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MODIFIED ESTIMATORS OF RENYI ENTROPY WITH APPLICATION IN TESTING EXPONENTIALITY BASED ON TRANSFORMED DATA

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Abstract: This paper deals with the estimation problem of Renyi entropy of a continuous random variable. Three estimators are obtained by correcting the coefficients of the Vasicek-type estimator of Renyi entropy. We perform a simulation study to compare these estimators with the Vasicek-type estimator. The results show that our estimators are better than the existing one in terms of bias and root mean squared error. We also introduce goodness of fit tests for testing exponentiality based on the estimated Renyi entropy of transformed data. By Monte Carlo simulation, the powers of the proposed tests under various alternatives are compared with the famous existing tests.

Key words: Renyi information; Entropy estimator; Goodness of fit test; Uniformity

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1. Introduction

Let X be an absolutely continuous random variable with distribution function (cdf), $F(x)$, and continuous density function (pdf), $f(x)$. The basic measure of uncertainty contained in random variable X is defined by Shannon (1948) as

$$H(f) = -\int_{-\infty}^{+\infty} f(x)\ln f(x)dx = -E[\ln f(X)]. \tag{1.1}$$

Entropy has been used in a wide range of problems, including characterization of probability distributions and goodness of fit tests, analysis of censored data and reliability theory.

By increasing applications of entropy measures, it seems necessary to find accurate and robust nonparametric estimations of these measures. The estimation of (1.1) has been discussed by many authors including Ahmad and Lin (1976), Vasicek (1976), Dudewicz and Van der Meulen (1981), Bowman (1992), Van Es (1992), Ebrahimi *et al.* (1994), Correa (1995), Yousefzadeh and Arghami (2008), Alizadeh (2010), Alizadeh and Arghami (2010) and Zamanzade and Arghami (2011).

Many researchers are also interested in developing entropy based goodness of fit tests including Vasicek (1976), Arizono and Ohta (1989), Bowman (1992), Ebrahimi *et al.* (1992), Choi *et al.* (2004), and Abbasnejad (2011a, 2011b).

After the Shannon work, interest has also increased in the applications of other measures of uncertainty, such as Renyi entropy , cumulative residual entropy and survival entropy (See Rao *et al.* (2004) and Abbasnejad *et al.* (2010)).

Renyi (1961) has generalized the entropy as

$$H_r(f) = -\frac{1}{r-1}\ln\int_{-\infty}^{+\infty} f^r(x)dx = -\frac{1}{r-1}\ln E[f^{r-1}(X)]. \tag{1.2}$$

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It is to be noted that, as $r \to 1$, (1.2) reduces to $H(f)$.

Wachowiak *et al.* (2005) have proposed an estimator of Renyi entropy which was based on the fact that using $p = F(x)$, (1.2) can be expressed as

$$H_r(f) = -\frac{1}{r-1} \ln \int_0^1 \left[\frac{dF^{-1}(p)}{dp}\right]^{1-r} dp.$$

Replacing the distribution function F by the empirical distribution function F_n, and using a difference operator instead of the differential operator, the estimator is constructed as

$$HV_{r,m,n} = -\frac{1}{r-1} \ln \left\{ \frac{1}{n} \sum_{i=1}^{n} \left[\frac{n}{2m}(X_{i+m:n} - X_{i-m:n}) \right]^{1-r} \right\}, \tag{1.3}$$

where $X_{1:n} \leq ... \leq X_{n:n}$, are order statistics of random sample $X_1, ..., X_n$, m is a positive integer, $m \leq \frac{n}{2}$, and $X_{i:n} = X_{1:n}$ if $i < 1$, $X_{i:n} = X_{n:n}$ if $i > n$.

In Section 2, we propose three modified estimators of Renyi entropy. In Section 3, we compare the proposed estimators with the existing estimators based on their mean squared errors (MSEs). In Section 4, a goodness of fit test for exponentiality is proposed based on Renyi distance. Transformations of the observations are used to turn the test of exponentiality into one of uniformity and a corresponding test based on Renyi entropy is given.

2. Three modified estimators

By rewriting equation (1.3) as

$$HV_{r,m,n} = -\frac{1}{r-1} \ln \left\{ \frac{1}{n} \sum_{i=1}^{n} \left[\frac{F_n(X_{i+m:n}) - F_n(X_{i-m:n})}{X_{i+m:n} - X_{i-m:n}} \right]^{r-1} \right\}, \tag{2.1}$$

we can see that inside the brackets in the equation (2.1) is the slope of the straight line that joins points $(F_n(X_{i+m:n}), X_{i+m:n})$ and $(F_n(X_{i-m:n}), X_{i-m:n})$. It is clear that $\frac{n}{2m}(X_{i+m:n} - X_{i-m:n})$ is not a good approximation for the slope when $i \leq m$ or $i \geq n-m+1$. The main idea is to prepare the better approximation for the slope and propose modified estimators of Renyi entropy.

