\} \cdot_* \overrightarrow{\widehat{l}_1} +_* \left\{-_* e^{a_2^{\bullet}} -_* B \cdot_* e^{\theta^{\bullet}}\right\} \cdot_* \overrightarrow{\widehat{l}_2} \\ &\quad +_* e^{x_1^{\bullet}} \cdot_* \overrightarrow{\widehat{l}_1} +_* e^{x_2^{\bullet}} \cdot_* \overrightarrow{\widehat{l}_2}\end{aligned} \tag{11}$$

Where $A = (e^{a_2} -_* e^{x_2})$ and $B = (e^{a_1} -_* e^{x_1})$.

Definition 3.3. In the formulation of the absolute velocity $\overrightarrow{V_a}$, the vector

$$\left\{-_* e^{a_1^{\bullet}} -_* A \cdot_* e^{\theta^{\bullet}}\right\} \cdot_* \overrightarrow{\widehat{l}_1} +_* \left\{-_* e^{a_2^{\bullet}} -_* B \cdot_* e^{\theta^{\bullet}}\right\} \cdot_* \overrightarrow{\widehat{l}_2}$$

represents the sliding velocity of the point $\widehat{X}$ and is denoted by $\overrightarrow{\widehat{V}_f}$. Where $A = (e^{a_2} -_* e^{x_2})$ and $B = (e^{a_1} -_* e^{x_1})$.

$$\overrightarrow{\widehat{V}_f} = \left\{-_* e^{a_1^{\bullet}} -_* A \cdot_* e^{\theta^{\bullet}}\right\} \cdot_* \overrightarrow{\widehat{l}_1} +_* \left\{-_* e^{a_2^{\bullet}} -_* B \cdot_* e^{\theta^{\bullet}}\right\} \cdot_* \overrightarrow{\widehat{l}_2} \tag{12}$$

represents the sliding velocity of the point $\widehat{X}$ and is denoted by $\overrightarrow{V_f}$. Hence, the absolute velocity can be decomposed into the sum of the sliding velocity and the relative velocity, yielding the relation

$$\overrightarrow{\widehat{V}_a} = \overrightarrow{\widehat{V}_f} +_* \overrightarrow{\widehat{V}_r}.$$

If the point is fixed in the moving plane L_*, then the relative velocity vanishes, that is $\overrightarrow{\widehat{V}_r} = 1$, and consequently the equality $\overrightarrow{\widehat{V}_a} = \overrightarrow{\widehat{V}_f}$ holds.

Theorem 3.1. The absolute velocity vector of a point can be expressed as the sum of its relative velocity vector and its sliding velocity vector. Accordingly, the following relationship is obtained:

$$\overrightarrow{\widehat{V}_a} = \overrightarrow{\widehat{V}_f} +_* \overrightarrow{\widehat{V}_r}.$$

Poles of Rotating and Orbit

In a multiplicative Lorentzian $B_1 = {}^{L_*}/_{L'_*}$ motion, if the sliding velocity $\overrightarrow{V_f}$ of a fixed point $\widehat{X}$ in L_* is zero at every instant t, then this point is fixed in both the multiplicative Lorentzian moving and the fixed planes. Such points are referred to as the multiplicative Lorentzian pole points of the motion.

Theorem 3.1.1 In a $B_1 = {}^{L_*}/_{L'_*}$ motion with multiplicative Lorentzian angular velocity different from one, there is a single point that is fixed in both planes at all times t.

Proof: Since the point $\widehat{X}$ is fixed in the multiplicative Lorentzian moving plane L_*, its relative velocity satisfies $\overrightarrow{\widehat{V}_r} = 1$. Similarly, because the same point $\widehat{X}$ is also fixed in the multiplicative Lorentzian fixed plane L'_*, its sliding velocity is $\overrightarrow{\widehat{V}_f} = 1$. Hence, for such points, the condition $\overrightarrow{\widehat{V}_r} = 1$ implies that $e^{x_1^{\circ}} = 1$ and $e^{x_2^{\circ}} = 1$.
Moreover, when $\overrightarrow{\widehat{V}_f} = 1$, it follows that

$$\left.\begin{aligned} e^{a_1^{\bullet}} +_* (e^{a_2} -_* e^{x_2}) \cdot_* e^{\theta^{\bullet}} = 1 \\ e^{a_2^{\bullet}} +_* (e^{a_1} -_* e^{x_1}) \cdot_* e^{\theta^{\bullet}} = 1 \end{aligned}\right\} \tag{13}$$

We assume that $e^{\theta^{\bullet}} \neq 1$ for the multiplicative Lorentzian motion. In other words, the multiplicative Lorentzian is not solely a translation. Then the two equations given in (13) always admit a unique solution. Denoting these solutions as p_1 and p_2, they can be expressed as

$$\left.\begin{aligned} e^{p_1} = e^{x_1} = e^{a_1} +_* e^{a_2^{\bullet}} /_* e^{\theta^{\bullet}} \\ e^{p_2} = e^{x_2} = e^{a_2} +_* e^{a_1^{\bullet}} /_* e^{\theta^{\bullet}} \end{aligned}\right\} \tag{14}$$
$$\overrightarrow{\widehat{O}\widehat{P}} = e^{p_1} \cdot_* \overrightarrow{\widehat{l}_1} +_* e^{p_2} \cdot_* \overrightarrow{\widehat{l}_2}$$

We call the point $\widehat{P} = (e^{p_1}, e^{p_2})$which corresponds to the position vector

$$\overrightarrow{\widehat{O}\widehat{P}} = e^{p_1} \cdot_* \overrightarrow{\widehat{l}_1} +_* e^{p_2} \cdot_* \overrightarrow{\widehat{l}_2}$$

the pole (or rotational pole), also known as the instantaneous center of rotation at time t. Consequently, the theorem below is obtained.

Theorem 3.1.2 For a motion with nonzero multiplicative angular velocity, there is exactly one point at every time t whose sliding velocity vanishes in the multiplicative sense; this point remains multiplicative Lorentzian fixed in both planes and is referred to as the pole point.

