Hacettepe Journal of Mathematics and Statistics
Volume 47 (6) (2018), 1417 – 1426

# $\oplus$-co-coatomically supplemented and co-coatomically semiperfect modules

Rafail Alizade*† and Serpil Güngör‡

**Abstract**

In this paper it is shown that a factor module of an $\oplus$-co-coatomically supplemented module is not in general $\oplus$-co-coatomically supplemented. If $M$ is $\oplus$-co-coatomically supplemented and $U$ is a fully invariant submodule of $M$, then $M/U$ is $\oplus$-co-coatomically supplemented. A ring $R$ is left perfect if and only if $R^{(\mathbb{N})}$ is an $\oplus$-co-coatomically supplemented $R$-module. A projective module $M$ is co-coatomically semiperfect if and only if $M$ is $\oplus$-co-coatomically supplemented. A ring is semiperfect if and only if every finitely generated free $R$-module is co-coatomically semiperfect.

**Keywords:** Co-coatomic submodule, $\oplus$-co-coatomically supplemented module, co-coatomically semiperfect module.

*Received* : 04.06.2014 *Accepted* : 18.11.2014 *Doi* : 10.15672/HJMS.20154413844

## 1. Introduction

Throughout this paper $\mathbb{N}$ is the set of all positive integers, $R$ is an associative ring with identity and all modules are *left* unitary $R$-modules (${}_RM$) unless otherwise stated. For any module $M$, $\mathrm{Rad}(M)$ denotes the radical of $M$. The Jacobson radical of ${}_RR$ is denoted by $\mathrm{Jac}(R)$. Let $U$ be a submodule of $M$. A submodule $V$ of $M$ is called a *supplement* of $U$ in $M$ if $V$ is minimal element in the set of submodules $L \leq M$ with $U + L = M$. $V$ is a supplement of $U$ in $M$ if and only if $U + V = M$ and $U \cap V \ll V$. A module $M$ is called *supplemented* if every submodule of $M$ has a supplement in $M$ (see [15, Section 41] or [5, Chapter 4]). Semisimple, artinian and hollow (in particular local) modules are supplemented. A module $M$ is called *coatomic* if every proper submodule of $M$ is contained in a maximal submodule (see [18]). Semisimple, finitely generated and hollow modules are coatomic modules. Let $N$ be a submodule of a module $M$. We say

*Yaşar University, Faculty of Science and Letter, Department of Mathematics, Selçuk Yaşar Kampüsü, Üniversite Cad., No:35-37, Ağaçlı Yol, 35100, Bornova, İzmir, Türkiye
Email: rafail.alizade@yasar.edu.tr

†Corresponding Author.

‡ Email: serpiltop@gmail.com

that $N$ is a *co-coatomic* submodule in $M$ if $M/N$ is coatomic. Since factor module of a coatomic module is coatomic, every submodule of semisimple, finitely generated and hollow modules is co-coatomic. A module $M$ is said to be *co-coatomically supplemented* if every co-coatomic submodule of $M$ has a supplement in $M$. A submodule $N$ of $M$ is called *cofinite* if $M/N$ is finitely generated. $M$ is called a *cofinitely supplemented* module if every cofinite submodule of $M$ has a supplement in $M$ (see [1]). Clearly a co-coatomically supplemented module is cofinitely supplemented and a coatomic module is co-coatomically supplemented if and only if it is a supplemented module. A module is said to be $\oplus$-*supplemented* if every submodule of $M$ has a supplement that is a direct summand of $M$. A module $M$ is called $\oplus$-*co-coatomically supplemented* if every co-coatomic submodule of $M$ has a supplement that is a direct summand of $M$. Obviously an $\oplus$-supplemented module is $\oplus$-co-coatomically supplemented. Hollow modules (in particular local modules) are $\oplus$-supplemented, so $\oplus$-co-coatomically supplemented. A module $M$ is called $\oplus$-cofinitely supplemented if every cofinite submodule of $M$ has a supplement that is a direct summand of $M$ (see [4]). Clearly an $\oplus$-co-coatomically supplemented module is $\oplus$-cofinitely supplemented.

In Section 2 , we show that a factor module of an $\oplus$-co-coatomically supplemented module need not be $\oplus$-co-coatomically supplemented by Example 2.1. If $M$ is $\oplus$-co-coatomically supplemented and $U$ is a fully invariant submodule of $M$, then $M/U$ is $\oplus$-co-coatomically supplemented. For any ring $R$, any finite direct sum of $\oplus$-co-coatomically supplemented $R$-modules is $\oplus$-co-coatomically supplemented, but any direct sum of $\oplus$-co-coatomically supplemented modules need not be $\oplus$-co-coatomically supplemented. We show that a ring $R$ is left perfect if and only if $R^{(\mathbb{N})}$ is an $\oplus$-co-coatomically supplemented $R$-module.