If in equation (2.1) we replace $F_n(X_{i-m:n})$ by $F_n(X_{1:n}) = \frac{1}{n}$ for $i \leq m$ and $F_n(X_{i+m:n})$ by $F_n(X_{n:n}) = 1$ for $i \geq n-m+1$, we obtain the first modified estimator of Renyi entropy as
− The first modified estimator

$$HE_{r,m,n} = -\frac{1}{r-1} \ln \left\{ \frac{1}{n} \sum_{i=1}^{n} \left[\frac{n}{c_i m}(X_{i+m:n} - X_{i-m:n}) \right]^{1-r} \right\},$$

where

$$c_i = \begin{cases} 1 + \frac{i-1}{m} & 1 \leq i \leq m \\ 2 & m+1 \leq i \leq n-m \\ 1 + \frac{n-i}{m} & n-m+1 \leq i \leq n. \end{cases}$$

Simulation results show that the bias of $HE_{r,m,n}$ is negative in most cases, that is, the first modified estimator underestimates the value of Renyi entropy. It may be because of the fact that we put too much weight on the extreme observations ($X_{i:n}$'s, $1 \leq i \leq m$ and $n-m+1 \leq i \leq n$). So replacing c_i's for $1 \leq i \leq m$ and $n-m+1 \leq i \leq n$ by smaller numbers could decrease the bias without affecting the standard deviation of the estimator. Therefore, we propose the second and third modified estimators as
− The second modified estimator

$$HA_{r,m,n} = -\frac{1}{r-1} \ln \left\{ \frac{1}{n} \sum_{i=1}^{n} \left[\frac{n}{a_i m}(X_{i+m:n} - X_{i-m:n}) \right]^{1-r} \right\},$$

where

$$a_i = \min_i c_i = \begin{cases} 1 & 1 \le i \le m \\ 2 & m+1 \le i \le n-m \\ 1 & n-m+1 \le i \le n. \end{cases}$$

− The third modified estimator

$$HZ_{r,m,n} = -\frac{1}{r-1} \ln \left\{ \frac{1}{n} \sum_{i=1}^{n} \left[\frac{n}{b_i m} (X_{i+m:n} - X_{i-m:n}) \right]^{1-r} \right\},$$

where

$$b_i = \begin{cases} \frac{i}{m} & 1 \le i \le m \\ 2 & m+1 \le i \le n-m \\ \frac{n-i+1}{m} & n-m+1 \le i \le n. \end{cases}$$

We choose the notations HE, HA and HZ because these estimators were obtained similar to modified estimators of Shannon entropy proposed by Ebrahimi *et al.* (1994), Alizadeh and Arghami (2010) and Zamanzade and Arghami (2011), respectively.

Monte Carlo studies show that these modifications reduce the bias and thus RMSE. In following theorem we show that the order of estimators of Renyi entropy.

THEOREM 1. *Let $X_{1:n}, ..., X_{n:n}$ be an ordered random sample from an unknown continuous distribution F with pdf f. Then*

$$HV_{r,m,n} \le HE_{r,m,n} \le HA_{r,m,n} \le HZ_{r,m,n}. \tag{2.2}$$

The second and third equalities hold if and only if $m = 1$.

PROOF. For $1 \le i \le m$, we have $c_i \le 2$, then

$$\left[\frac{n}{2m} (X_{i+m:n} - X_{i-m:n}) \right]^{1-r} \le \ (\ge) \ \left[\frac{n}{c_i m} (X_{i+m:n} - X_{i-m:n}) \right]^{1-r}, \qquad r < 1 \ (r > 1).$$

Similar result holds for $n-m+1 \le i \le n$ and so the first inequality of (2.2) follows by definitions of $HV_{r,m,n}$ and $HE_{r,m,n}$. Also, we can similarly prove the second and third inequalities.

In the next theorem, it is shown that the bias and mean squared error of $HE_{r,m,n}$, $HA_{r,m,n}$, $HZ_{r,m,n}$ and $HV_{r,m,n}$ are scale invariant.

THEOREM 2. *Let $H_r(X)$ and $H_r(Y)$ denote Renyi entropies of continuous random variables X and Y, respectively and $Y = bX$, where $b > 0$. Then:*
i) $E\left[Hi^Y_{r,m,n}\right] = E\left[Hi^X_{r,m,n}\right] + \ln b$,
ii) $Var\left[Hi^Y_{r,m,n}\right] = Var\left[Hi^X_{r,m,n}\right]$,
iii) $MSE\left[Hi^Y_{r,m,n}\right] = MSE\left[Hi^X_{r,m,n}\right]$,
for $i = 1, 2, 3, 4$, where the superscripts X and Y refer to the corresponding distribution and $H1 = HE_{r,m,n}$, $H2 = HA_{r,m,n}$, $H3 = HZ_{r,m,n}$, $H4 = HV_{r,m,n}$.