By means of the rotation pole $\widehat{P}$, the sliding velocity $\overrightarrow{V_f}$ of an arbitrary point $\widehat{X}$ can be reformulated in an alternative manner. To achieve this, we first evaluate the expressions obtained from Equation (13), namely

$$\left.\begin{array}{c} e^{a_1^{\bullet}} = (e^{p_2} -_* e^{a_2}) \cdot_* e^{\theta^{\bullet}} \\ e^{a_2^{\bullet}} = (-_* e^{a_1} +_* e^{p_1}) \cdot_* e^{\theta^{\bullet}} \end{array}\right\} \quad (15)$$

Substituting these results into equation (12) and performing the necessary multiplicative operations, we derive a new representation for the sliding velocity vector. Consequently, the multiplicative form of the sliding velocity of point $\hat{X}$ is expressed as

$$\overrightarrow{V_f} = e^{\theta^{\bullet}} \cdot_* \left\{ (e^{x_2} -_* e^{p_2}) \cdot_* \overrightarrow{\hat{1}_1} +_* (e^{x_1} -_* e^{p_1}) \cdot_* \overrightarrow{\hat{1}_2} \right\} \quad (16)$$

is obtained. This formulation provides an equivalent but structurally distinct expression for the sliding velocity in terms of the rotation pole coordinates and the multiplicative unit vectors.

Theorem 3.1.3 At every time t, the polar ray from the pole $\widehat{P}$ to the point $\widehat{X}$ is multiplicatively orthogonal to the sliding velocity vector $\overrightarrow{V_f}$.

Proof: The polar ray from the pole $\hat{P}$ to the point is given

$$\overrightarrow{PX} = (e^{x_1} -_* e^{p_1}) \cdot_* \overrightarrow{\hat{1}_1} +_* (e^{x_2} -_* e^{p_2}) \cdot_* \overrightarrow{\hat{1}_2}.$$

Noting that the expression represents

$$\overrightarrow{V_f} = e^{\theta^{\bullet}} \cdot_* \left\{ (e^{x_2} -_* e^{p_2}) \cdot_* \overrightarrow{\hat{1}_1} +_* (e^{x_1} -_* e^{p_1}) \cdot_* \overrightarrow{\hat{1}_2} \right\}$$

we conclude that

$$\langle \overrightarrow{V_f}, \overrightarrow{PX} \rangle_* = 1.$$

in this case, the vectors $\overrightarrow{V_f}$ and $\overrightarrow{PX}$ cannot both be classified as timelike vectors simultaneously.

Theorem 3.1.4. Let $\widehat{X}$ be a moving point in L and let $\widehat{P}$ denote its instantaneous center of rotation. Then

$$\left\| \overrightarrow{V_f} \right\|_* = e^{\theta^{\bullet}} \cdot_* \left\| \overrightarrow{PX} \right\|_*$$

Definition 3.1.1 In the multiplicative Lorentzian motion B_1, the locus of the pole point $\widehat{P}$ in the multiplicative Lorentzian moving plane L_* is called the multiplicative moving pole curve of the motion and is denoted by $(\widehat{P})$.On the other hand, the locus of the point P in the fixed multiplicative Lorentzian plane L'_* is referred to as the fixed multiplicative pole curve and is denoted by $(\widehat{P}')$.

The pole point $\widehat{P}$ is a moving point on the multiplicative Lorentzian moving plane L_*. Therefore, while tracing the pole curves $(\widehat{P})$ and $(\widehat{P}')$, the point $\widehat{P}$ possesses a velocity along each of these curves.

Theorem 3.1.5. At each instant t, the rotating pole point $\widehat{P}$ possesses identical velocities on the pole curves $(\widehat{P})$ and $(\widehat{P}')$ of the multiplicative Lorentzian fixed and moving planes. Under the assumption that the common tangent direction is not null, the two pole curves are tangent to one another at every instant of the motion.

The coincidence of the tangent directions of the moving and fixed pole curves indicates that the multiplicative Lorentzian pole curves roll on one another without multiplicative slipping at the instantaneous pole

Proof: Let $\overrightarrow{V_r}$ denote the tracing velocity of point $\widehat{X}$ along the curve associated with the pole $(\widehat{P})$ in the moving plane. Similarly, let $\overrightarrow{V_a}$ be the tracing velocity of the same point along the curve associated with the pole $(\widehat{P}')$ in the multiplicative Lorentzian fixed plane. Under the condition $\overrightarrow{V_f} = 1$, it follows that $\overrightarrow{V_a} = \overrightarrow{V_r}$.

Accelerations and Their Composition

Let $B_1 = {}^{L_*}/_{L'_*}$ denote the motion of the multiplicative Lorentzian plane L_* relative to the fixed multiplicative Lorentzian plane L'_*. Consider a point $\widehat{X}$ that is moving in L_* and therefore also in L'_*. Having derived the multiplicative velocity equations for $\widehat{X}$, we now proceed to derive the corresponding multiplicative acceleration equations.

Definition 3.3.1. The absolute acceleration vector of the point $\widehat{X}$ with respect to the fixed multiplicative Lorentzian plane L'_* is denoted by $\left(\overrightarrow{V_a}\right)^{\circ}$ Throughout this paper, this vector will be symbolized by $\vec{b}_a$.

Definition 3.3.2. The vector $\overrightarrow{b_r}$, defined as the multiplicative derivative of the relative velocity vector $\overrightarrow{V_r}$ of a point with respect to the moving multiplicative Lorentzian plane L_*, is referred to as the relative acceleration vector in L_*. In this work, it is denoted by $\overrightarrow{b_r}$.