In Section 3, we define co-coatomically semiperfect module. For an $R$-module $M$, a pair $(P, \pi)$ is a projective cover of $M$ in case $P$ is a projective $R$-module and $\pi : P \to M$ is a small epimorphism (see [2]). A module $M$ is called *semiperfect* if every factor module of $M$ has a projective cover (see [15]). A projective module is semiperfect if and only if it is $\oplus$-supplemented (see [10, Lemma 1.2.]). A module $M$ is called *co-coatomically semiperfect* if every coatomic factor module of $M$ has a projective cover. A projective module $M$ is co-coatomically semiperfect if and only if $M$ is $\oplus$-co-coatomically supplemented. An $R$-module $M$ is *cofinitely semiperfect* or briefly *cof-semiperfect* if every finitely generated factor module of $M$ has a projective cover (see [4]). Clearly a semiperfect module is co-coatomically semiperfect and a co-coatomically semiperfect module is cof-semiperfect. We show that a ring $R$ is semiperfect if and only if every finitely generated free $R$-module is co-coatomically semiperfect.

## 2. $\oplus$-co-coatomically supplemented modules

**2.1. Example.** [8, Example 2.2] Let $R$ be a commutative local ring which is not a valuation ring and let $n \geq 2$. By [14, Theorem 2], there exists a finitely presented indecomposable module $M = R^{(n)}/K$ which cannot be generated by fewer than $n$ elements. By [7, Corollary 1], $R^{(n)}$ is $\oplus$-co-coatomically supplemented. However, $M$ is not $\oplus$-cofinitely supplemented so it is not $\oplus$-co-coatomically supplemented (see [7, Proposition 2] and [13, Example 2.1]).

The above example shows that a factor module of an $\oplus$-co-coatomically supplemented module need not be $\oplus$-co-coatomically supplemented.

Let $M$ be a nonzero module and let $U$ be a fully invariant submodule of $M$, i.e. $f(U) \leq U$ for each $f \in End_R(M)$. If $M = M_1 \oplus M_2$, then $U = (U \cap M_1) \oplus (U \cap M_2)$ (see [6, Lemma 9.3] for abelian groups).

**2.2. Proposition.** *Let $M$ be a nonzero module and $U$ be a fully invariant submodule of $M$. If $M$ is $\oplus$-co-coatomically supplemented, then $M/U$ is $\oplus$-co-coatomically supplemented. Furthermore, if $U$ is a co-coatomic direct summand of $M$, then $U$ is also $\oplus$-co-coatomically supplemented.*

*Proof.* Suppose that $M$ is an $\oplus$-co-coatomically supplemented module and $L/U$ is a co-coatomic submodule of $M/U$. Therefore $M/L \cong (M/U)/(L/U)$ is coatomic. Since $M$ is $\oplus$-co-coatomically supplemented, there exist submodules $N$ and $N'$ of $M$ such that $M = N \oplus N'$, $M = N + L$ and $N \cap L \ll N$. Then $(N+U)/U$ is a supplement of $L/U$ in $M/U$ (see [15, 41.1(7)]). By hypothesis, $U$ is fully invariant, therefore $U = (U \cap N) \oplus (U \cap N')$ (see [6, Lemma 9.3]). Thus $U = (N+U) \cap (N'+U)$ and $M/U = ((N+U)/U) \oplus ((N'+U)/U)$. Hence $M/U$ is $\oplus$-co-coatomically supplemented.

Now suppose that $U$ is a co-coatomic direct summand of $M$. Then there exists a submodule $U'$ of $M$ such that $M = U \oplus U'$ and $U'$ is coatomic. Let $V$ be a co-coatomic submodule of $U$. Therefore $M/V = (U \oplus U')/V \cong (U/V) \oplus U'$ is coatomic as it is direct sum of two coatomic modules. Since $M$ is $\oplus$-co-coatomically supplemented, there exist submodules $K$ and $K'$ of $M$ such that $M = K \oplus K'$, $M = V + K$ and $V \cap K \ll K$. Thus $U = V + (U \cap K)$. Since $U$ is fully invariant, $U = (U \cap K) \oplus (U \cap K')$, and so $U \cap K$ is direct summand of $U$. Furthermore, $V \cap (U \cap K) = V \cap K \ll K$. Then $V \cap (U \cap K) \ll U \cap K$ (see [15, 19.3(5)]). Therefore $U \cap K$ is a supplement of $V$ in $U$ and it is a direct summand of $U$. Hence $U$ is $\oplus$-co-coatomically supplemented. □

**2.3. Corollary.** *Let $M$ be an $\oplus$-co-coatomically supplemented $R$-module. Then $M/\operatorname{Rad}(M)$ and $M/\operatorname{Soc}(M)$ are also $\oplus$-co-coatomically supplemented modules.*

Property (D3) for an $R$-module $M$ is the following: If $M_1$ and $M_2$ are direct summands of $M$ with $M = M_1 + M_2$, then $M_1 \cap M_2$ is also a direct summand of $M$.