PROOF. It is easy to see that

$$\begin{aligned} HE^Y_{r,m,n} &= -\frac{1}{r-1} \ln \left\{ \frac{1}{n} \sum_{i=1}^{n} \left[\frac{n}{c_i m} (Y_{i+m:n} - Y_{i-m:n}) \right]^{1-r} \right\} \\ &= -\frac{1}{r-1} \ln \left\{ \frac{1}{n} \sum_{i=1}^{n} b^{1-r} \left[\frac{n}{c_i m} (X_{i+m:n} - X_{i-m:n}) \right]^{1-r} \right\} = HE^X_{r,m,n} + \ln b, \end{aligned}$$

and (i), (ii) and (iii) are proved easily. The proof for $H2, H3$ and $H4$ is similar to $H1$.

To study the behavior of the proposed modified estimators of Renyi entropy, a simulation study was performed. We consider four estimators, Vasicek-type estimator $HV_{r,m,n}$, the first modified estimator $HE_{r,m,n}$, the second modified estimator $HA_{r,m,n}$ and the third modified estimator $HZ_{r,m,n}$.

For our simulation we have used normal, exponential and uniform distributions which are the same distributions considered in Correa (1995), Alizadeh (2010), Ebrahimi *et al.* (1994), Alizadeh and Arghami (2010) and Zamanzade and Arghami (2011).

TABLE 1. RMSE and bias of the estimators of Renyi entropy of the uniform distribution.

		RMSE				Bias			
n	r	HV_{rmn}	HE_{rmn}	HA_{rmn}	HZ_{rmn}	HV_{rmn}	HE_{rmn}	HA_{rmn}	HZ_{rmn}
5	0.2	0.6868	0.4063	**0.3354**	0.3938	-0.6038	-0.2417	**-0.0637**	0.2208
	0.5	0.4131	0.4210	**0.3440**	0.3827	-0.6314	-0.2602	**-0.0873**	0.1920
	0.8	0.7482	0.4420	**0.3585**	0.3741	-0.6675	-0.2846	**-0.1178**	0.1591
	1.2	0.8044	0.4736	0.3847	**0.3648**	-0.7246	-0.3215	-0.1644	**0.1147**
	1.5	0.8614	0.5142	0.4290	**0.3801**	-0.7706	-0.3491	-0.2009	**0.0843**
	2	0.9563	0.5713	0.4936	**0.3897**	-0.8542	-0.3991	-0.2656	**0.0341**
	5	1.3113	0.8114	0.7702	**0.5053**	-1.1691	-0.5892	-0.5112	**-0.1487**
10	0.2	0.3327	0.1953	**0.1655**	0.2110	-0.2939	-0.1105	**0.0568**	0.1235
	0.5	0.3668	0.2018	**0.1594**	0.1853	-0.3319	-0.1182	**0.0359**	0.0972
	0.8	0.4160	0.2012	**0.1613**	0.1780	-0.3796	-0.1219	**0.0121**	0.0326
	1.2	0.5026	0.2109	**0.1848**	0.1938	-0.4744	-0.1290	0.0821	**-0.0248**
	1.5	0.5233	0.2167	**0.1821**	0.2256	-0.4945	-0.1373	**0.0660**	-0.0711
	2	0.5745	0.2248	**0.1890**	0.3010	-0.5414	-0.1460	**0.0277**	-0.1461
	5	0.8310	**0.3085**	0.3151	0.5925	-0.7741	-0.2132	**-0.1491**	-0.4290
20	0.2	0.1914	**0.1002**	0.1073	0.2450	-0.1753	**-0.0542**	0.0739	0.2240
	0.5	0.2180	0.1022	**0.0949**	0.1990	-0.2030	-0.0564	**0.0508**	0.1772
	0.8	0.2452	0.1044	**0.0881**	0.1610	-0.2305	-0.0593	**0.0309**	0.1351
	1.2	0.3041	0.1083	**0.1038**	0.1228	-0.2913	-0.0645	**0.0576**	0.0765
	1.5	0.3279	0.1102	**0.0992**	0.1152	-0.3140	-0.0654	0.0398	**0.0356**
	2	0.3734	0.1140	**0.1028**	0.1335	-0.3549	-0.0714	**0.0109**	-0.0221
	5	0.6615	**0.1486**	0.2428	0.3501	-0.6065	**-0.1028**	-0.1507	-0.2632
30	0.2	0.1441	**0.0686**	0.0950	0.2863	-0.1344	**-0.0358**	0.0790	0.2757
	0.5	0.1647	**0.0702**	0.2587	0.2708	-0.1558	**-0.0372**	-0.1146	0.1069
	0.8	0.1889	**0.0706**	0.0706	0.1863	-0.1796	**-0.0380**	0.0412	0.1758
	1.2	0.2351	**0.0740**	0.0799	0.1365	-0.2269	**-0.0416**	0.0518	0.1194
	1.5	0.2549	**0.0765**	0.0747	0.1105	-0.2456	-0.0441	**0.0368**	0.0825
	2	0.2944	0.0783	**0.0749**	0.0951	-0.2817	-0.0475	**0.0104**	0.0271
	5	0.5626	**0.0979**	0.1955	0.2511	-0.5147	**-0.0661**	-0.1284	-0.1847

