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On generalized open sets

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Abstract

We introduce and explore generalized open sets namely γ-pre-open and γ-b-open sets in topological spaces. We also explore and investigate the properties and characterizations of γ-b-continuous function, γ-b-irresolute function and γ-b-open(closed) functions. Examples and counter examples are also provided for the existence of defined concepts.

Keywords: γ-open(closed), γ-interior(closure), γ-semi-open(closed), γ-semi-closure(interior), γ-pre-open(closed), γ-b-open(closed), γ-b-continuous function, γ-b-irresolute function, γ-b-open(closed) function.

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1. Introduction

Throughout the present paper, we will denote X as a topological space.

In [13], S. Kasahara defined an operation $\gamma : \tau \to$ P(X) as a function from τ to the power set of X such that V $\subseteq V^\gamma$, for each V $\in \tau$, where V^γ denotes the value of γ at V. The operations defined by γ(G) = G, γ(G) = cl(G) and γ(G) = intcl(G) are examples of operation γ. In [13], it is defined that a point $x \in A$ is said to be a γ-interior point of A, if there exists an open nbd N of x such that $N^\gamma \subseteq$ A and the set of all such points is denoted by int_γ(A). Thus int_γ (A) $= \{x \in A : x \in N \in \tau$ and $N^\gamma \subseteq A\} \subseteq A$. In [17], H. Ogata introduced that a point $x \in X$ is called a γ-closure point of A$\subseteq$ X, if $U^\gamma \cap A \neq \phi$, for each open nbd U of x. The set of all γ-closure points of A is called γ-closure of A and is denoted by cl_γ(A). A subset A of X is called γ-closed, if $cl_\gamma(A) \subseteq A$. Note that $cl_\gamma(A)$ is contained in every γ-closed superset of A. In [10], S. Hussain et. al defined that a subset A of a space X is said to be a γ-semi-open set, if there exists a γ-open set O such that $O \subseteq A \subseteq cl_\gamma(O)$. The set of all γ-semi-open sets is denoted by $SO_\gamma(X)$. A is γ-semi-closed iff $X - A$ is γ-semi-open in X. Moreover, A is γ-semi-open implies that $A \subseteq cl_\gamma(int_\gamma(A))$. In [2], B. Ahmad and S. Hussain defined γ-semi-closure as the intersection of all γ-semi-closed sets containing A and is denoted by $scl_\gamma(A)$. Also the

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γ-semi-interior of A denoted as $sint_\gamma(A)$ and is the union of γ-semi-open subsets of A. Moreover they discussed and investigated their properties. Many researches [1-20] have been done in recent years. In [14], Levine introduced the notion of semi-open sets in topological spaces. The notion of Levine has been generalized in different senses which generalized the several notions in classical topology.
In this paper, we introduce and explore generalized open sets namely γ-pre-open and γ-b-open sets in topological spaces. We also explore and investigate the properties and characterizations of γ-b-continuous function, γ-b-irresolute function and γ-b-open (closed) functions. Examples and counter examples are also provided for the existence of defined concepts.

2. Properties of generalized open sets

First we define the following.

2.1. Definition. Let X be a space and $A \subseteq X$. Then A is called a γ-pre-open set, if $A \subseteq int_\gamma(cl_\gamma(A))$. The family of all γ-pre-open sets in X will be denoted by $PO_\gamma(X)$.

2.2. Definition. Let X be a space and $A \subseteq X$. Then A is called a γ-pre-closed set, if A^c is γ-pre-open. Equivalently, A is called γ-pre-closed set, if $A \subseteq cl_\gamma(int_\gamma(A))$. The family of all γ-pre-closed sets in X will be denoted by $PC_\gamma(X)$.

Note that every γ-open(closed) set is γ-pre-open(closed) set. But the converse is not true in general as shown in the following example.

2.3. Example. Let X $= \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{c\}, \{a, c\}, \{a, b\}\}$ be topology on X. For $b \in X$, define an operation $\gamma : \tau \to P(X)$ by

$$\gamma(A) = A^\gamma = \begin{cases} A, & \text{if } b \in A \\ cl(A), & \text{if } b \notin A \end{cases}$$

Clearly, γ-open sets are $\phi, X, \{c\}, \{a, b\}$. Take $A = \{a, c\}$. Calculations show that A is γ-pre-open but not γ-open.