Accordingly, this relative acceleration vector satisfies the following relation:

$$\overrightarrow{b_r} = \left(\overrightarrow{V_r}\right)^{\circ} = (e^{x_1})^{\circ\circ} \cdot_* \overrightarrow{\hat{1}_1} +_* (e^{x_2})^{\circ\circ} \cdot_* \overrightarrow{\hat{1}_2}. \quad (17)$$

Now, let us take the multiplicative derivative of $\overrightarrow{V_a}$

$$\vec{b}_a = \left(\overrightarrow{V_a}\right)^{\circ} = \left(\overrightarrow{V_f} +_* \overrightarrow{V_r}\right)^{\circ}$$
$$= \left\{ -_* e^{p_2^{\bullet}} \cdot_* e^{\theta^{\bullet}} +_* M \cdot_* e^{(\theta^{\bullet})^2} +_* N \cdot_* e^{\theta^{\bullet\bullet}} \right\} \overrightarrow{\hat{1}_1}$$

$$+_*\{-_* e^{p_1^\bullet} \cdot_* e^{\theta^\bullet} +_* N \cdot_* e^{(\theta^\bullet)^2} +_* M \cdot_* e^{\theta^{\bullet\bullet}}\} \cdot_* \overrightarrow{\hat{I}_2}$$

$$+_* e^{2\theta^\bullet} \cdot_* (e^{x_1^\bullet} \cdot_* \overrightarrow{\hat{I}_2} +_* e^{x_2^\bullet} \cdot_* \overrightarrow{\hat{I}_1}) +_* (e^{x_1})^{\circ\circ} \cdot_* \overrightarrow{\hat{I}_1} +_* (e^{x_2})^{\circ\circ} \cdot_* \overrightarrow{\hat{I}_2}.$$

here, the relation

$$\overrightarrow{\hat{b}_c} = e^{2\theta^\bullet} \cdot_* (e^{x_2^\bullet} \cdot_* \overrightarrow{\hat{I}_1} +_* e^{x_1^\bullet} \cdot_* \overrightarrow{\hat{I}_2})$$

is called the Coriolis acceleration vector of the motion, while the relation

$$\overrightarrow{\hat{b}_f} = \{-_* e^{p_2^\bullet} \cdot_* e^{\theta^\bullet} +_* M \cdot_* e^{(\theta^\bullet)^2} +_* N \cdot_* e^{\theta^{\bullet\bullet}}\} \cdot_* \overrightarrow{\hat{I}_1}$$
$$+_*\{-_* e^{p_1^\bullet} \cdot_* e^{\theta^\bullet} +_* N \cdot_* e^{(\theta^\bullet)^2} +_* M \cdot_* e^{\theta^{\bullet\bullet}}\} \cdot_* \overrightarrow{\hat{I}_2}$$

is referred to as the sliding acceleration of the motion.

Moreover, the relation $\overrightarrow{\hat{b}_r} = (e^{x_1})^{\circ\circ} \cdot_* \overrightarrow{\hat{I}_1} +_* (e^{x_2})^{\circ\circ} \cdot_* \overrightarrow{\hat{I}_2}$ was previously defined as the relative acceleration vector. Here $M = (e^{x_1} -_* e^{p_1})$ and $N = (e^{x_2} -_* e^{p_2})$

Theorem 3.3.1. Consider a point $\hat{X}$ whose motion is defined on the plane L_* as a function of the parameter t the multiplicative components of its acceleration vector are governed by the following relation:

$$\overrightarrow{\hat{b}_a} = \overrightarrow{\hat{b}_f} +_* \overrightarrow{\hat{b}_c} +_* \overrightarrow{\hat{b}_r}.$$

Corollary 3.3.1. If the point remains fixed in L_*, then the sliding acceleration vector of $\overrightarrow{\hat{b}_f}$ is equal to the absolute acceleration vector of $\overrightarrow{\hat{b}_a}$. In simple terms,

$$\overrightarrow{\hat{b}_a} = \overrightarrow{\hat{b}_f}.$$

Proof: It is known that

$$\overrightarrow{\hat{b}_a} = \{-_* e^{p_2^\bullet} \cdot_* e^{\theta^\bullet} +_* M \cdot_* e^{(\theta^\bullet)^2} +_* N \cdot_* e^{\theta^{\bullet\bullet}}\} \cdot_* \overrightarrow{\hat{I}_1}$$
$$+_*\{-_* e^{p_1^\bullet} \cdot_* e^{\theta^\bullet} +_* N \cdot_* e^{(\theta^\bullet)^2} +_* M \cdot_* e^{\theta^{\bullet\bullet}}\} \cdot_* \overrightarrow{\hat{I}_2}$$
$$+_* e^{2\theta^\bullet} \cdot_* (e^{x_1^\bullet} \cdot_* \overrightarrow{\hat{I}_2} +_* e^{x_2^\bullet} \cdot_* \overrightarrow{\hat{I}_1}) +_* e^{x_1^{\bullet\bullet}} \cdot_* \overrightarrow{\hat{I}_1} +_* e^{x_2^{\bullet\bullet}} \cdot_* \overrightarrow{\hat{I}_2}.$$

Because point $\hat{X}$ is stationary, it can be deduced that

$$\overrightarrow{\hat{b}_a} = \{-_* e^{p_2^\bullet} \cdot_* e^{\theta^\bullet} +_* M \cdot_* e^{(\theta^\bullet)^2} +_* N \cdot_* e^{\theta^{\bullet\bullet}}\} \cdot_* \overrightarrow{\hat{I}_1}$$
$$+_* \left\{-_* e^{p_1^\bullet} \cdot_* e^{\theta^\bullet} +_* N \cdot_* e^{\theta^{\bullet 2}} +_* M \cdot_* e^{\theta^{\bullet\bullet}}\right\} \cdot_* \overrightarrow{\hat{I}_2} = \overrightarrow{\hat{b}_f}.$$

Here $M = (e^{x_1} -_* e^{p_1})$ and $N = (e^{x_2} -_* e^{p_2})$

as demonstrated,

$$\overrightarrow{\hat{b}_a} = \overrightarrow{\hat{b}_f}$$

Theorem3.3.2. The Coriolis acceleration vector associated with $\overrightarrow{\hat{b}_c}$ is orthogonal to the relative velocity vector $\overrightarrow{\hat{V}_r}$ in the multiplicative Lorentzian inner–product sense.