Table 2. RMSE and bias of the estimators of Renyi entropy of the standard exponential distribution

		RMSE				Bias			
n	r	HV_{rmn}	HE_{rmn}	HA_{rmn}	HZ_{rmn}	HV_{rmn}	HE_{rmn}	HA_{rmn}	HZ_{rmn}
5	0.2	1.7125	1.3761	1.2187	**0.9734**	-1.6157	-1.2536	-1.0791	**-0.7912**
	0.5	1.1792	0.8726	0.7503	**0.6044**	-1.0340	-0.6638	-0.4929	**-0.2128**
	0.8	0.9802	0.6958	0.6066	**0.5450**	-0.8144	-0.4324	-0.2663	**0.0102**
	1.2	0.8860	0.6227	**0.5679**	0.5721	-0.6920	-0.2872	**-0.1296**	0.1509
	1.5	0.8789	0.6149	**0.5747**	0.5952	-0.6686	-0.2417	**-0.0924**	0.1979
	2	0.9257	0.6422	**0.6163**	0.6298	-0.6875	-0.2218	**-0.0876**	0.2257
	5	1.1066	0.7653	0.7665	**0.7325**	-0.7975	-0.2100	**-0.1298**	0.2688
10	0.2	1.2732	1.1042	0.9345	**0.8402**	-1.2106	-1.0311	-0.8431	**-0.7182**
	0.5	0.7648	0.5887	0.4812	**0.4488**	-0.6663	-0.4392	-0.2840	**-0.2129**
	0.8	0.6061	0.4242	0.3732	**0.3725**	-0.4862	-0.1832	-0.0820	**-0.0599**
	1.2	0.5439	0.3688	0.3926	**0.3674**	-0.4061	-0.0161	0.1521	**-0.0129**
	1.5	0.5070	**0.3663**	0.4071	0.3836	-0.3631	0.0488	0.2029	**-0.0167**
	2	0.5173	**0.3776**	0.4361	0.4250	-0.3541	0.1126	0.2287	**-0.0425**
	5	0.6933	**0.4371**	0.4894	0.5747	-0.4967	0.1674	0.1547	**-0.1498**
20	0.2	1.0285	0.9132	0.7512	**0.5550**	-0.9872	-0.8635	-0.6828	**-0.4291**
	0.5	0.5356	0.3933	0.3258	**0.2913**	-0.4660	-0.2695	-0.1753	**-0.0020**
	0.8	0.3874	0.2756	0.2486	**0.2677**	-0.2994	-0.0362	**-0.0198**	0.0921
	1.2	0.3330	0.2868	0.2654	**0.2594**	-0.2293	0.1199	0.1133	**0.0923**
	1.5	0.3343	0.3068	0.2732	**0.2654**	-0.2234	0.1819	0.1219	**0.0742**
	2	0.3706	0.3356	0.2856	**0.2776**	-0.2476	0.2342	0.1206	**0.0474**
	5	0.6258	0.3729	**0.3489**	0.3633	-0.4712	0.2548	**0.0458**	-0.0615
30	0.2	0.9283	0.8210	0.6575	**0.4173**	-0.8952	-0.7789	-0.5979	**-0.2699**
	0.5	0.4407	0.3168	**0.2587**	0.2708	-0.3834	-0.1945	-0.1146	**0.1069**
	0.8	0.3027	0.2332	**0.2028**	0.2594	-0.2283	0.0288	**0.0140**	0.1602
	1.2	0.2563	0.2844	**0.2195**	0.2353	-0.1665	0.1846	**0.1033**	0.1306
	1.5	0.2575	0.3127	**0.2235**	0.2306	-0.1623	0.2349	**0.1071**	0.1104
	2	0.2995	0.3447	0.2336	**0.2309**	-0.1960	0.2837	0.0949	**0.0729**
	5	0.5644	0.3601	**0.2869**	0.2903	-0.4352	0.2820	0.0278	**-0.0208**

Table 3. RMSE and bias of the estimators of Renyi entropy of the standard normal distribution.