2.4. Theorem. *Let X be a space. Then*
(1) any union of γ-pre-open sets is γ-pre-open set.
(2) any intersection of γ-pre-closed sets is γ-pre-closed set.

Proof. (1) Suppose that $\{A_i : i \in I\}$ be a family of γ-pre-open sets. So, for any $i \in I$, $A_i \subseteq int_\gamma(cl_\gamma(A_i))$. This implies that $\bigcup_{i\in I} A_i \subseteq \bigcup_{i\in I} int_\gamma(cl_\gamma(A_i)) \subseteq int_\gamma(\bigcup_{i\in I} cl_\gamma(A_i)) \subseteq int_\gamma(cl_\gamma(\bigcup_{i\in I} A_i))$. So, $\bigcup A_i \subseteq int_\gamma(cl_\gamma(\bigcup_{i\in I} A_i))$.
(2) This directly follows by using (1) and taking complement. □

2.5. Definition. Let X be a space and $A \subseteq X$. Then
(1) γ-pre-closure of A is denoted by $pcl_\gamma(A)$ and is defined as the intersection of all γ-pre-closed subsets of X containing A. That is,
$pcl_\gamma(A) = \bigcap\{B : B \supseteq A \text{ with } B^c \in PO_\gamma(X)\}$.
(2) γ-pre-interior of A is denoted by $pint_\gamma(A)$ and is defined as the union of all γ-pre-open subsets of X contained in A. That is,
$pint_\gamma(A) = \bigcup\{B : B \subseteq A \text{ with } B \in PO\gamma(X)\}$.

The following theorem directly follows from the definitions and thus the proof is omitted.

2.6. Theorem. *Let X be a space and $A \subseteq X$. Then*
(1) $pcl_\gamma(A^c) = (pint_\gamma(A))^c$.
(2) $pint_\gamma(A^c) = (pcl_\gamma(A))^c$.
(3) A is γ-pre-open(closed) $\Leftrightarrow A = pint_\gamma(A)\{A = pcl_\gamma(A)\}$.

The proof of the following theorem is easy and therefore omitted.

2.7. Theorem. *Let X be a space and $A, B \subseteq X$. Then*
(1) $pcl_\gamma(\phi) = \phi$.
(2) $pcl_\gamma(A)$ *is γ-pre-closed in X.*
(3) If $A \subseteq B$, *then* $pcl_\gamma(A) \subseteq pcl_\gamma(B)$.
(4) $pcl_\gamma(pcl_\gamma(A)) = pcl_\gamma(A)$.
(5) $pcl_\gamma(A \cup B) \supseteq pcl_\gamma(A) \cup pcl_\gamma(B)$.
(6) $pcl_\gamma(A \cap B) \subseteq pcl_\gamma(A) \cap pcl_\gamma(B)$.

2.8. Lemma. [18] *Let X be a space and $A \subseteq X$. Then*
(1) $cl_\gamma(A) \cap U \subseteq cl_\gamma(A \cap U)$, *for any γ-open set U in X.*
(2) $int_\gamma(A \cup V) \subseteq int_\gamma(A) \cup V$, *for any γ-closed set V in X.*

2.9. Theorem. *Let X be a space and $A \subseteq X$. Then*
(1) $scl_\gamma(A) = A \cup int_\gamma(cl_\gamma(A))$.
(2) $sint_\gamma(A) = A \cap cl_\gamma(int_\gamma(A))$.
(3) $pcl_\gamma(A) = A \cup cl_\gamma(int_\gamma(A))$.
(4) $pint_\gamma(A) = A \cap int_\gamma(cl_\gamma(A))$.

Proof. (1) Since $int_\gamma(cl_\gamma(A)) \subseteq int_\gamma(cl_\gamma(scl_\gamma(A))) \subseteq scl_\gamma(A)$ and $A \cup scl_\gamma(A) = scl_\gamma(A) \supseteq A \cup int_\gamma(cl_\gamma(A))$. Then $A \cup int_\gamma(cl_\gamma(A)) \subseteq scl_\gamma(A)$.

