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POINT ESTIMATION OF PARAMETERS IN DISCRETE BURR DISTRIBUTION BASED ON TYPE I CENSORED SAMPLE

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Abstract: In this study, maximum likelihood, proportion and modified maximum likelihood estimators of discrete Burr distribution parameters based on Type I censored sample are obtained. Monte Carlo simulation study is performed to compare these estimators in terms of their biases and mean squared errors. A numerical example is also provided.

Key words: Discrete Burr distribution, Maximum likelihood, Modified maximum likelihood, Monte Carlo simulation, Proportion method.

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1. Introduction

In life testing experiments, a number of continuous models have been suggested and studied (see, for example, Lawless [6] and Sinha [12]). However, it is sometimes impossible or inconvenient to measure the life length of a device on a continuous scale. In practice, we come across situations, where lifetime of a device is considered to be a discrete random variable.

In the last two decades, well known discrete distributions like geometric and negative binomial have been employed to model lifetime data. However, there is a need to find more plausible discrete lifetime distributions to fit to various types of lifetime data. For this purpose, recently some popular continuous lifetime distributions are discretized.

Nakagawa [14] discretized the Weibull distribution. Stein and Dattero [15] discussed another discrete type of Weibull distribution. Roy ([10], [11]) considered discrete normal and Rayleigh distributions. Krishna and Pundir [4] proposed discrete Burr and Pareto distributions. Jazi et al. [1] studied discrete inverse Weibull distribution. These authors have also studied distributional and reliability properties of such discretized distributions.

Type I censoring is very common in nature. A life testing experiment may be terminated after a certain number of cycles or weeks prefixed by an experimenter due to various constraints. This gives rise to a natural Type I censored data set. As an example, Kulasekera [3] studied discrete Weibull distribution based on Type I censored data. In this paper, estimation problem for discrete Burr parameters are discussed under Type I censoring.

The paper is organized as follows: In Section 2, maximum likelihood (ML), modified maximum likelihood (MML) and method of proportion (MP) estimators of discrete Burr [4] parameters based on Type I censored sample are given. Simulation study is conducted to see the performance of these estimators in Section 3. Finally, a numerical example is given in Section 4.

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2. Discrete Burr Distribution

Krishna and Pundir [4] defined the probability mass function (pmf) and the cumulative distribution function (cdf) of the discrete Burr distribution with parameter q and β are given, respectively, by

$$f(x) = q^{\log\left(1+x^{\beta}\right)} - q^{\log\left(1+(1+x)^{\beta}\right)}, \quad x = 0, 1, 2, \ldots \tag{2.1}$$

$$F(x) = 1 - q^{\log\left(1+x^{\beta}\right)}, \quad x = 0, 1, 2, \ldots, \tag{2.2}$$

where $0 < q < 1$ and $\beta > 0$ are parameters. It will be denoted by DBD(q, β).

2.1. Estimation Under Type I Censoring

Method 1: Maximum Likelihood Estimation

Let $X_1^0, X_2^0, \ldots, X_n^0$ be i.i.d. observations from DBD(q, β) and let each observation be Type I censored by $L_1, \ldots, L_n$ respectively. Thus we observe

$$X_i = \min\left(X_i^0, L_i\right), \quad i = 1, 2, \ldots, n$$

along with the indicator variables (Kulasekera [3])

$$\delta_i = \begin{cases} 1, & if\ X_i = X_i^0 \\ 0, & otherwise. \end{cases}$$

Then the log likelihood function for this data is given by

$$\begin{aligned} \ell(q, \beta) &= \log\left\{\prod_{i=1}^{n} f(x_i)^{\delta_i} \left(1 - F(x_i)\right)^{1-\delta_i}\right\} \\ &= \sum_{i=1}^{n} \delta_i \log\left(f(x_i)\right) + \sum_{i=1}^{n} (1-\delta_i) \log\left(1 - F(x_i)\right) \\ &= \sum_{i=1}^{n} \delta_i \log\left(q^{\log\left(1+x_i^{\beta}\right)} - q^{\log\left(1+(1+x_i)^{\beta}\right)}\right) + \sum_{i=1}^{n} (1-\delta_i) \log\left(q^{\log\left(1+x_i^{\beta}\right)}\right) \end{aligned} \tag{2.3}$$