		RMSE				Bias			
n	r	HV_{rmn}	HE_{rmn}	HA_{rmn}	HZ_{rmn}	HV_{rmn}	HE_{rmn}	HA_{rmn}	HZ_{rmn}
5	0.2	1.3896	1.0480	0.8898	**0.6473**	-1.3281	-0.9648	-0.7901	**-0.5013**
	0.5	1.1210	0.7855	0.6455	**0.4614**	-1.0454	-0.6729	-0.5024	**-0.2198**
	0.8	1.0227	0.6893	0.5649	**0.4261**	-0.9366	-0.5529	-0.3872	**-0.1087**
	1.2	0.9696	0.6318	0.5253	**0.4223**	-0.8721	-0.4699	-0.3117	**-0.0342**
	1.5	0.9836	0.6363	0.5396	**0.4384**	-0.8752	-0.4569	-0.3051	**-0.0255**
	2	1.0201	0.6527	0.5734	**0.4636**	-0.8911	-0.4432	-0.3033	**-0.0147**
	5	1.2089	0.7571	0.7213	**0.5323**	-1.0301	-0.4678	-0.3740	**-0.0407**
10	0.2	0.9779	0.8178	0.6365	**0.5404**	-0.9431	-0.7772	-0.5804	**-0.4562**
	0.5	0.7116	0.5429	0.3980	**0.3470**	-0.6613	-0.4781	-0.2960	**-0.2041**
	0.8	0.6172	0.4328	0.3244	**0.3030**	-0.5552	-0.3471	-0.1870	**-0.1242**
	1.2	0.6402	0.3579	**0.2640**	0.3012	-0.5836	-0.2516	**-0.0280**	-0.1011
	1.5	0.6093	0.3370	**0.2637**	0.3098	-0.5485	-0.2119	**0.0091**	-0.1124
	2	0.5870	0.3053	**0.2736**	0.3396	-0.5188	-0.1574	0.0436	**-0.1371**
	5	0.6671	0.2960	**0.3338**	0.5147	-0.5627	-0.0763	**0.0356**	-0.2993
20	0.2	0.7726	0.6770	0.4919	**0.3034**	-0.7519	-0.6526	-0.4533	**-0.2110**
	0.5	0.4960	0.3728	0.2573	**0.2008**	-0.4623	-0.3251	-0.1795	**0.0047**
	0.8	0.3995	0.2768	0.1968	**0.1888**	-0.3577	-0.2105	-0.0797	**0.0412**
	1.2	0.3648	0.2123	**0.1802**	0.1828	-0.3165	-0.1118	**0.0187**	0.0314
	1.5	0.3342	0.1884	**0.1823**	0.1831	0.0395	-0.0676	0.0395	**0.0083**
	2	0.3121	**0.1783**	0.1876	0.2021	-0.2465	**-0.0140**	0.0484	-0.0405
	5	0.3431	0.2142	**0.2138**	0.3544	-0.2375	0.0950	**-0.0115**	-0.2168
30	0.2	0.6873	0.6070	0.4218	**0.2076**	-0.6714	-0.5879	-0.3893	**-0.0734**
	0.5	0.4146	0.3178	**0.1961**	0.1970	-0.3886	-0.2802	-0.1233	**0.1045**
	0.8	0.3117	0.2105	**0.1515**	0.1929	-0.2763	-0.1843	**-0.0381**	0.1212
	1.2	0.2626	0.1586	**0.1461**	0.1646	-0.2205	-0.0480	**0.0395**	0.0829
	1.5	0.2348	0.1504	**0.1500**	0.1546	-0.1851	**-0.0026**	0.0506	0.0520
	2	0.2099	0.1559	**0.1540**	0.1564	-0.1486	0.0500	0.0517	**0.0001**
	5	0.2316	0.2289	**0.1865**	0.2747	-0.1372	0.1692	**-0.0298**	-0.1659

Tables 1-3 contain the results of 10000 simulations (of samples of size 5,10,20 and 30) per each case, to obtain the RMSE (Root of Mean Squared Error) values of the four estimators at different sample sizes for each of the three considered distributions. The optimal choice of m (the value which results the minimum value of RMSE for given n) is still an open problem. Grzegorzewski and Wieczorkowski (1999) suggested the heuristics formula for choosing m for estimation of Shannon entropy as $m=[\sqrt{n}+0.5]$.

We perform the simulation study for all values of $m \leq n/2$ and choose the value of m which gives the minimum RMSE for most of the values of r for three distributions. Our simulation results show that the optimal choice of m for proposed modified estimators are as

- Vasicek-type estimator

$$m=\begin{cases} [\sqrt{n}+0.5]-1 & r<1 \\ [\sqrt{n}+0.5]+1 & r>1. \end{cases}$$

- The first modified estimator

$$m=[n/2].$$

- The second modified estimator

$$m=\begin{cases} [\sqrt{n}+0.5] & r<1 \\ [\sqrt{n}+0.5]+1 & r>1. \end{cases}$$

- The third modified estimator

$$m=\begin{cases} [\sqrt{n}+0.5] & n<10 \\ [\sqrt{n}+0.5]-1 & n\geq 10. \end{cases}$$

From Tables 1-3, we observe that mostly, the main competition is between $HA_{r,m,n}$ and $HZ_{r,m,n}$. The third modified estimator has less RMSE and absolute of bias than the second modified estimator for small values of n. However by increasing n, $HA_{r,m,n}$ behaves better than $HZ_{r,m,n}$. On the other hand, if we compare the estimators for small and large values of r, it is seen that generally for small values of r, $(r=0.2,0.5)$, $HZ_{r,m,n}$ behaves better than $HA_{r,m,n}$. Also, for $r=5$, $HE_{r,m,n}$ compete favorably with $HA_{r,m,n}$ and $HZ_{r,m,n}$.

