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On a second-order evolution inclusion

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Abstract

We study a class of second-order evolution inclusions and we obtain a sufficient condition for f-local controllability along a reference trajectory.

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1. Introduction

In this note we are concerned with the following problem

$$x'' \in A(t)x + F(t,x), \quad x(0) \in X_0, \quad x'(0) \in X_1, \tag{1.1}$$

where $F : [0,T] \times X \to \mathcal{P}(X)$ is a set-valued map, X is a separable Banach space, $X_0, X_1 \subset X$ and $\{A(t)\}_{t\geq 0}$ is a family of linear closed operators from X into X that generates an evolution system of operators $\{\mathcal{U}(t,s)\}_{t,s\in[0,T]}$.

The general framework of evolution operators $\{A(t)\}_{t\geq 0}$ that define problem (1.1) has been developed by Kozak ([14]) and improved by Henriquez ([12]). In several recent papers ([2-5], [8-11]) existence results and qualitative properties of solutions for problem (1.1) have been obtained by using several techniques.

The aim of the present paper is to obtain a sufficient condition for f-local controllability of inclusion (1.1). We denote by S_F be the set of all mild solutions of (1.1) and by $R_F(T)$ the reachable set of (1.1). If $y(.) \in S_F$ is a mild solution and if $f : \mathbf{R}^n \to \mathbf{R}^m$ is a locally Lipschitz function then we say that the differential inclusion (1.1) is *f-locally controllable* around $y(.)$ if $h(y(T)) \in \mathrm{int}(f(R_F(T)))$. In particular, if f is the identity map the above definitions reduces to the usual concept of local controllability of systems around a solution.

The proof of our result is based on an approach of Tuan ([16]). More precisely, we prove that inclusion (1.1) is f-locally controllable around the solution $y(.)$ if a certain variational inclusion is h-locally controllable

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around the null solution for every $h \in \partial f(z(T))$, where $\partial f(.)$ denotes Clarke's generalized Jacobian of the locally Lipschitz function f. The main tools in the proof of our result is a continuous version of Filippov's theorem for mild solutions of problem (1.1) obtained in [8] and a certain generalization of the classical open mapping principle in [17].

We note that similar results for other classes of differential inclusions may be found in our previous papers [6,7].

The paper is organized as follows: in Section 2 we present some preliminary results to be used in the sequel and in Section 3 we present our main results.

2. Preliminaries

Let us denote by I the interval $[0,T]$ and let X be a real separable Banach space with the norm $|.|$ and with the corresponding metric $d(.,.)$. Denote by $\mathcal{L}(I)$ the σ-algebra of all Lebesgue measurable subsets of I, by $\mathcal{P}(X)$ the family of all nonempty subsets of X and by $\mathcal{B}(X)$ the family of all Borel subsets of X. Recall that the Pompeiu-Hausdorff distance of the closed subsets $A, B \subset X$ is defined by

$$d_H(A,B) = \max\{d^*(A,B), d^*(B,A)\}, \quad d^*(A,B) = \sup\{d(a,B); a \in A\},$$

where $d(x,B) = \inf_{y \in B} d(x,y)$.

As usual, we denote by $C(I,X)$ the Banach space of all continuous functions $x(.): I \to X$ endowed with the norm $||x(.)||_C = \sup_{t\in I}||x(t)||$, by $L^1(I,X)$ the Banach space of all (Bochner) integrable functions $x(.): I \to X$ endowed with the norm $||x(.)||_1 = \int_I ||x(t)||dt$ and by $B(X)$ the Banach space of linear bounded operators on X.

In what follows $\{A(t)\}_{t\geq 0}$ is a family of linear closed operators from X into X that generates an evolution system of operators $\{\mathcal{U}(t,s)\}_{t,s\in I}$. By hypothesis the domain of $A(t)$, $D(A(t))$ is dense in X and is independent of t.