and the likelihood equations are given, respectively, by

$$\begin{aligned} \frac{\partial \ell(q, \beta)}{\partial q} &= \sum_{i=0}^{n} \frac{\delta_i}{q} \left(\frac{q^{\log\left(1+x_i^{\beta}\right)} \log\left(1+x_i^{\beta}\right) - q^{\log\left(1+x_i^{\beta}\right)} \log\left(1+(1+x_i)^{\beta}\right)}{q^{\log\left(1+x_i^{\beta}\right)} - q^{\log\left(1+(1+x_i)^{\beta}\right)}}\right) \\ &\quad + \sum_{i=0}^{n} \frac{(1-\delta_i) \log\left(1+x_i^{\beta}\right)}{q} = 0, \end{aligned} \tag{2.4}$$

$$\begin{aligned} \frac{\partial \ell(q, \beta)}{\partial \beta} &= \sum_{i=0}^{n} \delta_i \frac{q^{\log\left(1+x_i^{\beta}\right)} x_i^{\beta} \log(x_i) \log(q)}{1+x_i^{\beta}} \left(\left(q^{\log\left(1+x_i^{\beta}\right)} - q^{\log\left(1+(1+x_i)^{\beta}\right)}\right)\right)^{-1} \\ &\quad - \sum_{i=0}^{n} \delta_i \frac{q^{\log\left(1+(1+x_i)^{\beta}\right)} (1+x)^{\beta} \log(1+x_i) \log(q)}{\left(1+(1+x_i)^{\beta}\right)\left(\left(q^{\log\left(1+x_i^{\beta}\right)} - q^{\log\left(1+(1+x_i)^{\beta}\right)}\right)\right)} \\ &\quad + \sum_{i \neq 0}^{n} \frac{(1-\delta_i) x_i^{\beta} \log(x_i) \log(q)}{1+x_i^{\beta}} = 0. \end{aligned} \tag{2.5}$$

The solution of these likelihood equations will provide the ML estimators of q and β and it will be denoted by $\hat{q}$ and $\hat{\beta}$, respectively. Since Eq.(2.4) and Eq.(2.5) cannot be solved analytically for q and β, some numerical methods such as Newton's method must be employed.

Method 2: Method of proportions

Khan et al. [2] proposed a method of proportions to estimate the parameters of discrete Weibull distribution. Their method is applied to estimate the DBD(q,β) parameters.

Let $X_1, X_2, \ldots, X_n$ be the Type I censored sample from DBD(q,β) and define indicator function

$$\upsilon(X_i) = \begin{cases} 1, & X_i = 0 \\ 0, & X_i > 0 \end{cases} \tag{2.6}$$

Then $Y = \frac{1}{n}\sum_{i=1}^{n} \upsilon(X_i)$ denotes the proportion of 0's in the sample. It is clear that the proportion Y is an unbiased and a consistent estimator of the probability $f(0) = 1 - (q)^{\log(2)}$. Therefore, an estimator of q can be offered as

$$\tilde{q} = (1 - Y)^{1/\log(2)}. \tag{2.7}$$

Similarly, the proportion of 1's in the sample, denoted by Z, is also an unbiased and a consistent estimator of the probability $f(1) = q^{\log(2)} - q^{\log\left(1+2^{\beta}\right)}$. Hence the proportion estimator of β, is given by

$$\tilde{\beta} = \log\left(\exp\left\{\frac{\log\left(\tilde{q}^{\log(2)} - Z\right)}{\log(\tilde{q})}\right\} - 1\right)(\log(2))^{-1}, \tag{2.8}$$

where $\tilde{q}$ is given in Eq.(2.7). Note that if there is no observations equal to 0 in the sample, one can obtained proportion estimators by equating $f(1)$ and $f(2)$ to proportion of 1's and 2's in the sample, respectively. In this case, the performance of proportion estimators may be changed.