3. Testing exponentiality

3.1. Test statistics

In many situations such as life testing, when X is the life of a product, it is assumed that X has an exponential distribution. Suppose $X_1,...,X_n$ be a random sample from a continuous non-negative probability distribution function F with a density function f. we are interested in testing the hypothesis

$$H_0:\ f(x)=f_0(x;\theta),$$

where $f_0(x;\theta)=\theta e^{-\theta x},\ x>0$ and θ is unknown, against the $H_1:\ f(x)\neq f_0(x;\theta)$.

In order to obtain a test statistic, we use the following transformations.
Let X_1 and X_2 be two independent observations from a continuous distribution function F. It is shown by Kotz and Stentel (1983), that $W=\frac{X_1}{X_1+X_2}$ is distributed as $U(0,1)$ if and only if F is exponential. Also, Alizadeh and Arghami (2011) proved that $Y=\left|\frac{X_1-X_2}{X_1+X_2}\right|$ is distributed as $U(0,1)$ if and only if F is exponential.

Let $X_{1:n},...,X_{n:n}$ be an ordered random sample of size n. Consider the following transformations of the sample data as

$$W_{ij}=\frac{X_{i:n}}{X_{i:n}+X_{j:n}},\quad i\neq j,\ j=1,...,n,$$

$$V_{ij}=\frac{X_{i:n}-X_{j:n}}{X_{i:n}+X_{j:n}},\quad i>j,\ j=1,...,n.$$

Under the null hypothesis, each W_{ij} and V_{ij} have uniform distribution on $(0,1)$. Hence, to test H_0, we can transform X_i to W_{ij} or V_{ij} and test $H_0':\ g(u)=g_0(u)$, where $g_0(u)=1,\ 0<u<1$.

The asymmetric Renyi distance of g from g_0 is:

$$\begin{aligned} D_r(g,g_0) &= \frac{1}{r-1}\ln\int_0^1\left[\frac{g(u)}{g_0(u)}\right]^{r-1}g(u)du \\ &= \frac{1}{r-1}\ln\int_0^1 g^r(u)du=-H_r(g), \end{aligned} \tag{3.1}$$

where the $H_r(g)$ is the Renyi entropy of order r of distribution with density g. $D_r(g,g_0)\geq 0$ and the equality holds if and only if H_0' holds.

If a sample comes from an exponential distribution, $D_r(g,g_0)$ should be close to zero value and thus, large values of $D_r(g,g_0)$ lead us to reject the null hypothesis H_0' in favor of the alternative hypothesis H_1'.

To derive a test statistic by evaluating the information function (3.1), the density g must be completely specified, which is not operational. Thus, it is necessary to estimate the information function (3.1) from a sample.

Toward this end, we use four estimators of Renyi entropy $HV_{r,m,n}$, $HE_{r,m,n}$, $HA_{r,m,n}$ and $HZ_{r,m,n}$ based on the transformed data W_i for $i=1,2,...,n'$, where $n'=n(n-1)$ and V_i for $i=1,2,...,n''$, where $n''=n(n-1)/2$. Since the performance of the first and second transformations were similar in terms of power, we just consider the second transformation, that is, we have four test statistics as:

$$TV=HV_{r,m,n''}=-\frac{1}{r-1}\ln\left\{\frac{1}{n''}\sum_{i=1}^{n''}\left[\frac{n''}{2m}(V_{i+m:n''}-Y_{i-m:n''})\right]^{1-r}\right\},$$

$$TE=HE_{r,m,n''}=-\frac{1}{r-1}\ln\left\{\frac{1}{n''}\sum_{i=1}^{n''}\left[\frac{n''}{c_im}(V_{i+m:n''}-V_{i-m:n''})\right]^{1-r}\right\},$$

$$TA=HA_{r,m,n''}=-\frac{1}{r-1}\ln\left\{\frac{1}{n''}\sum_{i=1}^{n''}\left[\frac{n''}{a_im}(V_{i+m:n''}-V_{i-m:n''})\right]^{1-r}\right\},$$

$$TZ=HZ_{r,m,n''}=-\frac{1}{r-1}\ln\left\{\frac{1}{n''}\sum_{i=1}^{n''}\left[\frac{n''}{b_im}(V_{i+m:n''}-V_{i-m:n''})\right]^{1-r}\right\}.$$

We reject the null hypothesis for small values of the test statistics.

Critical points are determined by the quantiles of the distributions of the test statistics under hypothesis H_0'. Since the sampling distributions of test statistics are intractable, we determine the critical points using Monte Carlo method. Unfortunately, there is no choice criterion of r^* and m^* (the optimum values of r and m). In general they depends on the type of the alternatives. We suggest to choose $r^*=1.5$ based on simulation results. The results are given in Tables 4-8 for $r^*=1.5$ and different values of m.