Definition 2.1. ([12,14]) A family of bounded linear operators $\mathcal{U}(t,s) : X \to X$, $(t,s) \in \Delta := \{(t,s) \in I \times I; s \leq t\}$ is called an evolution operator of the equation

$$x''(t) = A(t)x(t) \tag{2.1}$$

if

i) For any $x \in X$, the map $(t,s) \to \mathcal{U}(t,s)x$ is continuously differentiable and

a) $\mathcal{U}(t,t) = 0$, $t \in I$.

b) If $t \in I, x \in X$ then $\frac{\partial}{\partial t}\mathcal{U}(t,s)x|_{t=s} = x$ and $\frac{\partial}{\partial s}\mathcal{U}(t,s)x|_{t=s} = -x$.

ii) If $(t,s) \in \Delta$, then $\frac{\partial}{\partial s}\mathcal{U}(t,s)x \in D(A(t))$, the map $(t,s) \to \mathcal{U}(t,s)x$ is of class C^2 and

a) $\frac{\partial^2}{\partial t^2}\mathcal{U}(t,s)x \equiv A(t)\mathcal{U}(t,s)x$.

b) $\frac{\partial^2}{\partial s^2}\mathcal{U}(t,s)x \equiv \mathcal{U}(t,s)A(t)x$.

c) $\frac{\partial^2}{\partial s \partial t}\mathcal{U}(t,s)x|_{t=s} = 0$.

iii) If $(t,s) \in \Delta$, then there exist $\frac{\partial^3}{\partial t^2 \partial s}\mathcal{U}(t,s)x$, $\frac{\partial^3}{\partial s^2 \partial t}\mathcal{U}(t,s)x$ and

a) $\frac{\partial^3}{\partial t^2 \partial s}\mathcal{U}(t,s)x \equiv A(t)\frac{\partial}{\partial s}\mathcal{U}(t,s)x$ and the map $(t,s) \to A(t)\frac{\partial}{\partial s}\mathcal{U}(t,s)x$ is continuous.

b) $\frac{\partial^3}{\partial s^2 \partial t}\mathcal{U}(t,s)x \equiv \frac{\partial}{\partial t}\mathcal{U}(t,s)A(s)x$.

As an example for equation (2.1) one may consider the problem (e.g., [12])

$$\frac{\partial^2 z}{\partial t^2}(t,\tau) = \frac{\partial^2 z}{\partial \tau^2}(t,\tau) + a(t)\frac{\partial z}{\partial t}(t,\tau), \quad t \in [0,T], \tau \in [0,2\pi],$$

$$z(t,0) = z(t,\pi) = 0, \quad \frac{\partial z}{\partial \tau}(t,0) = \frac{\partial z}{\partial \tau}(t,2\pi),\ t \in [0,T],$$

where $a(.) : I \to \mathbf{R}$ is a continuous function. This problem is modeled in the space $X = L^2(\mathbf{R}, \mathbf{C})$ of 2π-periodic 2-integrable functions from $\mathbf{R}$ to $\mathbf{C}$, $A_1 z = \frac{d^2 z(\tau)}{d\tau^2}$ with domain $H^2(\mathbf{R}, \mathbf{C})$ the Sobolev space of 2π-periodic functions whose derivatives belong to $L^2(\mathbf{R}, \mathbf{C})$. It is well known that$A_1$ is the infinitesimal generator of strongly continuous cosine functions $C(t)$ on X. Moreover, A_1 has discrete spectrum; namely the spectrum of A_1 consists of eigenvalues $-n^2$, $n \in \mathbf{Z}$ with associated eigenvectors $z_n(\tau) = \frac{1}{\sqrt{2\pi}} e^{in\tau}$, $n \in \mathbf{N}$. The set z_n, $n \in \mathbf{N}$ is an orthonormal basis of X. In particular, $A_1 z = \sum_{n\in\mathbf{Z}} -n^2 < z, z_n > z_n$, $z \in D(A_1)$. The cosine function is given by $C(t)z = \sum_{n\in\mathbf{Z}} \cos(nt) < z, z_n > z_n$ with the associated sine function $S(t)z = t < z, z_0 > z_0 + \sum_{n\in\mathbf{Z}^*} \frac{\sin(nt)}{n} < z, z_n > z_n$.

For $t \in I$ define the operator $A_2(t)z = a(t)\frac{dz(\tau)}{d\tau}$ with domain $D(A_2(t)) = H^1(\mathbf{R}, \mathbf{C})$. Set $A(t) = A_1 + A_2(t)$. It has been proved in [12] that this family generates an evolution operator as in Definition 1.