For the reverse inclusion, we note that, $int_\gamma(cl_\gamma(A \cup int_\gamma(cl_\gamma(A)))) = int_\gamma(cl_\gamma(A) \cup cl_\gamma(int_\gamma(cl_\gamma(A)))) \subseteq cl_\gamma(A) \cup int_\gamma(cl_\gamma(int_\gamma(cl_\gamma(A)))) = cl_\gamma(A) \cup int_\gamma(cl_\gamma(A)) = cl_\gamma(A)$. Thus $int_\gamma(cl_\gamma(A \cup int_\gamma(cl_\gamma(A)))) \subseteq int_\gamma(cl_\gamma(A)) \subseteq A \cup int_\gamma(cl_\gamma(A))$. Therefore $A \cup int_\gamma(cl_\gamma(A))$ is γ-semi-closed and hence $scl_\gamma(A) \subseteq A \cup int_\gamma(cl_\gamma(A))$. This proves that $scl_\gamma(A) = A \cup int_\gamma(cl_\gamma(A))$.
(2) The proof follows from (1) and by taking complement.
(3) We observe that $cl_\gamma(int_\gamma(A \cup cl_\gamma(int_\gamma(A)))) \subseteq cl_\gamma(int_\gamma(A) \cup cl_\gamma(int_\gamma(A))) =$
$= cl_\gamma(int_\gamma(A)) \subseteq A \cup cl_\gamma(int_\gamma(A))$, by Lemma 2.8. Therefore $A \cup cl_\gamma(int_\gamma(A))$ is γ-pre-closed and thus $pcl_\gamma(A) \subseteq A \cup cl_\gamma(int_\gamma(A))$. Also, $pcl_\gamma(A)$ is γ-pre-closed follows that $cl_\gamma(int_\gamma(A)) \subseteq cl_\gamma(int_\gamma(pcl_\gamma((A))) \subseteq pcl_\gamma(A)$. Thus $A \cup cl_\gamma(int_\gamma(A)) \subseteq pcl_\gamma(A)$.
(4) The proof follows from (3) and by taking complement. □

The following theorem follows from Theorem 2.9.

2.10. Theorem. *Let X be a space and $A \subseteq X$. Then*
(1) $scl_\gamma(sint_\gamma((A))) = sint_\gamma(A) \cup int_\gamma(cl_\gamma(int_\gamma(A)))$.
(2) $pcl_\gamma(pint_\gamma((A))) = pint_\gamma(A) \cup cl_\gamma(int_\gamma(A))$.

Now we define the following.

2.11. Definition. Let X be a space and $A \subseteq X$. Then A is called γ-b-open set, if $A \subseteq cl_\gamma(int_\gamma(A)) \cup int_\gamma(cl_\gamma(A))$. The set consisting of all γ-b-open sets will be denoted as $BO_\gamma(X)$.

2.12. Definition. Let X be a space and $A \subseteq X$. Then A is called γ-b-closed set, if $A \supseteq cl_\gamma(int_\gamma(A)) \cap int_\gamma(cl_\gamma(A))$. The set consisting of all γ-b-closed sets will be denoted as $BC_\gamma(X)$.

2.13. Theorem. *Let X be a space and $A \subseteq X$.*
(1) A is γ-b-open if and only if A^c is γ-b-closed
(2) If A is γ-pre-open(closed), then A is γ-b-open(closed).
(3) If A is γ-semi-open(closed), then A is γ-b-open(closed).

Proof. (1) This directly follows from the definitions.
(2) Given A is γ-pre-open. Then $A \subseteq int_\gamma(cl_\gamma(A))$. This follows that $A \subseteq int_\gamma(cl_\gamma(A)) \subseteq int_\gamma(cl_\gamma(A)) \cup int_\gamma(A) \subseteq int_\gamma(cl_\gamma(A)) \cup cl_\gamma(int_\gamma(A))$. Which implies that A is γ-b-open set. Similar proof follows for γ-closed sets.
(3) A is γ-semi-open implies that $A \subseteq cl_\gamma(int_\gamma(A))$. This follows that $A \subseteq cl_\gamma(int_\gamma(A)) \cup int_\gamma(A) \subseteq cl_\gamma(int_\gamma(A)) \cup int_\gamma(cl_\gamma(A))$. Hence A is γ-b-open set. Similar proof is for γ-semi-closed sets. This completes the proof. □

Note that every γ-open set is a γ-b-open set and a γ-pre-open set. But the converse is not true in general as shown in the following example. Moreover the following examples show that the converses of (2) and (3) are not true in general.