**2.4. Proposition.** *Let $M$ be an $\oplus$-co-coatomically supplemented module with property (D3). Then every co-coatomic direct summand of $M$ is $\oplus$-co-coatomically supplemented.*

*Proof.* Let $N$ be a co-coatomic direct summand of $M$ that is there exists a submodule $N'$ of $M$ such that $M = N \oplus N'$ and $N' \cong M/N$ is coatomic. Let $U$ be a co-coatomic submodule of $N$. Then

$$M/U = (N \oplus N')/U \cong (N/U) \oplus N'$$

is coatomic as it is a direct sum of two coatomic modules. Since $M$ is $\oplus$-co-coatomically supplemented, there exists a direct summand $V$ of $M$ such that

$$M = U + V \text{ and } U \cap V \ll V.$$

Hence

$$N = N \cap M = N \cap (U + V) = U + (N \cap V)$$

Since $M$ has property (D3), $N \cap V$ is a direct summand of $M$. Furthermore $N \cap V$ is a direct summand of $N$ because $N$ is a direct summand of $M$. Then $U \cap (N \cap V) = U \cap V$ is small in $N \cap V$ by [15, 19.3(5)]. Hence $N$ is $\oplus$-co-coatomically supplemented. □

A ring $R$ is called a left $V$-ring if every simple $R$-module is injective (see [15, p. 192]). A commutative ring $R$ is a $V$-ring if and only if $R$ is a von Neumann regular ring (see [15, 23.5]).

**2.5. Proposition.** *Over a $V$-ring $R$, a module $M$ is $\oplus$-co-coatomically supplemented if and only if $M$ is semisimple.*

*Proof.* ($\Leftarrow$) Clear.
($\Rightarrow$) Since $M$ is $\oplus$-co-coatomically supplemented, $M$ is $\oplus$-cofinitely supplemented and so cofinitely supplemented. Therefore $M/\operatorname{Soc}(M)$ has no maximal submodule by [1, Theorem 2.8] and [1, Proposition 3.6]. Since $R$ is a $V$-ring, $M/\operatorname{Soc}(M) = \operatorname{Rad}(M/\operatorname{Soc}(M)) = 0$ (see [15, 23.1]). Thus $M$ is semisimple. $\square$

Obviously an $\oplus$-supplemented module is supplemented and $\oplus$-co-coatomically supplemented module is co-coatomically supplemented. An $\oplus$-supplemented module is $\oplus$-co-coatomically supplemented module, but an $\oplus$-co-coatomically supplemented module need not be $\oplus$-supplemented in general by the following example.

**2.6. Example.** The $\mathbb{Z}$-module $\mathbb{Q}$ does not have any proper co-coatomic submodule. Thus $\mathbb{Q}$ is $\oplus$-co-coatomically supplemented. But $\mathbb{Z}$-module $\mathbb{Q}$ is not supplemented so it is not $\oplus$-supplemented (see [16, Theorem 3.1]).

An $\oplus$-co-coatomically supplemented module is $\oplus$-cofinitely supplemented but the example below shows that an $\oplus$-cofinitely supplemented module need not be $\oplus$-co-coatomically supplemented.

**2.7. Example.** [9, p. 282] Let $R$ denote the ring $K[[x]]$ of all power series $\sum\limits_{i=0}^{\infty} k_i x^i$ in an indeterminate $x$ and with coefficients from a field $K$. $R$ is a local ring. The $R$-module $R$ is supplemented so $R$ is semiperfect (see [15, 42.6]). Note that

$$\operatorname{Jac}(R) = \left\{ \sum_{i=1}^{\infty} k_i x^i \mid k_i \in K \right\} = Rx$$

is not $t$-nilpotent. Thus $R$ is not perfect (see [15, 43.9]). Since $R$ is semiperfect, $R/\operatorname{Jac}(R)$ is semisimple. Therefore ${}_RR^{(\mathbb{N})}/\operatorname{Rad}({}_RR^{(\mathbb{N})})$ is semisimple, so $\operatorname{Rad}({}_RR^{(\mathbb{N})})$ is a co-coatomic submodule of ${}_RR^{(\mathbb{N})}$. By [3, Theorem 1], $\operatorname{Rad}({}_RR^{(\mathbb{N})})$ does not have a supplement. Thus ${}_RR^{(\mathbb{N})}$ is not co-coatomically supplemented so it is not $\oplus$-co-coatomically supplemented. On the other hand, since $R$ is local, $R$-module $R$ is $\oplus$-supplemented and so $\oplus$-cofinitely supplemented. Any direct sum of ${}_RR$, in particular ${}_RR^{(\mathbb{N})}$ is $\oplus$-cofinitely supplemented by [4, Theorem 2.6].