Definition 2.2. A continuous mapping $x(.) \in C(I, X)$ is called a mild solution of problem (1.1) if there exists a (Bochner) integrable function $f(.) \in L^1(I, X)$ such that

$$f(t) \in F(t, x(t)) \quad a.e.\,(I), \tag{2.2}$$

$$x(t) = -\frac{\partial}{\partial s}\mathcal{U}(t,0)x_0 + \mathcal{U}(t,0)y_0 + \int_0^t \mathcal{U}(t,s)f(s)ds,\ t \in I. \tag{2.3}$$

We shall call $(x(.), f(.))$ a *trajectory-selection pair* of (1.1) if $f(.)$ verifies (2.2) and $x(.)$ is defined by (2.3).

Hypothesis H1. i) $F(.,.) : I \times X \to \mathcal{P}(X)$ has nonempty closed values and is $\mathcal{L}(I) \otimes \mathcal{B}(X)$ measurable.

ii) There exists $l(.) \in L^1(I, \mathbf{R}_+)$ such that, for any $t \in I, F(t,.)$ is $l(t)$-Lipschitz in the sense that

$$d_H(F(t, x_1), F(t, x_2)) \le l(t)|x_1 - x_2| \quad \forall\, x_1, x_2 \in X.$$

Hypothesis H2. Let S be a separable metric space, $X_0, X_1 \subset X$ are closed sets, $a_0(.) : S \to X_0$, $a_1(.) : S \to X_1$ and $c(.) : S \to (0, \infty)$ are given continuous mappings.

The continuous mappings $g(.) : S \to L^1(I, X)$, $y(.) : S \to C(I, X)$ are given such that

$$(y(s))''(t) = A(t)y(s)(t) + g(s)(t), \quad y(s)(0) \in X_0, \quad (y(s))'(0) \in X_1.$$

and there exists a continuous function $q(.) : S \to L^1(I, \mathbf{R}_+)$ such that

$$d(g(s)(t), F(t, y(s)(t))) \le q(s)(t) \quad a.e.\,(I),\ \forall\, s \in S. \tag{2.4}$$

Theorem 2.3. *([10]) Assume that Hypotheses H1 and H2 are satisfied.*

Then there exist $M > 0$ and the continuous functions $x(.) : S \to L^1(I, X)$, $h(.) : S \to C(I, X)$ such that for any $s \in S$ $(x(s)(.), h(s)(.))$ is a trajectory-selection of (1.1) satisfying for any $(t, s) \in I \times S$

$$x(s)(0) = a_0(s), \quad (x(s))'(0) = a_1(s),$$

$$|x(s)(t) - y(s)(t)| \le M[c(s) + |a_0(s) - y(s)(0)| + |a_1(s) - (y(s))'(0)| + \int_0^t q(s)(u)du]. \tag{2.5}$$

In what follows we assume that $X = \mathbf{R}^n$.

A closed convex cone $C \subset \mathbf{R}^n$ is said to be *regular tangent cone* to the set X at $x \in X$ ([16]) if there exists continuous mappings $q_\lambda : C \cap B \to \mathbf{R}^n$, $\forall \lambda > 0$ satisfying

$$\lim_{\lambda\to 0+} \max_{v \in C\cap B} \frac{|q_\lambda(v)|}{\lambda} = 0,$$

$$x + \lambda v + q_\lambda(v) \in X \quad \forall \lambda > 0, v \in C \cap B,$$

where B is the closed unit ball in $\mathbf{R}^n$.

We recall, also, some well known intrinsic tangent cones in the literature (e.g. [1]); namely, the *contingent*, the *quasitangent* and *Clarke's tangent cones*, defines, respectively, by

$$\begin{aligned} &K_xX = \{v \in \mathbf{R}^n; \quad \exists\, s_m \to 0+,\; x_m \in X : \tfrac{x_m - x}{s_m} \to v\} \\ &Q_xX = \{v \in \mathbf{R}^n; \quad \forall\, s_m \to 0+, \exists x_m \in X : \tfrac{x_m - x}{s_m} \to v\} \\ &C_xX = \{v \in \mathbf{R}^n; \forall\, (x_m, s_m) \to (x, 0+),\; x_m \in X,\; \exists\, y_m \in X : \tfrac{y_m - x_m}{s_m} \to v\}. \end{aligned}$$

In is known that, unlike K_xX, Q_xX, the cone C_xX is convex and one has $C_xX \subset Q_xX \subset K_xX$.