Proof:

$$\langle \overrightarrow{\hat{b}_c}, \overrightarrow{\hat{V}_r} \rangle_*$$
$$= \langle (e^{2\theta^\bullet} \cdot_* e^{x_2^\bullet}) \cdot_* \overrightarrow{\hat{I}_1} +_* (e^{2\theta^\bullet} \cdot_* e^{x_1^\bullet}) \cdot_* \overrightarrow{\hat{I}_2}), \left(e^{x_1^\bullet} \cdot_* \overrightarrow{\hat{I}_1} +_* e^{x_2^\bullet} \cdot_* \overrightarrow{\hat{I}_2}\right)\rangle_*$$
$$= (e^{2\theta^\bullet} \cdot_* e^{x_2^\bullet}) \cdot_* (e^{x_1^\bullet}) -_* (e^{2\theta^\bullet} \cdot_* e^{x_1^\bullet}) \cdot_* (e^{x_2^\bullet})$$
$$= e^{2\theta^\bullet} \cdot_* (e^{x_2^\bullet x_1^\bullet - x_1^\bullet x_2^\bullet})$$
$$= e^{2\theta^\bullet} \cdot_* e^{0}$$
$$= e^{0}$$
$$= 1$$

We now consider a general $B_1 = {}^{L_*}/_{L'_*}$ type motion and examine the points at time t where the drag acceleration becomes zero in a multiplicative sense. From the condition $\overrightarrow{\hat{b}_f} = 1$, we obtain the following results.

$$\left.\begin{aligned} M \cdot_* e^{(\theta^\bullet)^2} +_* N \cdot_* e^{\theta^{\bullet\bullet}} &= e^{\theta^\bullet} \cdot_* e^{p_2^\bullet} \\ M \cdot_* e^{\theta^{\bullet\bullet}} +_* N \cdot_* e^{(\theta^\bullet)^2} &= e^{\theta^\bullet} \cdot_* p_1^\bullet \end{aligned}\right\} \quad (18)$$

When $e^{\theta^\bullet} \neq 1$, the resulting system of equations is non-homogeneous in terms of the variables $M = (e^{x_1} -_* e^{p_1})$ and $N = (e^{x_2} -_* e^{p_2})$. A necessary condition for the solvability of Equation (18) is that the determinant of its coefficient matrix satisfies $\Delta \neq 1$.

$$\Delta = \begin{vmatrix} e^{(\theta^\bullet)^2} & e^{\theta^{\bullet\bullet}} \\ e^{\theta^{\bullet\bullet}} & e^{(\theta^\bullet)^2} \end{vmatrix}_*$$
$$= e^{(\theta^\bullet(t))^4 -_* (\theta^{\bullet\bullet}(t))^2} \neq 1.$$

Because the condition $(\theta^\bullet(t))^4 \neq (\theta^{\bullet\bullet}(t))^2$, holds for all t,the corresponding system has a unique solution.

$$M = (e^{x_1} -_* e^{p_1}) = \begin{vmatrix} e^{\theta^\bullet} \cdot_* e^{p_2^\bullet} & e^{\theta^{\bullet\bullet}} \\ e^{\theta^\bullet} \cdot_* e^{p_1^\bullet} & e^{(\theta^\bullet)^2} \end{vmatrix}_*$$
$$= e^{\theta^\bullet} \cdot_* \left(e^{p_2^\bullet} \cdot_* e^{(\theta^\bullet)^2} -_* e^{p_1^\bullet} \cdot_* e^{\theta^{\bullet\bullet}}\right) /_* \Delta$$
$$N = (e^{x_2} -_* e^{p_2}) = \begin{vmatrix} e^{(\theta^\bullet)^2} & e^{\theta^\bullet} \cdot_* e^{p_2^\bullet} \\ e^{\theta^{\bullet\bullet}} & e^{\theta^\bullet} \cdot_* e^{p_1^\bullet} \end{vmatrix}_*$$
$$= e^{\theta^\bullet} \cdot_* \left(e^{p_1^\bullet} \cdot_* e^{(\theta^\bullet)^2} -_* e^{p_2^\bullet} \cdot_* e^{\theta^{\bullet\bullet}}\right) /_* \Delta$$

hence, the coordinates of the acceleration pole can be expressed as

$$e^{x_1} = e^{q_1} = e^{p_1} +_* \{e^{\theta^\bullet} \cdot_* \left(e^{p_2^\bullet} \cdot_* e^{(\theta^\bullet)^2} -_* e^{p_1^\bullet} \cdot_* e^{\theta^{\bullet\bullet}}\right) /_* \Delta\}$$
$$e^{x_2} = e^{q_2} = e^{p_2} +_* \{e^{\theta^\bullet} \cdot_* \left(e^{p_1^\bullet} \cdot_* e^{(\theta^\bullet)^2} -_* e^{p_2^\bullet} \cdot_* e^{\theta^{\bullet\bullet}}\right) /_* \Delta\}.$$

Example 3.3.1.
Let $\theta(t) = t$, $a_1(t) = 1$ and $a_2(t) = t$. Then, the coordinates of the pole point are obtained as

$$e^{p_1} = e^1 +_* e^1 /_* e^1 = e^2$$

and

$$e^{p_2} = e^t +_* e^0 /_* e^1 = e^t$$

Therefore, the pole point associated with the motion

$B_1 = {}^{L_*}/_{L'_*}$ is $\hat{P} = (e^{p_1}, e^{p_2}) = (e^2, e^t)$.

Next, we determine the coordinates of the acceleration pole. Substituting the given functions into the acceleration-pole formulas, we obtain

$$e^{q_1} = e^2 +_* \{e^1 \cdot_* (e^1 \cdot_* e^1 -_* e^0 \cdot_* e^0) /_* e^1\} = e^3$$

And

$$e^{q_2} = e^t +_* \{e^1 \cdot_* (e^0 \cdot_* e^1 -_* e^1 \cdot_* e^0) /_* e^1\} = e^t.$$

Hence, the coordinates of the acceleration pole are given by

$\hat{Q} == (e^{q_1}, e^{q_2}) = (e^3, e^t)$.