By the example above, it is seen that arbitrary direct sum of $\oplus$-co-coatomically supplemented modules need not be $\oplus$-co-coatomically supplemented.

To prove that a finite sum of $\oplus$-co-coatomically supplemented modules is an $\oplus$-co-coatomically supplemented module, we will use the following lemma.

**2.8. Lemma.** *Let $M$ be an $R$-module and $N, U$ be submodules of $M$ such that $N$ is co-caotomically supplemented, $U$ is co-coatomic and $N + U$ has a supplement $A$ in $M$. Then $N \cap (U + A)$ has a supplement $B$ in $N$ and $A + B$ is a supplement of $U$ in $M$.*

*Proof.* Let $A$ be a supplement of $N+U$ in $M$. Then $M = N+U+A$ and $(N+U)\cap A \ll A$. Note that

$$N/(N \cap (U + A)) \cong (N + U + A)/(U + A) = M/(U + A) \cong (M/U)/((U + A)/U)$$

is coatomic. Therefore $N \cap (U + A)$ is a co-coatomic submodule of $N$. Since $N$ is co-coatomically supplemented, $N \cap (U+A)$ has a supplement $B$ in $N$, i.e. $N \cap (U+A)+B = N$ and $B \cap (U + A) \ll B$. Then

$$M = N + U + A = U + A + B.$$

$$U \cap (A + B) \le (A \cap (U + B)) + (B \cap (U + A)) \le (A \cap (U + N)) + (B \cap (U + A)) \ll A + B$$

Hence $A + B$ is a supplement of $U$ in $M$. $\square$

**2.9. Proposition.** *For any ring $R$, any finite direct sum of $\oplus$-co-coatomically supplemented $R$-modules is $\oplus$-co-coatomically supplemented.*

*Proof.* Let $n$ be a positive integer and $M = M_1 \oplus \cdots \oplus M_n$ where $M_i$ is co-coatomically supplemented for each $1 \leq i \leq n$. To prove that $M$ is $\oplus$-co-coatomically supplemented it is sufficient to prove the case when $n = 2$. Therefore let $M = M_1 \oplus M_2$ and $L$ be any co-coatomic submodule of $M$. Then $M = M_1 + M_2 + L$ such that $M_1 + M_2 + L$ has a supplement 0 in $M$. Consider the submodule $M_2 \cap (M_1 + L)$ of $M_2$.

$$M_2/(M_2 \cap (M_1 + L)) \cong (M_1 + M_2 + L)/(M_1 + L) = M/(M_1 + L).$$

Since $M_1 + L$ is a co-coatomic submodule of $M$, $M_2 \cap (M_1 + L)$ is a co-coatomic submodule of $M_2$. Since $M_2$ is an $\oplus$-co-coatomically supplemented module, $M_2 \cap (M_1 + L)$ has a supplement $H$ in $M_2$ such that $H$ is a direct summand of $M_2$. $H$ is a supplement of $M_1 + L$ in $M$ by Lemma 2.8. Now consider the submodule $M_1 \cap (L + H)$ of $M_1$.

$$M_1/(M_1 \cap (L + H)) \cong (M_1 + L + H)/(L + H) = M/(L + H).$$

Since $L + H$ is a co-coatomic submodule of $M$, $M_1 \cap (L + H)$ is a co-coatomic submodule of $M_1$. Since $M_1$ is $\oplus$-co-coatomically supplemented, $M_1 \cap (L + H)$ has a supplement $K$ in $M_1$ such that $K$ is a direct summand of $M_1$. Again by Lemma 2.8, we obtain $H + K$ is a supplement of $L$ in $M$. It follows that $H + K = H \oplus K$ is a direct summand of $M$ since $H$ is a direct summand of $M_2$ and $K$ is a direct summand of $M_1$. Thus $M = M_1 \oplus M_2$ is $\oplus$-co-coatomically supplemented. $\square$

**2.10. Proposition.** *Let $M$ be an indecomposable $R$-module. The following are equivalent:*

(1) *Every co-coatomic submodule of $M$ has a supplement that is a direct summand.*
(2) *Every maximal submodule of $M$ has a supplement that is a direct summand.*
(3) *$M$ is radical or $M$ is local.*