2.14. Example. Let X $= \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$ be topology on X. For $b \in X$, define an operation $\gamma : \tau \to P(X)$ by

$$\gamma(A) = A^\gamma = \begin{cases} A, & \text{if } b \in A \\ cl(A), & \text{if } b \notin A \end{cases}.$$

Clearly, γ-open sets are $\phi, X, \{b\}, \{a, b\}, \{a, c\}$. Let $A = \{b, c\}$, calculations show that A is γ-b-open and γ-pre-open but not γ-open and γ-semi-open set.

2.15. Example. Let X $= \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{a, b\}\}$ be topology on X. For $b \in X$, define an operation $\gamma : \tau \to P(X)$ by

$$\gamma(A) = A^\gamma = \begin{cases} cl(A), & \text{if } b \in A \\ A, & \text{if } b \notin A \end{cases}.$$

Clearly, γ-open sets are $\phi, X, \{a\}$ and γ-semi-open sets are $\phi, X, \{a\}, \{a, b\}, \{a, c\}$. Let $A = \{b, c\}$. Calculations show that the set A is γ-b-open but neither γ-pre-open nor γ-semi-open.

2.16. Proposition. *Let X be a space and $A \subseteq X$. If A is γ-b-open with $int_\gamma(A) = \phi$. Then A is γ-pre-open.*

2.17. Proposition. *Let X be a space and $A \subseteq X$.*
(1) The union of any collection of γ-b-open sets is γ-b-open.
(2) The intersection of any collection of γ-b-closed sets is γ-b-closed.
(3) The intersection of a γ-open and a γ-b-open set is a γ-b-open set.

Proof. (1) Suppose $\{A_i : i \in I\}$ be a collection of γ-b-open sets. Therefore, for any i, $A_i \subseteq cl_\gamma(int_\gamma(A_i)) \cup int_\gamma(cl_\gamma(A_i))$. This follows that $\bigcup_{i\in I} A_i \subseteq \bigcup_{i\in I}\{cl_\gamma(int_\gamma(A_i)) \cup int_\gamma(cl_\gamma(A_i))\} \subseteq cl_\gamma(int_\gamma(\bigcup_{i\in I} A_i)) \cup int_\gamma(cl_\gamma(\bigcup_{i\in I} A_i))$. This implies that $\bigcup_{i\in I} A_i$ is γ-b-open sets.
(2) This follows from (1) and by taking complement.
(3) Let A be any γ-b-open set and B a γ-open set. Consider $B \cap A \subseteq B \cap \{cl_\gamma(int_\gamma(A)) \cup int_\gamma(cl_\gamma(A))\} = \{B \cap int_\gamma(cl_\gamma(A))\} \cup \{B \cap cl_\gamma(int_\gamma(A))\} \subseteq \{cl_\gamma(int_\gamma(B)) \cap cl_\gamma(int_\gamma(A))\} \cup \{int_\gamma(cl_\gamma(B)) \cap int_\gamma(cl_\gamma(A))\} \subseteq cl_\gamma(int_\gamma(A \cap B)) \cup int_\gamma(cl_\gamma(A \cap B))$. This implies that $B \cap A$ is a γ-b-open set. □

2.18. Proposition. *Let X be a space and $A \subseteq X$. Then the following statements are equivalent.*
(1) A is γ-b-open.
(2) $A \subseteq pcl_\gamma(pint_\gamma(A))$.
(3) $A = pint_\gamma(A) \cup sint_\gamma(A)$.