Moving Coordinate System

In this section, we consider a multiplicative Lorentzian plane A moving with respect to the multiplicative Lorentzian planes L_* and L_*'. We consider three multiplicative Lorentzian planes, denoted by L_*, L_*' and A, where one of them is fixed while the other two are in motion. Accordingly, we investigate the motion of the multiplicative Lorentzian orthogonal frame $\{\hat{B}; \overrightarrow{\hat{a}_1}, \overrightarrow{\hat{a}_2}\}$ associated with the plane (A) relative to the coordinate frames $\{\hat{O}'; \overrightarrow{\hat{l}'_1}, \overrightarrow{\hat{l}'_2}\}$ and $\{\hat{O}; \overrightarrow{\hat{l}_1}, \overrightarrow{\hat{l}_2}\}$.

$$\left.\begin{aligned} \overrightarrow{\hat{a}_1} &= \cosh_*\theta \cdot_* \overrightarrow{\hat{l}_1} +_* \sinh_*\theta \cdot_* \overrightarrow{\hat{l}_2} \\ \overrightarrow{\hat{a}_2} &= \sinh_*\theta \cdot_* \overrightarrow{\hat{l}_1} +_* \cosh_*\theta \cdot_* \overrightarrow{\hat{l}_2} \end{aligned}\right\}$$

and

$$\overrightarrow{\hat{O}\hat{B}} = \vec{\hat{b}} = e^{b_1} \cdot_* \overrightarrow{\hat{a}_1} +_* e^{b_2} \cdot_* \overrightarrow{\hat{a}_2}, \; e^{b_1}, e^{b_2} \in \mathbb{R}_* .$$

where, $\vec{\hat{b}} = (e^{b_1}, e^{b_2}) \in \mathbb{R}^2_*$,

Taking the multiplicative differential of $\overrightarrow{\hat{a}_1}$ we obtain

$$\begin{aligned} d_*\overrightarrow{\hat{a}_1} &= (e^{\sinh\theta d\theta} \cdot_* \overrightarrow{\hat{l}_1} +_* e^{\cosh\theta d\theta} \cdot_* \overrightarrow{\hat{l}_2}) \\ &= (e^{\sinh\theta d\theta} \cdot_* \overrightarrow{\hat{l}_1} +_* e^{\cosh\theta d\theta} \cdot_* \overrightarrow{\hat{l}_2}) \\ &= (e^{\sinh\theta} \cdot_* \overrightarrow{\hat{l}_1} +_* e^{\cosh\theta} \cdot_* \overrightarrow{\hat{l}_2}) \cdot_* e^{d\theta} \\ &= (\sinh_*\theta \cdot_* \overrightarrow{\hat{l}_1} +_* \cosh_*\theta \cdot_* \overrightarrow{\hat{l}_2}) \cdot_* e^{d\theta} \end{aligned}$$

Hence,

$$d_*(\overrightarrow{\hat{a}_1}) = \overrightarrow{\hat{a}_2} \cdot_* e^{d\theta} \tag{19}$$

A similar argument yields

$$d_*(\overrightarrow{\hat{a}_2}) = \overrightarrow{\hat{a}_1} \cdot_* e^{d\theta} \tag{20}$$

Taking the multiplicative differential of $\vec{\hat{b}}$, we have

$$\begin{aligned} d_*(\vec{\hat{b}}) = \; &\overrightarrow{\hat{a}_1} \cdot_* d_*(e^{b_1}) +_* e^{b_1} \cdot_* d_*(\overrightarrow{\hat{a}_1}) \\ &+_* \overrightarrow{\hat{a}_2} \cdot_* d_*(e^{b_2}) +_* e^{b_2} \cdot_* d_*(\overrightarrow{\hat{a}_2}) \end{aligned} \tag{21}$$

Substituting Eqs. (19) and (20) into Eq. (21), we obtain

$$\begin{aligned} d_*(\vec{\hat{b}}) &= \overrightarrow{\hat{a}_1} \cdot_* d_*(e^{b_1}) +_* \overrightarrow{\hat{a}_2} \cdot_* e^{b_1} \cdot_* e^{d\theta} \\ &\quad +_* \overrightarrow{\hat{a}_2} \cdot_* d_*(e^{b_2}) +_* \overrightarrow{\hat{a}_1} \cdot_* e^{b_2} \cdot_* e^{d\theta} \\ &= \overrightarrow{\hat{a}_1} \cdot_* (d_*(e^{b_1}) +_* e^{b_2} \cdot_* e^{d\theta}) \\ &\quad +_* \overrightarrow{\hat{a}_2} \cdot_* (d_*(e^{b_2}) +_* e^{b_1} \cdot_* e^{d\theta}) \end{aligned} \tag{22}$$

These relations represent the equations of motion of the plane A relative to L_*.
In a similar manner, the equations describing the motion of the plane A with respect to the plane L_*' can be expressed as

$$\left.\begin{aligned} \overrightarrow{\hat{a}_1} &= \cosh_*\theta' \cdot_* \overrightarrow{\hat{l}'_1} +_* \sinh_*\theta' \cdot_* \overrightarrow{\hat{l}'_2} \\ \overrightarrow{\hat{a}_2} &= \sinh_*\theta' \cdot_* \overrightarrow{\hat{l}'_1} +_* \cosh_*\theta' \cdot_* \overrightarrow{\hat{l}'_2} \end{aligned}\right\}$$

And

$$\overrightarrow{\hat{O}'\hat{B}} = \overrightarrow{\hat{b}'} = e^{b'_1} \cdot_* \overrightarrow{\hat{a}_1} +_* e^{b'_2} \cdot_* \overrightarrow{\hat{a}_2}.$$

In a similar manner, the motion equations of A relative to the stationary multiplicative Lorentzian plane L_*' are obtained as follows:

$$d'_*(\overrightarrow{\hat{a}_1}) = \overrightarrow{\hat{a}_2} \cdot_* e^{d\theta'} \tag{22}$$

$$d'_*(\overrightarrow{\hat{a}_2}) = \overrightarrow{\hat{a}_1} \cdot_* e^{d\theta'} \tag{23}$$

$$\begin{aligned} d'_*(\overrightarrow{\hat{b}'}) = \; &\overrightarrow{\hat{a}_1} \cdot_* (d_*(e^{b'_1}) +_* e^{b'_2} \cdot_* e^{d\theta'}) \\ &+_* \overrightarrow{\hat{a}_2} \cdot_* (d_*(e^{b'_2}) +_* e^{b'_1} \cdot_* e^{d\theta'}) \end{aligned} \tag{24}$$