*Proof.* (1) $\Rightarrow$ (2) Since every maximal submodule is co-coatomic it is clear.
(2) $\Rightarrow$ (3) If $M$ is not radical, that is there is a maximal submodule $N$ of $M$, then $N$ has a supplement $K$ that is a direct summand. Since $M$ is indecomposable either $K = 0$ or $K = M$. If $K = 0$ then $M = N + K = N$. Contradiction. If $K = M$ then $N = N \cap K \ll M$, therefore $N$ is the largest proper submodule of $M$, so $M$ is local.
(3) $\Rightarrow$ (1) Let $N$ be a co-coatomic submodule of $M$. If $N \neq M$ then $M/N$ has a maximal submodule, therefore $M$ has also a maximal submodule, that is $M$ is not a radical module. Then $M$ is local and therefore $\oplus$-supplemented. Thus every co-coatomic submodule has a supplement that is a direct summand in $M$. $\square$

**2.11. Corollary.** *Let $M$ be an indecomposable $R$-module such that* $\operatorname{Rad}(M) \neq M$. *$M$ is $\oplus$-co-coatomically supplemented if and only if $M$ is local.*

A module $M$ is called $\Sigma$-*selfprojective* if for each index set $I$, the module $M^{(I)}$ is selfprojective (see [17]).

**2.12. Remark.** For an $R$-module $M$, if $M$ is $\Sigma$-selfprojective and $U \leq \operatorname{Rad}(M)$, then the following holds: $U$ has a supplement in $M$, so $U$ is small in $M$ [17, Satz 4.1]. Clearly ${}_RR^{(\mathbb{N})}$ is $\Sigma$-selfprojective and $\operatorname{Rad}({}_RR^{(\mathbb{N})}) \leq \operatorname{Rad}({}_RR^{(\mathbb{N})})$, therefore if $\operatorname{Rad}({}_RR^{(\mathbb{N})})$ has a supplement in ${}_RR^{(\mathbb{N})}$ then $\operatorname{Rad}({}_RR^{(\mathbb{N})}) \ll {}_RR^{(\mathbb{N})}$.

**2.13. Theorem.** *A ring $R$ is left perfect if and only if $R^{(\mathbb{N})}$ is an $\oplus$-co-coatomically supplemented $R$-module.*

*Proof.* ($\Rightarrow$) By [12, Proposition 4.8 and Theorem 4.41], ${}_RR^{(\mathbb{N})}$ is $\oplus$-supplemented and so $\oplus$-co-coatomically supplemented.
($\Leftarrow$) Let $M$ denote the $R$-module $R^{(\mathbb{N})}$. Since $M$ is $\oplus$-co-coatomically supplemented, it is $\oplus$-cofinitely supplemented and so cofinitely supplemented. Thus ${}_RR$ is cofinitely supplemented (see [1, Lemma 2.1]). Therefore ${}_RR$ is supplemented since it is finitely generated. Therefore $R/\operatorname{Jac}(R)$ is semisimple by [15, 42.6]. It follows that ${}_RR^{(\mathbb{N})}/\operatorname{Rad}({}_RR^{(\mathbb{N})})$ is semisimple. Thus $\operatorname{Rad}({}_RR^{(\mathbb{N})})$ is co-coatomic in ${}_RR^{(\mathbb{N})}$. By the assumption, $\operatorname{Rad}({}_RR^{(\mathbb{N})})$ has a supplement in ${}_RR^{(\mathbb{N})}$ that is a direct summand. By Remark 2.12, $\operatorname{Rad}({}_RR^{(\mathbb{N})}) \ll {}_RR^{(\mathbb{N})}$. Therefore, since $R/\operatorname{Jac}(R)$ is semisimple, $R$ is left perfect by [15, 43.9]. □

**2.14. Corollary.** *The following are equivalent for a ring $R$:*
(1) *$R$ is left perfect.*
(2) *The $R$-module $R^{(\mathbb{N})}$ is $\oplus$-supplemented.*
(3) *The $R$-module $R^{(\mathbb{N})}$ is $\oplus$-co-coatomically supplemented.*

*Proof.* (1) $\Leftrightarrow$ (2) By [11, Theorem 2.10].
(2) $\Rightarrow$ (3) Clear.
(3) $\Rightarrow$ (1) By Theorem 2.13. □

Let $R$ be a commutative ring. An $R$-module $M$ is called a multiplication module if every submodule of $M$ is of the form $IM$ for some ideal $I$ of $R$. Let $M$ be an $\oplus$-cofinitely supplemented multiplication module with $\operatorname{Rad}(M) \ll M$, then $M$ can be written as an irredundant sum of local direct summands of $M$ (see [13, Theorem 2.7]).