The results in the next section will be expressed, in the case when the mapping $f(.) : X \subset \mathbf{R}^n \to \mathbf{R}^m$ is locally Lipschitz at x, in terms of the Clarke generalized Jacobian, defined by ([11])

$$\partial f(x) = \mathrm{co}\{\lim_{i\to\infty} f'(x_i); \quad x_i \to x, \quad x_i \in X \backslash J_f\},$$

where J_f is the set of points at which f is not differentiable.

Corresponding to each type of tangent cone, say τ_xX one may introduce (e.g. [1]) a *set-valued directional derivative* of a multifunction $G(.) : X \subset \mathbf{R}^n \to \mathcal{P}(\mathbf{R}^n)$ (in particular of a single-valued mapping) at a point $(x, y) \in \mathrm{graph}(G)$ as follows

$$\tau_yG(x; v) = \{w \in \mathbf{R}^n;\ (v, w) \in \tau_{(x,y)}\mathrm{graph}(G)\}, \quad \in \tau_xX.$$

We recall that a set-valued map, $A(.) : \mathbf{R}^n \to \mathcal{P}(\mathbf{R}^n)$ is said to be a *convex* (respectively, closed convex) *process* if $\mathrm{graph}(A(.)) \subset \mathbf{R}^n \times \mathbf{R}^n$ is a convex (respectively, closed convex) cone. For the basic properties of convex processes we refer to [1], but we shall use here only the above definition.

Hypothesis H3. i) *Hypothesis H1 is satisfied and $X_0, X_1 \subset \mathbf{R}^n$ are closed sets.*

ii) $(y(.), g(.)) \in C(I, \mathbf{R}^n) \times L^1(I, \mathbf{R}^n)$ *is a trajectory-selection pair of (1.1) and a family* $L(t, .) : \mathbf{R}^n \to \mathcal{P}(\mathbf{R}^n)$, $t \in I$ *of convex processes satisfying the condition*

$$L(t, u) \subset Q_{g(t)}F(t, .)(y(t); u) \quad \forall\, u \in dom(P(t, .)),\ a.e.\ t \in I \tag{2.6}$$

is assumed to be given.

The family of convex processes in Hypothesis H3 defines the variational inclusion

$$v'' \in A(t)v + L(t, v). \tag{2.7}$$

Remark 2.4. We point out that Hypothesis H3 is not restrictive, since for any set-valued map $F(., .)$, one may find an infinite number of families of convex processes $L(t, .)$, $t \in I$, satisfying condition (2.6). Any family of closed convex subcones of the quasitangent cones, $\overline{L}(t) \subset Q_{(y(t),g(t))}graph(F(t, .))$, defines the family of closed convex processes

$$L(t, u) = \{v \in \mathbf{R}^n;\ (u, v) \in \overline{L}(t)\}, \quad u, v \in \mathbf{R}^n,\ t \in I$$

that satisfy condition (2.6). For example one may take an "intrinsic" family of such closed convex process given by Clarke's convex-valued directional derivatives $C_{g(t)}F(t, .)(y(t); .)$.

Since $F(t, .)$ is assumed to be Lipschitz a.e. on I, the quasitangent directional derivative is given by ([1])

$$Q_{g(t)}F(t, .)((y(t); u)) = \{w \in \mathbf{R}^n;\ \lim_{\theta\to 0+} \frac{1}{\theta} d(g(t) + \theta w, F(t, y(t) + \theta u)) = 0\}. \tag{2.8}$$

In what follows $B_{\mathbf{R}^n}$ denotes the closed unit ball in $\mathbf{R}^n$ and 0_n denotes the null element in $\mathbf{R}^n$. Consider $f : \mathbf{R}^n \to \mathbf{R}^m$ an arbitrary given function.

Definition 2.5. Differential inclusion (1.1) is said to be *f-locally controllable* around $y(.)$ if $f(y(T)) \in \mathrm{int}(f(R_F(T)))$.

In particular, differential inclusion (1.1) is said to be *locally controllable* around the solution $y(.)$ if $y(T) \in \mathrm{int}(R_F(T))$.