For convenience, we adopt the notation

$$\begin{aligned} &e^{d\theta} = k \\ &e^{d\theta'} = k', \quad k, k' \in \mathbb{R}_* \\ &d_*(e^{b_1}) +_* e^{b_2} \cdot_* e^{d\theta} = k_1, \\ &d_*(e^{b_2}) +_* e^{b_1} \cdot_* e^{d\theta} = k_2, \quad k_1, k_2 \in \mathbb{R}_* \\ &d_*(e^{b'_1}) +_* e^{b'_2} \cdot_* e^{d\theta'}) = k'_1 \\ &d_*(e^{b'_2}) +_* e^{b'_1} \cdot_* e^{d\theta'} = k'_2, \quad k'_1, k'_2 \in \mathbb{R}_* \end{aligned}$$

Furthermore, to simplify the notation, the multiplicative differential of B with respect to L_*' will be denoted by

$$d'_*(\vec{\hat{b}})$$

instead of $d'_*(\overrightarrow{\hat{b}'})$.

Definition 4.1. For $1 \le j \le 2$ the quantities k_j, k'_j, k and k' are called the Pfaffian forms in the multiplicative

Lorentzian sense associated with a one-parameter multiplicative Lorentzian motion.

By means of these Pfaffian forms, the multiplicative differential equations describing the motion of A with respect to the plane L_* can be written in the compact form

$$\left.\begin{aligned} d_*\overrightarrow{\hat{a}_1} &= \overrightarrow{\hat{a}_2}\cdot_* k \\ d_*\overrightarrow{\hat{a}_2} &= \overrightarrow{\hat{a}_1}\cdot_* k \\ d_*\vec{\hat{b}} &= \overrightarrow{\hat{a}_1}\cdot_* k_1 +_* \overrightarrow{\hat{a}_2}\cdot_* k_2 \end{aligned}\right\} \quad (25)$$

Similarly, the derivative equations describing the motion of A with respect to the plane L_*' are obtained as

$$\left.\begin{aligned} d_*'(\overrightarrow{\hat{a}_1}) &= \overrightarrow{\hat{a}_2}\cdot_* k' \\ d_*'(\overrightarrow{\hat{a}_2}) &= \overrightarrow{\hat{a}_1}\cdot_* k' \\ d_*'(\vec{\hat{b}}) &= \overrightarrow{\hat{a}_1}\cdot_* k' +_* \overrightarrow{\hat{a}_2}\cdot_* k' \end{aligned}\right\} \quad (26)$$

Next, consider a point $\hat{X} = (e^{x_1}, e^{x_2})$in plane A. Then,

$$\overrightarrow{\hat{B}\hat{X}} = \vec{\hat{b}} = e^{x_1}\cdot_* \overrightarrow{\hat{a}_1} +_* e^{x_2}\cdot_* \overrightarrow{\hat{a}_2}$$
$$\vec{\hat{x}} = \overrightarrow{\hat{O}\hat{X}} = \overrightarrow{\hat{O}\hat{X}} + \overrightarrow{\hat{B}\hat{X}} = \vec{\hat{b}} +_* e^{x_1}\cdot_* \overrightarrow{\hat{a}_1} +_* e^{x_2}\cdot_* \overrightarrow{\hat{a}_2}$$
$$\vec{\hat{x}}' = \overrightarrow{\hat{O}'\hat{X}} = \overrightarrow{\hat{O}'\hat{B}} + \overrightarrow{\hat{B}\hat{X}} = \vec{\hat{b}}' +_* e^{x_1}\cdot_* \overrightarrow{\hat{a}_1} +_* e^{x_2}\cdot_* \overrightarrow{\hat{a}_2}.$$

Using the multiplicative differential relations given in equation (25), the

variation of $\hat{X}$ with respect to L_* is obtained as
$\hat{X}$ in L_*

$$\begin{aligned} d_*(\vec{\hat{x}}) &= (d_*(e^b) +_* d_*(\overrightarrow{\hat{a}_1})\cdot_* e^{x_1} +_* \overrightarrow{\hat{a}_1}\cdot_* d_*(e^{x_1}) \\ &\quad +_* d_*(\overrightarrow{\hat{a}_2})\cdot_* e^{x_2} +_* \overrightarrow{\hat{a}_2}\cdot_* d_*(e^{x_2}) \\ &= d_*(e^b) +_* \overrightarrow{\hat{a}_2}\cdot_* k\cdot_* e^{x_1} +_* \overrightarrow{\hat{a}_1}\cdot_* d_*(e^{x_1}) \\ &\quad +_* \overrightarrow{\hat{a}_1}\cdot_* k\cdot_* e^{x_2} +_* \overrightarrow{\hat{a}_2}\cdot_* d_*(e^{x_2}) \\ &= \overrightarrow{\hat{a}_1}\cdot_* k_1 +_* \overrightarrow{\hat{a}_1}\cdot_* k_2 +_* \overrightarrow{\hat{a}_2}\cdot_* k\cdot_* e^{x_1} \\ &\quad +_* \overrightarrow{\hat{a}_1}\cdot_* d_*(e^{x_1}) +_* \overrightarrow{\hat{a}_1}\cdot_* k\cdot_* e^{x_2} +_* \overrightarrow{\hat{a}_2}\cdot_* d_*(e^{x_1}) \\ &= \overrightarrow{\hat{a}_1}\cdot_* (d_*(e^{x_1}) +_* k_1 +_* e^{x_2}\cdot_* k) \\ &\quad +_* \overrightarrow{\hat{a}_2}\cdot_* (d_*(e^{x_2}) +_* k_2 +_* e^{x_1}\cdot_* k) \end{aligned} \quad (27)$$

Accordingly, the relative velocity vector is given by

$$\overrightarrow{\hat{V}_r} = d_*(\vec{\hat{x}}) /_* d_* t$$

When $\overrightarrow{\hat{V}_r} = 1$, the point $\hat{X}$ is stationary with respect to L_*. Hence, the condition characterizing the fixed points of L_* can be obtained. Indeed, if $\overrightarrow{\hat{V}_r} = 1$ then it follows that $d_*(\vec{\hat{x}}) = 1$.