**2.15. Proposition.** *Let $R$ be a commutative ring and $M$ be a multiplication $R$-module. If $M$ is an $\oplus$-co-coatomically supplemented module with $\operatorname{Rad}(M) \ll M$, then $M$ can be written as an irredundant sum of local direct summands of $M$.*

*Proof.* Since every $\oplus$-co-coatomically supplemented module is $\oplus$-cofinitely supplemented the proof is clear by [13, Theorem 2.7]. □

## 3. Co-coatomically semiperfect modules

**3.1. Definition.** Let $M$ be an $R$-module. $M$ is called co-coatomically semiperfect if every coatomic factor module of $M$ has a projective cover.

The following proposition gives a characterization of a projective $\oplus$-co-coatomically supplemented module.

**3.2. Proposition.** *Let $M$ be a projective $R$-module. Then $M$ is co-coatomically semiperfect if and only if $M$ is $\oplus$-co-coatomically supplemented.*

*Proof.* ($\Rightarrow$) Let $N$ be a co-coatomic submodule of $M$. Then $M/N$ is coatomic. By hypothesis, there exists a projective cover $\pi : P \to M/N$. Let $\sigma : M \to M/N$ be canonical epimorphism. Since $M$ is projective there exists a homomorphism $f : M \to P$ such that the diagram

$$\begin{tikzcd} & M \arrow[dl, dotted, "f"'] \arrow[d, "\sigma"] \\ P \arrow[r, "\pi"'] & M/N \end{tikzcd}$$

is commutative, i.e. $\pi \circ f = \sigma$. Since $\pi$ is a small epimorphism, $f$ is epic by [15, 19.2]. Since $P$ is projective, $f$ splits, i.e. there exists a homomorphism $g : P \to M$ such that $f \circ g = 1_P$ by [9, 3.9.3]. Thus $\pi = \pi \circ f \circ g = \sigma \circ g$. It follows that $M = \ker f \oplus g(P)$

and $\ker f \leq N$, so $M = N + g(P)$. Let $\mu = \sigma\mid_{g(P)}: g(P) \to M/N$. Then $\pi = \mu \circ g$ and therefore $\mu$ is epic since $\pi$ is epimorphism. Furthermore, since $\pi$ is a small epimorphism, $\mu$ is also a small epimorphism by [15, 19.3]. Therefore $\ker \mu = N \cap g(P) \ll g(P)$. Thus $g(P)$ is a supplement of $N$.
($\Leftarrow$) Let $M/N$ be a coatomic factor module of $M$. Since $M$ is $\oplus$-co-coatomically supplemented, there exists submodules $K$ and $K'$ such that $M = K \oplus K'$, $M = N + K$ and $N \cap K \ll K$. Since $M$ is projective, $K$ is projective. Therefore $\sigma \circ i : K \to M/N$ is an epimorphism and $\ker \sigma \circ i = N \cap K \ll K$ for the inclusion homomorphism $i : K \to M$ and the canonical epimorphism $\sigma : M \to M/N$. □

An $R$-module $M$ is cofinitely semiperfect or briefly cof-semiperfect if every finitely generated factor module of $M$ has a projective cover.

A co-coatomically semiperfect module is cof-semiperfect but converse need not be true by Example 2.7 since the projective $R$-module $R^{(\mathbb{N})}$ in that example is $\oplus$-cofinitely supplemented but not $\oplus$-co-coatomically supplemented.

A submodule $U$ of an $R$-module $M$ has *ample supplements* in $M$ if, for every submodule $V$ of $M$ with $U + V = M$, there exists a supplement $V'$ of $U$ with $V' \leq V$ (see [5, p. 237]). A module $M$ is called *co-coatomically amply supplemented* if every co-coatomic submodule of $M$ has ample supplements in $M$. Clearly a co-coatomically amply supplemented module is co-coatomically supplemented.

Let $M$ be an $R$-module and $N$ be a submodule of $M$. $N$ is called lie above a direct summand of $M$ if there is a decomposition $M = K \oplus K'$ such that $K \leq N$ and $K' \cap N \ll K'$.

**3.3. Proposition.** *Let $M$ be a projective module. Then the following are equivalent:*

(1) *$M$ is co-coatomically semiperfect.*
(2) *$M$ is $\oplus$-co-coatomically supplemented.*
(3) *Each co-coatomic submodule of $M$ lies above a direct summand of $M$.*
(4) *$M$ is co-coatomically amply supplemented by supplements which have projective covers.*
(5) *$M$ is co-coatomically supplemented by supplements which have projective covers.*