Finally a key tool in the proof of our results is the following generalization of the classical open mapping principle due to Warga ([17]).

For $k \in \mathbf{N}$ we define

$$\Sigma_k := \{\beta = (\beta_1, ..., \beta_k);\quad \sum_{i=1}^{k} \beta_i \leq 1,\quad \beta_i \geq 0,\ i = 1, 2, ..., k\}.$$

Lemma 2.6. *([17]) Let $\delta \leq 1$, let $g(.) : \mathbf{R}^n \to \mathbf{R}^m$ be a mapping that is C^1 in a neighborhood of 0_n containing $\delta B_{\mathbf{R}^n}$. Assume that there exists $\beta > 0$ such that for every $\theta \in \delta\Sigma_n$, $\beta B_{\mathbf{R}^m} \subset g'(\theta)\Sigma_n$. Then, for any continuous mapping $\varphi : \delta\Sigma_n \to \mathbf{R}^m$ that satisfies $\sup_{\theta \in \delta\Sigma_n} |g(\theta) - \varphi(\theta)| \leq \frac{\delta\beta}{32}$ we have $\varphi(0_n) + \frac{\delta\beta}{16} B_{\mathbf{R}^m} \subset \varphi(\delta\Sigma_n)$.*

3. The main result

In order to prove our result we assume that Hypothesis H3 is satisfied, C_0 is a regular tangent cone to X_0 at $y(0)$ and C_1 is a regular tangent cone to X_1 at $y'(0)$. We denote by S_L the set of all solutions of the differential inclusion

$$w'' \in A(t)w + L(t, w),\quad w(0) \in C_0,\quad w'(0) \in C_1$$

and by $R_L(T) = \{x(T);\ x(.) \in S_L\}$ its reachable set at time T.

Theorem 3.1. *Assume that Hypothesis H3 is satisfied and let $f : \mathbf{R}^n \to \mathbf{R}^m$ be a Lipschitz function with m its Lipschitz constant.*

Then, differential inclusion (1.1) is f-locally controllable around the solution $y(.)$ if

$$0_m \in int(hR_L(T))\quad \forall h \in \partial f(y(T)). \tag{3.1}$$

Proof. $hR_L(T)$ is a convex cone, thus, by (3.1), it follows that $hR_L(T) = \mathbf{R}^m$ $\forall h \in \partial f(y(T))$. Taking into account that the set $\partial f(y(T))$ is compact (e.g., [11]), we have that for every $\gamma > 0$ there exist $k \in \mathbf{N}$ and $w_j \in R_L(T)$ $j = 1, 2, ..., k$ such that

$$\gamma B_{\mathbf{R}^m} \subset h(w(\Sigma_k))\quad \forall h \in \partial f(y(T)), \tag{3.2}$$

with

$$w(\Sigma_k) = \{w(\beta) := \sum_{j=1}^{k} \beta_j w_j,\quad \beta = (\beta_1, ..., \beta_k) \in \Sigma_k\}.$$

Using an usual separation theorem we deduce the existence of $\gamma_1, r_1 > 0$ such that for all $h \in L(\mathbf{R}^n, \mathbf{R}^m)$ with $d(h, \partial f(y(T))) \leq r_1$ we have

$$\gamma_1 B_{\mathbf{R}^m} \subset h(w(\Sigma_k)). \tag{3.3}$$

Since $w_j \in R_L(T)$, $j = 1, ..., k$, there exist $(w_j(.), q_j(.))$, $j = 1, ..., k$ trajectory-selection pairs of (2.7) such that $w_j = w_j(T)$, $j = 1, ..., k$. We note that $\gamma > 0$ can be taken small enough such that $|w_j(0)| \leq 1$, $j = 1, ..., k$.

Define

$$w(t, s) = \sum_{j=1}^{k} s_j w_j(t),\quad \overline{q}(t, s) = \sum_{j=1}^{k} s_j q_j(t),\quad \forall s = (s_1, ..., s_k) \in \mathbf{R}^k.$$

Obviously, $w(., s) \in S_L$, $\forall s \in \Sigma_k$.