Hence,

$$\left.\begin{aligned} d_*(e^{x_1}) +_* k_1 +_* e^{x_2}\cdot_* k = 1 \\ (d_*(e^{x_2}) +_* k_2 +_* e^{x_1}\cdot_* k = 1 \end{aligned}\right\}$$

which, after simplification, yields

$$\left.\begin{aligned} d_*(e^{x_1}) &= -_* k_1 -_* e^{x_2}\cdot_* k \\ (d_*(e^{x_2}) &= -_* k_2 -_* e^{x_1}\cdot_* k \end{aligned}\right\} \quad (28)$$

Equation (28) gives the necessary and sufficient condition for the point $\hat{X}$ to be fixed in L_*. Similarly, the variation of $\hat{X}$ relative to L_*' can be multiplicative differential. To simplify the notation, we write $d'_*(\vec{\hat{x}})$ instead of $d_*(\vec{\hat{x}'})$

Hence, $d'_*(\vec{\hat{x}})$ can be expressed as

$$\begin{aligned} d'_*(\vec{\hat{x}}) &= \overrightarrow{\hat{a}_1}\cdot_* (d_*(e^{x_1}) +_* k_1' +_* e^{x_2}\cdot_* k') \\ &\quad +_* \overrightarrow{\hat{a}_2}\cdot_* (d_*(e^{x_2}) +_* k_2' +_* e^{x_1}\cdot_* k). \end{aligned} \quad (29)$$

From this, the absolute velocity vector is given by

$$\overrightarrow{\hat{V}_a} = d'_* \hat{x} /_* d_* t.$$

If $\overrightarrow{\hat{V}_a} = 1$ then the point $\hat{X}$ is stationary with respect to L_*'. In other words

$$\overrightarrow{\hat{V}_a} = 1 \text{ then } d'_* \hat{x} = 1.$$

Therefore, the following relations are obtained

$$\left.\begin{aligned} d_*(e^{x_1}) &= -_* k_1' -_* e^{x_2}\cdot_* k' \\ (d_*(e^{x_2}) &= -_* k_2' -_* e^{x_1}\cdot_* k' \end{aligned}\right\}. \quad (30)$$

Equation (30) gives the necessary condition for the point $\hat{X}$ to be stationary in L_*'.

If the point $\hat{X}$ remains stationary in L_*, then the vector

$$\overrightarrow{\hat{V}_f} = d_{*f} \hat{x} /_* d_* t$$

is called the sliding velocity. Here, $d_{*f}\hat{x}$ expresses the multiplicative change of the point $\hat{X}$ relative to the frame L_*'.

Substituting the relations in (28) into (29), is obtained as

$$\begin{aligned} d_{*f}\hat{x} &= \overrightarrow{\hat{a}_1}\cdot_* (-_* k_1 -_* e^{x_2}\cdot_* k +_* k_1' +_* e^{x_2}\cdot_* k') \\ &\quad +_* \overrightarrow{\hat{a}_2}\cdot_* (-_* k_2 -_* e^{x_1}\cdot_* k +_* k_2' +_* e^{x_1}\cdot_* k') \\ &= \overrightarrow{\hat{a}_1}\cdot_* ((k_1' -_* k_1) -_* e^{x_2}\cdot_* (k -_* k')) \\ &\quad +_* \overrightarrow{\hat{a}_2}\cdot_* ((k_2' -_* k_2) -_* e^{x_1}\cdot_* (k -_* k')). \end{aligned} \quad (31)$$

Theorem 3.4.1. When the point $\hat{X}$ is fixed in the multiplicative Lorentz plane L_*, the vector $d_{*f}\hat{x}$ is obtained. Adding this vector to the vector $d_*(\vec{x})$ given in Equation (27) yields the vector $d'_*(\vec{x})$.

Proof:

$$\begin{aligned} d_{*_f}\hat{x} +_* d_*(\vec{x}) &= \overrightarrow{\hat{a}_1} \cdot_* \{(k'_1 -_* k_1) -_* e^{x_2} \cdot_* (k -_* k')\} \\ &\quad +_* \overrightarrow{\hat{a}_2} \cdot_* \{(k'_2 -_* k_2) -_* e^{x_1} \cdot_* (k -_* k')\} \\ &\quad +_* \overrightarrow{\hat{a}_1} \cdot_* (d_*(e^{x_1}) +_* k_1 +_* e^{x_2} \cdot_* k) \\ &\quad +_* \overrightarrow{\hat{a}_2} \cdot_* (d_*(e^{x_2}) +_* k_2 +_* e^{x_1} \cdot_* k) \\ &= \overrightarrow{\hat{a}_1} \cdot_* \{(k'_1 -_* k_1 -_* e^{x_2} \cdot_* k +_* e^{x_2} \cdot_* k' +_* d_*(e^{x_1}) +_* k_1 +_* e^{x_2} \cdot_* k)\} \\ &\quad +_* \overrightarrow{\hat{a}_2} \cdot_* \{k'_2 -_* k_2 -_* e^{x_1} \cdot_* k +_* e^{x_1} \cdot_* k' +_* d_*(e^{x_2}) +_* k_2 +_* e^{x_1} \cdot_* k\} \\ &= \overrightarrow{\hat{a}_1} \cdot_* \{d_*(e^{x_1}) +_* k'_1 +_* e^{x_2} \cdot_* k'\} \\ &\quad +_* \overrightarrow{\hat{a}_2} \cdot_* \{d_*(e^{x_2}) +_* k'_2 +_* e^{x_1} \cdot_* k'\} \\ &= d'_*(\vec{x}). \end{aligned}$$

Determination of the Rotation Pole

The rotation pole $\hat{P}$ is characterized by a sliding velocity of 1, where the absolute velocity equals the relative velocity. Accordingly, by taking

$$d_{*_f}\hat{x} = 1$$

and letting $\overrightarrow{\hat{B}\hat{P}} = e^{p_1} \cdot_* \overrightarrow{\hat{a}_1} +_* e^{p_2} \cdot_* \overrightarrow{\hat{a}_2}$

equation (31) yields the following system:

$$\left.\begin{aligned} (k'_1 -_* k_1) -_* e^{x_2} \cdot_* (k -_* k') = 1 \\ (k'_2 -_* k_2) -_* e^{x_1} \cdot_* (k -_* k') = 1 \end{aligned}\right\} \quad (32)$$

Solving Equation (32) yields the coordinates of the pole point as

$$\left.\begin{aligned} e^{x_1} = e^{p_1} = (k'_2 -_* k_2) /_* (k -_* k') \\ e^{x_2} = e^{p_2} = (k'_1 -_* k_1) /_* (k -_* k') \end{aligned}\right\} \quad (33)$$

which represent the coordinates of the pole point.