Method 3: Modified Maximum Likelihood Estimation

Let $X_1, X_2, \ldots, X_n$ be Type I censored sample from DBD(q,β). The ML estimators of q and β are obtained by solving Eq. (2.4) and Eq. (2.5) for q and β. An iterative method such as Newton-Raphson should be used to get the ML estimator. In order to avoid this process, we used Kulasekera's [3] approximation to obtain the ML estimators in the following fashion. Let us assume $L_i > 0$, $i = 1, 2, \ldots, n$. Let $f_1(q,\beta) = LHS$ of (2.4) and $f_2(q,\beta) = LHS$ of (2.5). Then to obtain the ML estimators of q and β, following equations should be solved:

$$\left.\begin{array}{l} f_1(q,\beta) = 0 \\ f_2(q,\beta) = 0 \end{array}\right\} \tag{2.9}$$

This system of equations cannot be solved analytically. Now, instead of solving Eq.'s (2.9), one can solve the system

$$\left.\left(f_1(q,\beta) + (\hat{q}_* - q)\frac{\partial f_1(q,\beta)}{\partial q} + \left(\hat{\beta}_* - \beta\right)\frac{\partial f_1(q,\beta)}{\partial \beta}\right)\right|_{\substack{q=\tilde{q} \\ \beta=\tilde{\beta}}} = 0 \tag{2.10}$$

$$\left.\left(f_2(q,\beta) + (\hat{q}_* - q)\frac{\partial f_2(q,\beta)}{\partial q} + \left(\hat{\beta}_* - \beta\right)\frac{\partial f_2(q,\beta)}{\partial \beta}\right)\right|_{\substack{q=\tilde{q} \\ \beta=\tilde{\beta}}} = 0, \tag{2.11}$$

TABLE 1. Mean square errors of estimators DBD ($q=0.3, \beta=0.5$)

		ML		MP		$UMML$		MML	
L	n	$\hat{q}$	$\hat{\beta}$	$\tilde{q}$	$\tilde{\beta}$	$\hat{q}_{**}$	$\hat{\beta}_{**}$	$\hat{q}_{*}$	$\hat{\beta}_{*}$
3	100	0.0024	0.0161	0.0024	0.0266	0.0025	0.0169	0.0024	0.0204
3	300	0.0008	0.0063	0.0008	0.0097	0.0008	0.0063	0.0008	0.0061
3	500	0.0005	0.0038	0.0005	0.0058	0.0005	0.0038	0.0006	0.0052
5	100	0.0023	0.0118	0.0023	0.0262	0.0024	0.0123	0.0026	0.0152
5	300	0.0008	0.0042	0.0008	0.0095	0.0008	0.0042	0.0008	0.0042
5	500	0.0005	0.0024	0.0005	0.0057	0.0005	0.0024	0.0006	0.0029
7	100	0.0024	0.0116	0.0024	0.0256	0.0024	0.0113	0.0027	0.0135
7	300	0.0008	0.0037	0.0008	0.0093	0.0008	0.0037	0.0007	0.0036
7	500	0.0005	0.0022	0.0005	0.0056	0.0005	0.0022	0.0004	0.0022
∞	100	0.0024	0.0116	0.0024	0.0254	0.0024	0.0112	0.0024	0.0124
∞	300	0.0008	0.0024	0.0008	0.0091	0.0008	0.0024	0.0006	0.0038
∞	500	0.0005	0.0015	0.0005	0.0055	0.0005	0.0015	0.0005	0.0015

where $\tilde{q}$ and $\tilde{\beta}$ are the proportion estimators. This procedure gives us MML estimators which are obtained explicitly by

$$\hat{\beta}_* = \left. \frac{-f_2(q,\beta)\frac{\partial f_1(q,\beta)}{\partial q} + \frac{\partial f_2(q,\beta)}{\partial q} f_1(q,\beta) - \frac{\partial f_2(q,\beta)}{\partial q}\frac{\partial f_1(q,\beta)}{\partial \beta}\beta + \frac{\partial f_2(q,\beta)}{\partial \beta}\frac{\partial f_1(q,\beta)}{\partial q}\beta}{\frac{\partial f_2(q,\beta)}{\partial \beta}\frac{\partial f_1(q,\beta)}{\partial q} - \frac{\partial f_2(q,\beta)}{\partial q}\frac{\partial f_1(q,\beta)}{\partial \beta}} \right|_{\substack{q=\tilde{q} \\ \beta=\tilde{\beta}}} \tag{2.12}$$

and

$$\hat{q}_* = -\left. \frac{f_1(q,\beta) - \frac{\partial f_1(q,\beta)}{\partial q} q + \frac{\partial f_1(q,\beta)}{\partial \beta}\hat{\beta}_* - \frac{\partial f_1(q,\beta)}{\partial \beta}\beta}{\frac{\partial f_1(q,\beta)}{\partial q}} \right|_{\substack{q=\tilde{q} \\ \beta=\tilde{\beta}}} . \tag{2.13}$$