*Proof.* (1) $\Leftrightarrow$ (2) By Proposition 3.2.
(2) $\Rightarrow$ (3) Let $N$ be a co-coatomic submodule of $M$. Since $M$ is $\oplus$-co-coatomically supplemented, there exist submodules $K$ and $K'$ of $M$ such that $M = N+K$, $N\cap K \ll K$ and $M = K \oplus K'$. Since $M$ is projective there exists a submodule $K'' \leq N$ such that $M = K'' \oplus K$ (see [15, 41.14]).
(3) $\Rightarrow$ (2) Clear.
(1) $\Rightarrow$ (4) Let $N$ be a co-coatomic submodule of $M$ and $M = N + L$ for some submodule $L$ of $M$. Let $(P, f)$ be a projective cover of $M/N$. Since $P$ is projective and $M/N \cong L/(N \cap L)$, there exists a homomorphism $g : P \to L$. Since $\operatorname{Ker} f \ll P$ and $g(\operatorname{Ker} f) = \operatorname{Im} g \cap N \cap L = \operatorname{Im} g \cap N$, $\operatorname{Im} g \cap N = \operatorname{Im} g \cap N \cap L \ll \operatorname{Im} g$. $\operatorname{Im} g + (N \cap L) = L$ since $f$ is an epimorphism. Therefore $\operatorname{Im} g$ is a supplement of $N \cap L$ in $L$. $M = N + L = N + \operatorname{Im} g + (N \cap L) = \operatorname{Im} g + N$ and $\operatorname{Im} g \cap N \ll \operatorname{Im} g$, i.e. $\operatorname{Im} g$ is a supplement of $N$ in $M$ and $\operatorname{Im} g$ is contained in $L$. Since $\operatorname{Ker} g \leq \operatorname{Ker} f$ and $\operatorname{Ker} f \ll P$, $P$ is a projective cover of $\operatorname{Im} g$.
(4) $\Rightarrow$ (5) Clear.
(5) $\Rightarrow$ (1) Let $N$ be a co-coatomic submodule of $M$ and $L$ be a supplement of $N$ in $M$. Then $L$ is a small cover of $L/(N \cap L)$. Therefore every projective cover of $L$ is also a projective cover of $L/(N \cap L)$. Since $M/N \cong L/(N \cap L)$, $M/N$ has a projective cover. Thus $M$ is co-coatomically semiperfect. □

**3.4. Proposition.** *Every homomorphic image of a co-coatomically semiperfect module is co-coatomically semiperfect.*

*Proof.* Let $f : M \to N$ be a homomorphism and let $M$ be a co-coatomically semiperfect module. Let $f(M)/U$ be a coatomic factor module of $f(M)$. There is an epimorphism

$$\sigma : M \to f(M)/U,\ m \mapsto f(m) + U.$$

Since $M$ is co-coatomically semiperfect,

$$M/f^{-1}(U) \cong f(M)/U$$

that is $f(M)/U$ has a projective cover. Thus $f(M)$ is co-coatomically semiperfect. □

**3.5. Corollary.** *Every factor module of a co-coatomically semiperfect module is co-coatomically semiperfect.*

**3.6. Corollary.** *Let $M$ be a projective module. If $M$ is $\oplus$-co-coatomically supplemented then every factor module of an $\oplus$-co-coatomically supplemented module is also $\oplus$-co-coatomically supplemented.*

*Proof.* By Corollary 3.5. □

**3.7. Proposition.** *Every small cover of a co-coatomically semiperfect module is co-coatomically semiperfect.*

*Proof.* Let $N$ be a co-coatomically semiperfect module, $f : M \to N$ be a small epimorphism and $L$ be a co-coatomic submodule of $M$. Then $N/f(L)$ is an epimorphic image of $M/L$ under the epimorphism

$$\overline{f} : M/L \to N/f(L),\ \overline{f}(m + L) = f(m) + f(L)$$

Note that $\ker \overline{f} \ll M/L$ since $\ker f \ll M$. Therefore $N/f(L)$ is coatomic since $M/L$ is coatomic. By hypothesis, $N/f(L)$ has a projective cover, say $\pi : P \to N/f(L)$. Since $P$ is projective there exists a homomorphism $g : P \to M/L$ such that the following diagram is commutative

$$\begin{tikzcd}
& P \arrow[dl, dotted, "g"'] \arrow[d, "\pi"] \\
M/L \arrow[r, "\overline{f}"'] & N/f(L)
\end{tikzcd}$$

i.e. $\overline{f} \circ g = \pi$. Thus $g$ is epic by [15, 19.2] and since $\pi$ is small, $g$ is small by [15, 19.3]. Hence $P$ is a projective cover of the module $M/L$. □

**3.8. Corollary.** *If $K \ll M$ and $M/K$ is co-coatomically semiperfect then $M$ is co-coatomically semiperfect.*

**3.9. Corollary.** *Let $\pi : P \to M$ be a projective cover of a module $M$. Then the following statements are equivalent:*

(1) *$M$ is co-coatomically semiperfect.*
(2) *$P$ is co-coatomically semiperfect.*
(3) *$P$ is $\oplus$-co-coatomically supplemented.*

*Proof.* (1) $\Rightarrow$ (2) By Proposition 3.7.
(2) $\Rightarrow$ (1) By Proposition 3.4.
(2) $\Leftrightarrow$ (3) By Proposition 3.2. □

Let $M$ be an $R$-module. Then an $R$-module $N$ is called (finitely) $M$-generated if it is a homomorphic image of a (finite) direct sum of copies of $M$ (see [5, 1.1.]).