TABLE 4. Critical values of the TV, $\alpha = 0.05$.

	m									
n	5	10	15	20	25	30	35	40	45	50
5	-0.8413									
6	-0.7413									
7	-0.6825	-0.5689								
8	-0.6380	-0.5099								
9	-0.6180	-0.4673	-0.4581							
10	-0.5976	-0.4516	-0.4181	-0.4150						
15	-0.4782	-0.3765	-0.3259	-0.3045	-0.2916	-0.2901	-0.2938	-0.3017	-0.3120	-0.3201
20	-0.3996	-0.3222	-0.2790	-0.2568	-0.2422	-0.2396	-0.2362	-0.2311	-0.2311	-0.2351
30	-0.2925	-0.2461	-0.2243	-0.2050	-0.1855	-0.1812	-0.1794	-0.1725	-0.1698	-0.1733
40	-0.2476	-0.2085	-0.1841	-0.1751	-0.1608	-0.1544	-0.1455	-0.1372	-0.1388	-0.1362
50	-0.2137	-0.1766	-0.1618	-0.1549	-0.1419	-0.1332	-0.1322	-0.1258	-0.1156	-0.1181

TABLE 5. Critical values of the TE, $\alpha = 0.05$.

	m									
n	5	10	15	20	25	30	35	40	45	50
5	-0.6312									
6	-0.5767									
7	-0.5493	-0.3957								
8	-0.5230	-0.3679								
9	-0.5263	-0.3485	-0.3092							
10	-0.5018	-0.3466	-0.2903	-0.2670						
15	-0.4361	-0.3193	-0.2588	-0.2213	-0.2016	-0.1882	-0.1801	-0.1789	-0.1743	-0.1676
20	-0.3730	-0.2814	-0.2330	-0.2089	-0.1844	-0.1747	-0.1632	-0.1529	-0.1444	-0.1417
30	-0.2808	-0.2269	-0.1986	-0.1747	-0.1549	-0.1467	-0.1411	-0.1330	-0.1284	-0.1259
40	-0.2409	-0.1957	-0.1690	-0.1539	-0.1395	-0.1325	-0.1228	-0.1127	-0.1128	-0.1060
50	-0.2095	-0.1692	-0.1496	-0.1403	-0.1256	-0.1172	-0.1142	-0.1066	-0.0980	-0.0976

TABLE 6. Critical values of the TA, $\alpha = 0.05$.

	m									
n	5	10	15	20	25	30	35	40	45	50
5	-0.4885									
6	-0.4790									
7	-0.4639	-0.2449								
8	-0.4587	-0.2544								
9	-0.4664	-0.2576	-0.1769							
10	-0.4559	-0.2682	-0.1821	-0.1215						
15	-0.4060	-0.2777	-0.2052	-0.1557	-0.1196	-0.0931	-0.0728	-00543	-0.0318	-0.0070
20	-0.3555	-0.2534	-0.1987	-0.1675	-0.1362	-0.1194	-0.0973	-0.0814	-0.0656	-0.0551
30	-0.2734	-0.2117	-0.1789	-0.1523	-0.1311	-0.1192	-0.1095	-0.0979	-0.0897	-0.0830
40	-0.2367	-0.1866	-0.1568	-0.1396	-0.1232	-0.1147	-0.1027	-0.0917	-0.0900	-0.0821
50	-0.2071	-0.1630	-0.1420	-0.1281	-0.1128	-0.1038	-0.0997	-0.0922	-0.0820	-0.0807

TABLE 7. Critical values of the TZ, $\alpha = 0.05$.

	m									
n	5	10	15	20	25	30	35	40	45	50
5	-0.2095									
6	-0.2913									
7	-0.3101	0.0257								
8	-0.3314	-0.0510								
9	-0.3634	-0.0952	0.0573							
10	-0.3661	-0.1303	0.0012	0.1408						
15	-0.3608	-0.1985	-0.1159	-0.0412	0.0164	0.0643	0.1171	0.1692	0.2251	0.2913
20	-0.3320	-0.2028	-0.1406	-0.0986	-0.0552	-0.0288	0.0057	0.0373	0.0652	0.0911
30	-0.2617	-0.1841	-0.1426	-0.1131	-0.0883	-0.0738	-0.0559	-0.0424	-0.0290	-0.0151
40	-0.2300	-0.1712	-0.1348	-0.1128	-0.0942	-0.0854	-0.0648	-0.0576	-0.0524	-0.0410
50	-0.2035	-0.1538	-0.1253	-0.1088	-0.0941	-0.0797	-0.0711	-0.0655	-0.0558	-0.0523

3.2. Power Study

For power comparisons, we choose the competitor tests from the existing exponentiality tests discussed in Henze and Meintanis (2005). The test statistics of competitor tests are as follows. We

will denote the scaled observations $x_i/\bar{x}$ by y_i.
(1) The Kolmogorov-Smirnov test statistic (D'Agostino and Stephens, 1986):

$$\begin{aligned} KS &= \sup_{x\geq 0} |F_n(x) - (1-\exp(-x))| \\ &= \max\left\{ \max_{1\leq i\leq n}\left[\frac{i}{n} - z_{(i)}\right], \max_{1\leq i\leq n}\left[z_{(i)} - \frac{i-1}{n}\right]\right\}, \end{aligned}$$

where $z_i = (1-\exp(-y_i))$.