Proof. (1) $\Rightarrow$ (3) A is γ-b-open implies that, $A \subseteq cl_\gamma(int_\gamma(A)) \cup int_\gamma(cl_\gamma(A))$. Then Theorem 2.9 implies that $pint_\gamma(A) \cup sint_\gamma(A) = (A \cap int_\gamma(cl_\gamma(A))) \cup (A \cap cl_\gamma(int_\gamma(A))) = A \cap (int_\gamma(cl_\gamma(A)) \cup (cl_\gamma(int_\gamma(A)))) = A$.

(3) $\Rightarrow$ (2) Using Theorems 2.9 and 2.10, we get, $A = pint_\gamma(A) \cup sint_\gamma(A) = pint_\gamma(A) \cup (A \cap cl_\gamma(int_\gamma(A))) \subseteq pint_\gamma(A) \cup cl_\gamma(int_\gamma(A)) = pcl_\gamma(pint_\gamma(A))$.
(2) $\Rightarrow$ (1) Using Theorems 2.9 and 2.10, we have, $A \subseteq pcl_\gamma(pint_\gamma(A)) = (pint_\gamma(A)) \cup cl_\gamma(int_\gamma(A)) \subseteq int_\gamma(cl_\gamma(A)) \cup cl_\gamma(int_\gamma(A))$. This implies that A is γ-b-open. This completes the proof. □

2.19. Remark. Above theorem follows that every γ-b-open set may be expressed as the union of γ-pre-open and γ-semi-open sets.

2.20. Definition. Let X be a space and $A \subseteq X$. Then
(1) γ-b-closure of A denoted $bcl_\gamma(A)$ and is defined as the smallest γ-b-closed set B such that $A \subseteq B$.
(2) γ-b-interior of A denoted $bint_\gamma(A)$ and is defined as the largest γ-b-open set B such that $B \subseteq A$.

Note that if $A \subseteq B$, then $bcl_\gamma(A) \subseteq bcl_\gamma(B)$ and $bint_\gamma(A) \subseteq bint_\gamma(B)$.

2.21. Theorem. *Let X be a space and $A \subseteq X$. Then*
(1) $bcl_\gamma(A) = scl_\gamma(A) \cap pcl_\gamma(A)$.
(2) $bint_\gamma(A) = sint_\gamma(A) \cup pint_\gamma(A)$.
(3) $bcl_\gamma(A) = A \cup \{int_\gamma(cl_\gamma(A)) \cap cl_\gamma(int_\gamma(A))\}$.
(4) $bint_\gamma(A) = A \cap \{int_\gamma(cl_\gamma(A)) \cup cl_\gamma(int_\gamma(A))\}$.
(5) $bcl_\gamma(int_\gamma(A)) = int_\gamma(bcl_\gamma(A)) = int_\gamma(cl_\gamma(int_\gamma(A)))$.
(6) $bint_\gamma(cl_\gamma(A)) = cl_\gamma(bint_\gamma(A)) = cl_\gamma(int_\gamma(cl_\gamma(A)))$.
(7) $bcl_\gamma(sint_\gamma(A)) = scl_\gamma(sint_\gamma(A))$.
(8) $bint_\gamma(scl_\gamma(A)) = sint_\gamma(scl_\gamma(A))$.
(9) $sint_\gamma(bcl_\gamma(A)) = scl_\gamma(A) \cap cl_\gamma(int_\gamma(A))$.
(10) $scl_\gamma(bint_\gamma(A)) = sint_\gamma(A) \cup int_\gamma(cl_\gamma(A))$.
(11) $pint_\gamma(bcl_\gamma(A)) = bcl_\gamma(pint_\gamma(A)) = pint_\gamma(pcl_\gamma(A))$.
(12) $pcl_\gamma(bint_\gamma(A)) = bint_\gamma(pcl_\gamma(A)) = bcl_\gamma(pint_\gamma(A))$.
(13) $sint_\gamma(A) \cap pint_\gamma(A) \subseteq bint_\gamma(A)$.