**3.10. Lemma.** *Let $M$ be a projective module. If $M$ is semiperfect then every finitely $M$-generated module is co-coatomically semiperfect. The converse holds if $M$ is finitely generated.*

*Proof.* Let $N$ be a finitely $M$-generated module. Since $M$ is semiperfect, $M$ is $\oplus$-supplemented. Therefore $M$ is $\oplus$-co-coatomically supplemented. By Proposition 2.9, a finite direct sum of $M$, i.e. for any finite set $\Lambda$, $M^{(\Lambda)}$ is also $\oplus$-co-coatomically supplemented. Therefore $M^{(\Lambda)}$ is co-coatomically semiperfect by Proposition 3.2. By Corollary 3.5, $N$ is co-coatomically semiperfect. Conversely, suppose that $M$ is finitely generated and so it is coatomic. By hypothesis, $M$ is co-coatomically semiperfect. Therefore $M$ is semiperfect. $\square$

**3.11. Proposition.** *For a ring $R$, the following statements are equivalent:*

(1) *$R$ is semiperfect.*
(2) *Every finitely generated free $R$-module is semiperfect.*
(3) *Every finitely generated free $R$-module is co-coatomically semiperfect.*

*Proof.* (1) $\Leftrightarrow$ (2) By [10, Lemma 1.2] and [11, Theorem 2.1.].
(1) $\Leftrightarrow$ (3) By Lemma 3.10. $\square$

## References


[1] Alizade, R., Bilhan, G., and Smith, P. F. *Modules whose maximal submodules have supplements.* Communications in Algebra, 29(6):2389–2405, 2001.

[2] Anderson, F. and Fuller, K. *Rings and Categories of Modules.* Springer, 1992.

[3] Büyükaşık, E. and Lomp, C. *Rings whose modules are weakly supplemented are perfect. applications to certain ring extensions.* Mathematica Scandinavica, 105:25–30, 2009.

[4] Çalışıcı, H. and Pancar, A. *⊕-cofinitely supplemented modules.* Czechoslovak Mathematical Journal, 54(129):1083–1088, 2004.

[5] Clark, J., Lomp, C., Vanaja, N., and Wisbauer, R. *Lifting Modules.* Birkhäuser Verlag, 2006.

[6] Fuchs, L. *Infinite Abelian Groups, Vol. I.* New York: Academic Press, 1970.

[7] Idelhadj, A. and Tribak, R. *A dual notion of cs-modules generalization.* Algebra and Number Theory, Lecture Notes in Pure and Appl. Math., Marcel Dekker, New York, 208:149–155, 2000.

[8] Idelhadj, A. and Tribak, R. *On some properties of ⊕-supplemented modules.* International Journal of Mathematics and Mathematical Sciences, 69:4373–4387, 2003.

[9] Kasch, F. *Modules and Rings.* London Mathematical Society, 1982.

[10] Keskin, D., Harmancı, A., and Smith, P. F. *On ⊕-supplemented modules.* Acta Mathematica Hungaria, 83(1-2):161–169, 1999.

[11] Keskin, D., Smith, P. F., and Xue, W. *Rings whose modules are ⊕-supplemented.* Journal of Algebra, 218:470–487, 1999.

[12] Mohamed, S. H. and Müller, B. J. *Continuous and Discrete Modules.* London Mathematical Society Lecture Notes Series, Cambridge Univ. Press, Cambridge, UK, 1990.

[13] Wang, Y. and Sun, Q. *A note on ⊕-cofinitely supplemented modules.* International Journal of Mathematics and Mathematical Sciences, 2007:10836–5 pages, 2007.

[14] Warfield Jr., R. B. *Decomposability of finitely presented modules.* Proceedings of the American Mathematical Society, 25(1):167–172, 1970.

[15] Wisbauer, R. *Foundations of Modules and Rings.* Gordon and Breach, 1991.

[16] Zöschinger, H. *Komplementierte moduln über dedekindringen.* Journal of Algebra, 29:42–56, 1974.

[17] Zöschinger, H. *Moduln die in jeder erweiterung ein komplement haben.* Mathematica Scandinavica, 35:267–287, 1974.

[18] Zöschinger, H. and Rosenberg, F. A. *Koatomare moduln.* Mathematische Zeitschrift, 170(3):221–232, 1980.