Revised estimators: Following Lee *et al.* [5] and Tiku and Vaughan [15] and Tiku and Akkaya [14], we can update the estimators (2.12) and (2.13) by replacing $\tilde{q}$ and $\tilde{\beta}$ with $\hat{q}_*$ and $\hat{\beta}_*$. This process needs to be repeated only a few times until the estimators stabilize sufficiently enough. These estimators are called updated modified ML(UMML) estimators denoted by $\hat{q}_{**}$ and $\hat{\beta}_{**}$. After simulation studies, it is observed that UMML estimators are almost identical to ML estimators. It should be pointed out that the UMML estimators are almost the same as ML estimators and it is not necessary to use any iteration method unlike the ML estimators from Eqs (2.4) and (2.5) are obtained.

3. Simulation Study and Conclusion

In this section, ML, MP, UMML and MML estimators are compared in terms of MSE and Bias under Type I censoring and complete sample cases by using Monte Carlo Simulation. 10000 trials are used to get the results given in Table 1-4. Simulation is performed for ($n=100,300,500$), $L_i=L=3,5,7, \quad i=1,...,n$, and different values of parameters

TABLE 2. Mean square errors of estimators DBD ($q=0.5, \beta=1.5$)

		ML		MP		$UMML$		MML	
L	n	$\hat{q}$	$\hat{\beta}$	$\tilde{q}$	$\tilde{\beta}$	$\hat{q}_{**}$	$\hat{\beta}_{**}$	$\hat{q}_{*}$	$\hat{\beta}_{*}$
3	100	0.0032	0.0878	0.0032	0.1252	0.0031	0.0860	0.0072	0.0686
3	300	0.0011	0.0323	0.0011	0.0402	0.0011	0.0323	0.0028	0.0519
3	500	0.0006	0.0150	0.0006	0.0229	0.0006	0.0120	0.0010	0.0282
5	100	0.0032	0.0831	0.0032	0.1249	0.0032	0.0792	0.0055	0.0395
5	300	0.0011	0.0256	0.0011	0.0396	0.0011	0.0256	0.0010	0.0243
5	500	0.0006	0.0149	0.0006	0.0254	0.0006	0.0149	0.0010	0.0237
7	100	0.0031	0.0748	0.0031	0.1210	0.0035	0.0772	0.0052	0.0287
7	300	0.0011	0.0245	0.0011	0.0400	0.0011	0.0245	0.0012	0.0250
7	500	0.0007	0.0150	0.0007	0.0236	0.0007	0.0150	0.0009	0.0202
∞	100	0.0030	0.0722	0.0031	0.1190	0.0035	0.0779	0.0044	0.0280
∞	300	0.0011	0.0236	0.0011	0.0388	0.0011	0.0236	0.0009	0.0226
∞	500	0.0006	0.0141	0.0006	0.0235	0.0006	0.0141	0.0008	0.0146

TABLE 3. Bias of estimators DBD ($q=0.3, \beta=0.5$)

		ML		MP		$UMML$		MML	
L	n	$\hat{q}$	$\hat{\beta}$	$\tilde{q}$	$\tilde{\beta}$	$\hat{q}_{**}$	$\hat{\beta}_{**}$	$\hat{q}_{*}$	$\hat{\beta}_{*}$
3	100	0.0071	0.0466	0.0071	0.0463	0.0065	0.0447	-0.0075	0.0395
3	300	0.0005	0.0033	0.0005	0.0029	0.0005	0.0033	-0.0012	0.0028
3	500	0.0005	0.0015	0.0005	0.0016	0.0005	0.0015	-0.0010	-0.0014
5	100	0.0018	0.0140	0.0018	0.0322	0.0015	0.0132	-0.0013	0.0134
5	300	0.0005	0.0031	0.0005	0.0010	0.0005	0.0031	-0.0010	0.0024
5	500	0.0004	0.0013	0.0004	0.0013	0.0004	0.0013	-0.0009	-0.0013
7	100	0.0020	0.0116	0.0020	0.0135	0.0012	0.0083	-0.0010	0.0111
7	300	0.0002	0.0019	0.0010	0.0040	0.0002	0.0019	-0.0009	0.0021
7	500	0.0004	0.0012	0.0004	0.0016	0.0004	0.0012	-0.0008	0.0010
∞	100	0.0016	0.0113	0.0016	0.0115	0.0011	0.0079	-0.0010	-0.0083
∞	300	0.0010	0.0013	0.0010	0.0040	0.0010	0.0013	-0.0009	-0.0018
∞	500	0.0003	0.0010	0.0004	0.0015	0.0003	0.0010	-0.0007	0.0010