Proof. (1) The inclusion $bcl_\gamma(A) \subseteq scl_\gamma(A) \cap pcl_\gamma(A)$ is clear. For reverse inclusion, $bcl_\gamma(A)$ is γ-b-closed implies that $bcl_\gamma(A) \supseteq int_\gamma(cl_\gamma(bcl_\gamma(A))) \cap cl_\gamma(int_\gamma(bcl_\gamma(A))) \supseteq int_\gamma(cl_\gamma(A)) \cap cl_\gamma(int_\gamma(A))$. Therefore, Theorem 2.9 implies that
$bcl_\gamma(A) \supseteq A \cup (int_\gamma(cl_\gamma(A)) \cap cl_\gamma(int_\gamma(A)) = scl_\gamma(A) \cap pcl_\gamma(A)$.
(2) The proof follows from (1) and by taking complement.
(3 - 12) These follow using (1), (2), definitions and Theorems 2.9 and 2.10.
(13) $sint_\gamma(A) \cup pint_\gamma(A) \subseteq \{A \cap cl_\gamma(int_\gamma(A))\} \cup \{A \cap int_\gamma(cl_\gamma(A))\} = A \cap \{cl_\gamma(int_\gamma(A)) \cup int_\gamma(cl_\gamma(A))\} = bint_\gamma(A)$. □

2.22. Proposition. *Let X be a space and $A \subseteq X$. Then*

$$bint_\gamma(bcl_\gamma(A))) = bcl_\gamma(bint_\gamma(A)).$$

Proof. The proof follows from Theorems 2.9, 2.10 and 2.21. □

2.23. Theorem. *Let X be a space and A and B are any subsets of X. Then*
(1) $bcl_\gamma(\phi) = \phi$, $bint_\gamma(\phi) = \phi$.
(2) $bcl_\gamma(bcl_\gamma(A)) = A$, $bint_\gamma(bint_\gamma(A)) = A$.
(3) A is γ-b-open (γ-b-closed) iff $bint_\gamma(A) = A (bcl_\gamma(A) = A)$.
(4) $bcl_\gamma(A) \cup bcl_\gamma(B) \subseteq bcl_\gamma(A \cup B)\{bint_\gamma(A) \cup bint_\gamma(B) \subseteq bint_\gamma(A \cup B)\}$.
(5) $bcl_\gamma(A) \cap bcl_\gamma(B) \supseteq bcl_\gamma(A \cap B)\{bint_\gamma(A) \cap bint_\gamma(B) \supseteq bint_\gamma(A \cap B)\}$.

Proof. (1) and (2) are clear from the definitions.
(3) Let $A = bint_\gamma(A) = \cup\{B : \text{B is a } \gamma\text{-b-open and } B \subseteq A\}$. This follows that $A \in \cup\{B : \text{B is } \gamma\text{-b-open and } B \subseteq A\}$, which implies that A is γ-b-open. For the reverse inclusion, let A be γ-b-open. Then $A \in \cup\{B : \text{B is a } \gamma\text{-b-open and } B \subseteq A\}$. Now $A \supseteq B$ implies that $A = \cup\{B : \text{B is } \gamma\text{-b-open and } B \subseteq A\} = bint_\gamma(A)$.
Similar proof for γ-b-closed set.
(4) and (5) directly follows by using the property that, if $A \subseteq B$, then $bcl_\gamma(A) \subseteq bcl_\gamma(B)$ and $bint_\gamma(A) \subseteq bint_\gamma(B)$. □

3. γ-b-continuity

3.1. Definition. Let X and Y are two spaces. A function $f : X \to Y$ is called γ-b-continuous, if for each γ-open set V in Y, $f^{-1}(V)$ is γ-b-open in X.

3.2. Example. Let X $= \{a, b, c\}$, $\tau_X = \{\phi, X, \{a\}, \{c\}, \{a, c\}, \{a, b\}\}$ be topology on X. For $b \in X$, define an operation $\gamma : \tau \to P(X)$ by

$$\gamma(A) = A^\gamma = \begin{cases} A, & \text{if } b \in A \\ cl(A), & \text{if } b \notin A \end{cases}.$$

Clearly, γ-open sets in X are $\phi, X, \{c\}, \{a, b\}$. Calculations show that γ-b-open sets in X are $\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}$. Also let Y $= \{a, b, c\}$,
$\tau_Y = \{\phi, Y, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$ be topology on Y. For $b \in Y$, define an operation $\gamma : \tau \to P(Y)$ by

$$\gamma(A) = A^\gamma = \begin{cases} cl(A), & \text{if } b \in A \\ A, & \text{if } b \notin A \end{cases}.$$

Clearly, γ-open sets in Y are $\phi, Y, \{a\}, \{b\}, \{a, c\}$. The function $f : X \to Y$ defined by $f(a) = b, f(b) = c$ and $f(c) = a$. Simple calculations show that f is γ-b-continuous function.