From the definition of C_0 and C_1 we find that for every $\varepsilon > 0$ there exists a continuous mapping $o_\varepsilon : \Sigma_k \to \mathbf{R}^n$ such that

$$y(0) + \varepsilon w(0, s) + o_\varepsilon(s) \in X_0,\quad y'(0) + \varepsilon \frac{\partial w}{\partial t}(0, s) + o_\varepsilon(s) \in X_1 \tag{3.4}$$

$$\lim_{\varepsilon\to 0+}\max_{s\in\Sigma_k}\frac{|o_\varepsilon(s)|}{\varepsilon}=0. \tag{3.5}$$

Define

$$\rho_\varepsilon(s)(t):=\frac{1}{\varepsilon}\mathrm{d}(\overline{q}(t,s),F(t,y(t)+\varepsilon w(t,s))-g(t)),$$

$$d(t):=\sum_{j=1}^{k}[||q_j(t)||+l(t)||w_j(t)||],\quad t\in I.$$

Then, for every $s\in\Sigma_k$ one has

$$\begin{aligned}&\rho_\varepsilon(s)(t)\leq|\overline{q}(t,s)|+\tfrac{1}{\varepsilon}\mathrm{d}_H(0_n,F(t,y(t)+\varepsilon w(t,s))-g(t))\leq|\overline{q}(t,s)|+\\&\tfrac{1}{\varepsilon}\mathrm{d}_H(F(t,y(t)),F(t,y(t)+\varepsilon w(t,s)))\leq|\overline{q}(t,s)||+l(t)||w(t,s)||\leq d(t).\end{aligned} \tag{3.6}$$

Next, if $s_1,s_2\in\Sigma_k$ one has

$$\begin{aligned}&|\rho_\varepsilon(s_1)(t)-\rho_\varepsilon(s_2)(t)|\leq|\overline{q}(t,s_1)-\overline{q}(t,s_2)|+\tfrac{1}{\varepsilon}\mathrm{d}_H(F(t,y(t)+\varepsilon w(t,s_1)),\\&F(t,y(t)+\varepsilon w(t,s_2)))\leq|s_1-s_2|.\max_{j=\overline{1,k}}[|q_j(t)|+l(t)|w_j(t)|],\end{aligned}$$

thus $\rho_\varepsilon(.)(t)$ is Lipschitz with a Lipschitz constant not depending on ε.

At the same time, from (2.8) it follows that

$$\lim_{\varepsilon\to 0}\rho_\varepsilon(s)(t)=0\quad a.e.\,(I),\quad\forall s\in\Sigma_k$$

and hence

$$\lim_{\varepsilon\to 0+}\max_{s\in\Sigma_k}\rho_\varepsilon(s)(t)=0\quad a.e.\ (I). \tag{3.7}$$

Lebesgue's dominated convergence theorem, (3.6) and (3.7) imply that

$$\lim_{\varepsilon\to 0+}\int_0^T\max_{s\in\Sigma_k}\rho_\varepsilon(s)(t)dt=0. \tag{3.8}$$

From (3.4), (3.5), (3.8) and the upper semicontinuity of the Clarke generalized Jacobian we can find $\varepsilon_0,e_0>0$ such that

$$\max_{s\in\Sigma_k}\frac{||o_{\varepsilon_0}(s)||}{\varepsilon_0}+\int_0^T\max_{s\in\Sigma_k}\rho_{\varepsilon_0}(s)(t)\mathrm{d}t\leq\frac{\gamma_1}{2^8m^2}, \tag{3.9}$$

$$\varepsilon_0 w(T,s)\leq\frac{e_0}{2}\quad\forall s\in\Sigma_k. \tag{3.10}$$

We define

$$y(s)(t):=y(t)+\varepsilon_0 w(t,s),\quad g(s)(t):=g(t)+\varepsilon_0\overline{q}(t,s)\quad s\in\mathbf{R}^k,$$

$$a_0(s):=y(0)+\varepsilon_0 w(0,s)+o_{\varepsilon_0}(s),\quad a_1(s):=y'(0)+\varepsilon_0\frac{\partial w}{\partial t}(0,s)+o_{\varepsilon_0}(s),\ s\in\mathbf{R}^k,$$

and we apply Theorem 2.3 in order to obtain that there exists a continuous function $x(.):\Sigma_k\to C(I,\mathbf{R}^n)$ such that for any $s\in\Sigma_k$ the function $x(s)(.)$ is a mild solution of the differential inclusion $x''\in A(t)x+F(t,x)$, $x(s)(0)=a_0(s)$, $(x(s))'(0)=a_1(s)$ $\forall s\in\Sigma_k$ and one has

$$||x(s)(T)-y(s)(T)||\leq\frac{\varepsilon_0\gamma_1}{2^6m}\quad\forall s\in\Sigma_k. \tag{3.11}$$