CONCLUSIONS

In this study, one-parameter multiplicative Lorentzian motions in the multiplicative Lorentzian plane were investigated by means of multiplicative calculus. Fundamental kinematic concepts, including velocity components, the law of velocities, and the relationships among different velocity vectors, were established. Furthermore, the acceleration components of the motion were derived, and the relations among absolute, relative, sliding, and Coriolis accelerations were obtained. In addition, several definitions and theorems concerning multiplicative Lorentzian motions, pole points, and acceleration poles were presented. To extend the kinematic framework, a multiplicative Lorentzian moving frame was introduced and its differential equations were derived. The associated multiplicative Pfaffian forms were defined, providing a compact representation of the motion and a convenient tool for expressing the differential relations of moving systems. Moreover, the relative motions among three multiplicative Lorentzian planes were examined, thereby extending the classical moving-frame methodology to the multiplicative Lorentzian setting.

REFERENCES

[1] Grossman M., Katz R., Non-Newtonian calculus, Lee Press, Piegon Cove, Massachusetts, 1972.

[2] Stanley D. A., Multiplicative calculus, Primus IX, 4, 310, 1999.

[3] Campbell D., Multiplicative calculus and student projects, PRIMUS, Vol.9,Issue 4,327-332,2007. https://doi.org/10.1080/10511979908965938

[4] Grossman M., Bigeometric calculus: A system with a scale-free Derivative, Archimedes Foundation, Massachusetts, 1983.

[5] Grossman M., An introduction to non-Newtonian calculus, Int. J. Math. Educ. Sci. Technol. Vol.10, No:(4), 525-528, 1979.

[6] Grossman J.,Grossman M., Katz R., The first systems of weighted differential and integral calculus, University of Michigan, 1981.

[7] Grossman J., Meta-Calculus: Differential and Integral, University of Michigan, 1981.

[8] .Bashirov A.E, Kurpınar E. M., Ozyapici A., Multiplicative calculus and its applications, J. Math. Anal. Appl. 337, 36, 2008.

[9] Bashirov A.E., Rıza M., On complex multiplicative differentiation, TWMS J. App. Eng. Math. 1, 85, 2011.

[10] Bashirov A.E, .Mısırlı E., Tandoğdu Y., Ozyapıcı A.,On modeling with multiplicative differential equations, Appl. Math. J. Chinese Univ. 26(4), 425, 2011.

[11] Tekin S., Başar F., Certain sequence spaces over the non-Newtonian complex field, Abstr. Appl. Anal, Article ID 739319, 11 pages, 2013.

[12] Türkmen C., Başar F., Some basic results on the sets of sequences with geometric calculus, Commun. Fac. Fci. Univ. Ank. Series A1. 61(2), 17, 2012.

[13] Uzer A., Multiplicative type complex calculus as an alternative to the classical calculus, Comput. Math. Appl. 60, 2725, 2010.

[14] Çakmak A.F. Başar F., Some sequence spaces and matrix transformations in multiplicative sense, TWMS J. Pure Appl. Math. , 6(1), 27, 2015.

[15] Boruah K., Hazarika B., Some basic properties of G-calculus and its applications in numerical analysis, arXiv:1607.07749v1, 24 July 2016.

[16] Boruah K., Hazarika B., G- Calculus, TWMS J. App. Eng. Math. 8, 94, 2018.

[17] Boruah K., Hazarika B., Application of geometric calculus in numerical analysis and difference sequence spaces, arXiv:1603.09479v1, 2016.

[18] Çakmak A.F. Başar F., On classical sequence spaces and non-Newtonian calculus, J. Inequal, Appl. Art., ID 932734, 12pages. 2012.

[19] Gürefe Y., Multiplicative differential equations and its applications, Ph.D. in Department of Mathematics, 2013.

[20] Misirli E., Gürefe Y., Multiplicative Adams Bashforth–Moulton methods, Numer Algor, 57, 425, 2011.

[21] Aslan S., Bekar M., Yaylı Y., Geometric 3-space and multiplicative quaternions, International Journal 1 of Geometric Methods in Modern Physics, 20:9,2023.

[22] Nurkan S., Gürgil K.İ., Karacan M.K., Vector properties of geometric calculus, Math. Meth. Appl. Sci. 1–20,2023.

[23] Georgiev S.G., Zennir K., Multiplicative differential calculus (1st ed.), Chapman and Hall/CRC, New York, 2022.

[24] Es H., On The 1-Parameter motions With multiplicative calculus, Journal of Science and Arts, 2(59), 2022

[25] Es H., Plane kinematics in homothetic multiplicative calculus, Journal of Universal Mathematics, 7(1), 37-47, (2024).

[26] Es H., Plane kinematics in lorentzian homothetic multiplicative calculus, Journal of Universal Mathematics, 7(2), 64-74, (2024).

[27] Es H., Geometric Interpretation of multiplicative calculus in lorentzian space, Journal of Science and Arts, Vol.24, İssue 3, pp. 613-626, 2024.

[28] Has A., Yılmaz B., On non-Newtonian helices in multiplicativeeEuclidean space E3, Fundamentals of Contemporary Mathematical Sciences,Vol.6, Issue2, 196-217, 2025.

[29] Has A., Yılmaz B., A non-Newtonian conics in multiplicative analytic geometry, Turkish Journal of Mathematics, 48(5), 2024.

[30] Has A., Yılmaz B., Yildirim H., A non-Newtonian perspective on multiplicative lorentz- minkowski space L_*^3 , Math. Meth. Appl. Sci., 47:13875–13888, 2024.