3.3. Theorem. *Suppose that $f : X \to Y$ is a function from a space X to a space Y. Then f is γ-b-continuous if and only if for each γ-closed set B in Y, $f^{-1}(B)$ is γ-b-closed in X.*

Proof. ($\Rightarrow$) Suppose that f is γ-b-continuous and B be any γ-closed set in Y. Then $(Y - B)$ is γ-b-open in Y. Therefore, $f^{-1}(Y - B)$ is γ-b-open in X. This implies that $X - f^{-1}(Y - B) = f^{-1}(Y) - f^{-1}(Y - B) = f^{-1}(Y - (Y - B)) = f^{-1}(B)$. This follows that $f^{-1}(B)$ is γ-b-closed in X.
($\Leftarrow$) Let V be a γ-open set in Y, we show that $f^{-1}(V)$ is γ-b-open in X. Since V^c is γ-closed in Y, then by our supposition, $f^{-1}(V^c)$ is γ-b-closed in X. This follows that $f^{-1}(V) = (f^{-1}(V^c))^c$ is a γ-b-open set in X. This completes the proof. □

3.4. Theorem. *Let X and Y are two spaces. A function $f : X \to Y$ is γ-b-continuous, if for each $x \in X$ and each γ-open set B such that $f(x) \in B$, there exists a γ-b-open set $A \in X$ such that $f(A) \subseteq B$.*

Proof. Suppose that f is γ-b-continuous. Therefore for γ-open set B in Y, $f^{-1}(B)$ is γ-b-open in X. We prove that for each γ-open set B containing $f(x)$, there exists γ-b-open set A in X such that $x \in A$ and $f(A) \subseteq B$. Let $x \in f^{-1}(B)$ and $A = f^{-1}(B)$. Then $x \in A$ and $f(A) \subseteq ff^{-1}(B) \subseteq B$, where B is γ-open.
Conversely, let B be γ-open set in Y. We prove that inverse image of γ-open set in Y is γ-b-open set in X. Let $x \in f^{-1}(B)$. Then $f(x) \in B$. Thus there exists γ-b-open set A_x such that $x \in A_x$ and $f(A_x) \in B$. Then $x \in A_x \subseteq f^{-1}(B)$ and $f^{-1}(B) = \bigcup_{x \in f^{-1}(B)} A_x$ implies $f^{-1}(B)$ is γ-b-open set in X. This proves that f is γ-b-continuous. □

3.5. Definition. Let X and Y are two spaces. A function $f : X \to Y$ is called γ-b-irresolute, if for each γ-b-closed set B in Y, $f^{-1}(B)$ is γ-b-closed in X.

The following theorem directly follows from above definition.

3.6. Theorem. *Suppose that $f : X \to Y$ is a function from space X to space Y. Then f is γ-b-irresolute function, if for each γ-b-open set U in Y, $f^{-1}(U)$ is a γ-b-open set in X.*

3.7. Theorem. *Let X and Y are two spaces. If a function $f : X \to Y$ is γ-b-irresolute, then it is a γ-b-continuous function.*

Proof. Suppose that the function f is γ-b-irresolute and B be γ-closed in Y. Then B is γ-b-closed in Y. This implies that $f^{-1}(B)$ be γ-b-closed in X. Hence f is γ-b-continuous mapping. □

3.8. Definition. Let X and Y are two spaces. A function $f : X \to Y$ is called γ-b-open(closed), if for each γ-open(closed) set A in X, $f(A)$ is γ-b-open(closed) in Y.

3.9. Theorem. *Let X and Y are topological spaces and function $f : X \to Y$ be a functions. Then the following statements are equivalent:*
(1) f is γ-b-open.
(2) For each set B of Y and for each γ-open set A in X such that $f^{-1}(B) \subseteq A$, there is a γ-b-open set U of Y such that $B \subseteq U$ and $f^{-1}(U) \subseteq A$.