From Table 1-4, It is observed that MSE and Bias of estimators decrease when n increases as expected for all estimators discussed here. When L (censoring bound) increases then bias and MSE decrease as also expected for all estimators. ML and UMML estimators of β are better than the others in terms of MSE in almost all sample size and censoring schemes. All estimator of parameter q have the same performance in terms of MSE criterion. From Table 3 and 4, the bias of estimators decrease when L or n increases.

4. A real data example

Here, we consider the recordings of Phyo [9] of the total number of carious teeth among the four deciduous molars in a sample of 100 children 10 and 11 years old. Symmetry between right and left molars is presumed and only the right molars are considered with a time unit of two years. The data are given in Table 5.

Table 4. Bias of estimators DBD ($q = 0.5, \beta = 1.5$)

		ML		MP		$UMML$		MML	
L	n	$\hat{q}$	$\hat{\beta}$	$\tilde{q}$	$\tilde{\beta}$	$\hat{q}_{**}$	$\hat{\beta}_{**}$	$\hat{q}_{*}$	$\hat{\beta}_{*}$
3	100	0.0023	0.0366	0.0013	0.0360	-0.0031	0.0320	-0.0027	0.0415
3	300	0.0007	0.0123	0.0007	0.0115	0.0007	0.0123	-0.0014	-0.0221
3	500	-0.0002	0.0091	-0.0002	0.0086	-0.0002	0.0091	0.0009	0.0182
5	100	0.0011	0.0362	0.0011	0.0354	0.0012	0.0319	-0.0015	-0.0302
5	300	0.0005	0.0137	0.0005	0.0117	0.0005	0.0137	-0.0013	0.0140
5	500	0.0005	0.0085	0.0005	0.0070	0.0005	0.0085	-0.0009	-0.0074
7	100	0.0006	0.0307	0.0006	0.0344	0.0010	0.0266	-0.0014	0.0220
7	300	0.0003	0.0118	0.0010	0.0102	0.0003	0.0118	-0.0009	0.0123
7	500	0.0002	0.0086	0.0006	0.0071	0.0002	0.0084	-0.0005	-0.0073
∞	100	0.0005	0.0284	0.0004	0.0310	0.0009	0.0211	-0.0012	0.0116
∞	300	-0.0002	0.0096	-0.0002	0.0063	-0.0002	0.0096	-0.0004	0.0071
∞	500	0.0002	0.0084	0.0002	0.0059	0.0002	0.0084	-0.0004	0.0069

Table 5. Total number of carious teeth

Total number of carious teeth(x)	0	1	2	3	4	Total
Frequency	64	17	10	6	3	100

Table 6. Estimations of Discrete Burr parameters for real data in Table 5

	Complete data		$L = 2$		$L = 3$	
	q	β	q	β	q	β
ML	0.2312	1.4408	0.2322	1.4820	0.2314	1.4124
MP	0.2290	1.0604	0.2290	1.0604	0.2290	1.0604
MML	0.2163	1.3240	0.2197	1.3529	0.2195	1.3125
UMML	0.2312	1.4408	0.2322	1.4820	0.2314	1.4124

The ML, MP, MML and UMML estimates of q and β based on the real data set given in Table 6.

In Table 7, we also reported the χ^2 goodness of fit test results for discrete Burr distribution based on the ML, MP and MML estimates of q and β. Since the MP method has smaller χ^2value, so it provides a better fit.

TABLE 7. χ^2 goodness of fit test results.

Total number of carious teeth(x_i)	o_i	Using MLE	Using MP	Using MMLE
0	64	63.7677	64	65.3979
1	17	21.6015	17	19.9364
2	10	7.1432	6.9645	6.8449
3	6	3.0354	3.5924	3.0225
4	3	1.5326	2.1226	1.5739
		$\chi^2 = 6.4239$	$\chi^2 = 3.2994$	$\chi^2 = 6.1422$
		Sig=0.0403	Sig=0.1921	Sig=0.0464

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