We define

$$f_0(x):=\int_{\mathbf{R}^n}f(x-ay)\chi(y)dy,\quad x\in\mathbf{R}^n,$$

$$\psi(s):=f_0(y(T)+\varepsilon_0 w(T,s)),$$

where $\chi(.) : \mathbf{R}^n \to [0,1]$ is a C^∞ function with the support contained in $B_{\mathbf{R}^n}$ that satisfies $\int_{\mathbf{R}^n} \chi(y)\mathrm{d}y = 1$ and $a = \min\{\frac{e_0}{2}, \frac{\varepsilon_0\gamma_1}{2^6 m}\}$.

Hence $f_0(.)$ is of class C^∞ and verifies

$$||f(x) - f_0(x)|| \le m \cdot a, \tag{3.12}$$

$$f_0'(x) = \int_{\mathbf{R}^n} f'(x - ay)\chi(y)dy. \tag{3.13}$$

In particular,

$$f_0'(x) \in \overline{\text{co}}\{f'(u); \quad ||u - x|| \le a, \quad f'(u) \text{ exists}\},$$

$$\psi'(s)\mu = f_0'(y(T) + \varepsilon_0 w(T,s))\varepsilon_0 w(T,\mu) \quad \forall \mu \in \Sigma_k.$$

If we denote $h(s) := f_0'(y(T) + \varepsilon_0 w(T,s))$, then $\psi'(s)\mu = h(s)\varepsilon_0 w(T,\mu)\ \forall \mu \in \Sigma_k$.

Taking into account, again, the upper semicontinuity of the Clarke generalized Jacobian we obtain

$$d(h(s), \partial f(z(T))) = d(f_0'(y(T) + \varepsilon_0 w(T,s)), \partial f(y(T))) \le \sup\{d(f_0'(u), \partial f(y(T)));$$
$$||u - y(T)|| \le ||u - (y(T) + \varepsilon_0 w(T,s))|| + ||\varepsilon_0 w(t,s)|| \le e_0, \quad f'(u) \text{ exists}\} < r_1.$$

The last inequality together with (3.3) gives

$$\gamma_1 B_{\mathbf{R}^m} \subset h(s)w(\Sigma_k).$$

and therefore

$$\varepsilon_0\gamma_1 B_{\mathbf{R}^m} \subset h(s)\varepsilon_0 w(\Sigma_k) = h(s)\varepsilon_0 w(T,\mu) = \psi'(s)\mu, \quad \forall \mu \in \Sigma_k,$$

i.e.,

$$\varepsilon_0\gamma_1 B_{\mathbf{R}^m} \subset \psi'(s)\Sigma_k.$$

Finally, for $s \in \Sigma_k$, we put $\varphi(s) = f(x(s)(T))$.

Obviously, $\varphi(.)$ is continuous and from (3.11), (3.12), (3.13) one may write

$$|\varphi(s) - \psi(s)| = |f(x(s)(T)) - f_0(y(s)(T))| \le |f(x(s)(T)) - f(y(s)(T))| +$$
$$|f(y(s)(T)) - f_0(y(s)(T))| \le m|x(s)(T) - y(s)(T)| + m \cdot a \le \frac{\varepsilon_0\gamma_1}{64} + \frac{\varepsilon_0\gamma_1}{64} = \frac{\varepsilon_0\gamma_1}{32}.$$

It remains to apply Lemma 2.6 and to find that

$$f(x(0_k)(T)) + \frac{\varepsilon_0\gamma_1}{16} B_{\mathbf{R}^m} \subset \varphi(\Sigma_k) \subset f(R_F(T)).$$

Finally, $|f(y(T)) - f(x(0_k)(T))| \le \frac{\varepsilon_0\gamma_1}{64}$, so we have $f(z(T)) \in \text{int}(f(R_F(T)))$, which completes the proof.

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