Proof. (1) $\Rightarrow$ (2) Suppose that f be a γ-b-open function. Let $B \subseteq Y$ and A be a γ-open set in X such that $f^{-1}(B) \subseteq A$. Therefore $U = f(A)$ is a γ-b-open set in Y with $B \subseteq U$ and $f^{-1}(U) \subseteq A$.
(2) $\Rightarrow$ (1) Let $B \subseteq X$ be a γ-closed set. This implies that $f^{-1}(f(B^c)) \subseteq B^c$, where B^c is a γ-open set. Using (2), there is a γ-b-open set $U \subseteq Y$ with $f(B^c) \subseteq U$ and $f^{-1}(U) \subseteq A = f^{-1}(f(B^c))$. Thus $U \subseteq f(f^{-1}(f(B^c))) \subseteq f(B^c) \subseteq U$. This follows that $f(B^c) = U$ is γ-b-open set. Hence f is γ-b-open function. □

3.10. Theorem. *Let X and Y are two spaces and $f : X \to Y$ be a function. Then the following statements are equivalent:*
(1) f is γ-b-closed.
(2) For any $B \subseteq X$, $bcl_\gamma(f(B)) \subseteq f(bcl_\gamma(B))$.

Proof. (1) $\Rightarrow$ (2) Let $B \subseteq X$ and since f is γ-b-closed , then $f(bcl_\gamma(B))$ is γ-b-closed in Y. Thus $f(B) \subseteq f(bcl_\gamma(B))$ implies that $bcl_\gamma(f(B)) \subseteq f(bcl_\gamma(B))$.
(2) $\Rightarrow$ (1) Let A be a γ-b-closed set in X. By (2), we have $f(A) \subseteq bcl_\gamma(f(A)) \subseteq f(bcl_\gamma(A)) \subseteq f(A)$. Hence $f(A) = bcl_\gamma(f(A))$. This follows that f is γ-b-closed. Hence the proof. □

3.11. Theorem. *Let X, Y and Z are three spaces and $f : X \to Y$, $g : Y \to Z$ be two functions with $gof : X \to Z$ is a γ-b-closed function. If g is injective and γ-b-irresolute, then f is a γ-b-closed function.*

Proof. Suppose that $U \subseteq X$ be a γ-closed set. Then $gof(U)$ is γ-b-closed set in Z. This implies that $g^{-1}(gof(U)) = f(U)$ is a γ-b-closed set in Y. Thus g is γ-b-irresolute and injective implies that f is γ-b-closed function. □

3.12. Definition. A net $(x_\alpha)_{\alpha \in I}$ is called γ-b-converges to a point x in a space X, if for any γ-b-open set U and $x \in U$, there exists $\alpha_0 \in I$ with $\alpha \geq \alpha_0$ implies $x_\alpha \in U$.

3.13. Definition. A filterbase Γ in X, γ-b-converges to a point x in a space X, if for any γ-b-open set U with $x \in U$, there exists $F \in \Gamma$ such that $F \subseteq U$.

3.14. Definition. A filterbase Γ in X, γ-b-accumulates to a point x in a space X, if for any γ-b-open set U with $x \in U$ and any $F \in \Gamma$, $F \cap U \neq \phi$.

The following theorem directly follows from the above definitions:

3.15. Theorem. *If a filterbase Γ in X, γ-b-converges to a point x in a space X, then Γ γ-b-accumulates to x.*

3.16. Theorem. *Let X and Y are two spaces and $f : X \to Y$ be a function. Then the following statements are equivalent:*
(1) f is γ-b-irresolute.
(2) The net $(f(x_i))_{i\in I}$ of Y, γ-b-converges to $f(x)$ in Y, for each x in X and each net $(x_i)_{i\in I}$ in X, which γ-b-converges to x.
(3) $f(\Gamma)$ γ-b-converges to $f(x)$, for each $x \in X$ and each filterbase Γ, which γ-b-converges to x.

Proof. The proof is easy and thus omitted. □

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