(2) The Cramer-von Mises test statistic (D'Agostino and Stephens, 1986):

$$CV = \frac{1}{12n} + \sum_{i=1}^{n}\left(\frac{2i-1}{n} - z_{(i)}\right)^2.$$

(3) The Watson test statistic:

$$W = T_n - n(\bar{z} - \frac{1}{2})^2.$$

(4) The Anderson-Darling test statistic:

$$AD = -n - \sum_{i=1}^{n}\frac{2i-1}{n}\left[\ln z_{(i)} + \ln(1 - z_{(n-i+1)})\right].$$

(5) The test statistic of D'Agostino and Stephens (1986) which is based on the normalized spacings $D_i = (n+1-i)(X_{i:n} - X_{i-1:n}), i = 1,...,n, X_{0:n} = 0$:

$$S = 2n - \frac{2}{n}\sum_{i=1}^{n} i y_{(i:n)}.$$

(6) The test statistic of Cox and Oakes (1984):

$$CO = n + \sum_{i=1}^{n}(1-y_i)\ln(y_i).$$

(7) The test statistic of Epps and Pulley (1986):

$$EP = \sqrt{48n}\left(\frac{1}{n}\sum_{i=1}^{n}\exp(-y_i) - \frac{1}{2}\right).$$

(8) The test statistic of Baringhaus and Henze (1991) which is based on the empirical Laplace transform:

$$BH = n\int_0^{\infty}\left[(1+t)\psi_n'(t) + \psi_n(t)\right]^2 \exp(-at)dt,$$

where $\psi_n(t) = 1/n\sum_{i=1}^{n}\exp(-ty_i)$ is the empirical Laplace transform of the exponential distribution. Their suggested value of a is 2.5.

(9) The test statistic of Ebrahimi *et al.* (1992) which is based on Kullback-Leibler information:

$$KL = \frac{\exp(H_{mn})}{\exp(\ln\bar{x} + 1)},$$

where $H_{mn}=\frac{1}{n}\sum_{i=1}^{n}\ln\left[\frac{n}{2m}(X_{i+m:n}-X_{i-m:n})\right]$ is Vasicek's (1976) entropy estimator of Shannon entropy. We have taken $m=3$, following the recommendation of Ebrahimi *et al.* (1992).

(10) The test statistic of Henze (1993) which is based on the empirical Laplace transform:

$$H=n\int_0^\infty\left[\psi_n(t)-\frac{1}{1+t}\right]^2\exp(-at)dt,$$

where $\psi_n(t)=1/n\sum_{i=1}^{n}\exp(-ty_i)$ is the empirical Laplace transform of the exponential distribution. His suggested value of a is 2.5.

(11) The Kolmogorov-Smirnov and Cramer-von Mises-type statistics proposed by Baringhaus and Henze (2000):

$$\begin{aligned}\overline{KS}&=\sqrt{n}\sup_{t\geq 0}\left|\frac{1}{n}\sum_{i=1}^{n}\min(y_i,t)-\frac{1}{n}\sum_{i=1}^{n}I_{\{y_i\leq t\}}\right|,\\ \overline{CV}&=n\int_0^\infty\left[\frac{1}{n}\sum_{i=1}^{n}\min(y_i,t)-\frac{1}{n}\sum_{i=1}^{n}I_{\{y_i\leq t\}}\right]^2\exp(-t)dt,\end{aligned}$$

where $I_{\{A\}}$ is the indicator of the event A.

(12) The test statistics proposed by Henze and Meintanis (2005) which are based on the empirical characteristic function:

$$\begin{aligned}T_{n,a}^{(1)}=&\frac{a}{n}\sum_{j,k=1}^{n}\left[\frac{1}{a^2+y_{jk-}^2}+\frac{1}{a^2+y_{jk+}^2}\right]\\ &-\frac{2a}{n^2}\sum_{j,k=1}^{n}\sum_{l=1}^{n}\left[\frac{1}{a^2+(y_{jk-}-y_l)^2}+\frac{1}{a^2+(y_{jk-}+y_l)^2}\right]\\ &+\frac{a}{n^3}\sum_{j,k=1}^{n}\sum_{l=1}^{n}\left[\frac{1}{a^2+(y_{jk-}-y_{lm-})^2}+\frac{1}{a^2+(y_{jk-}+y_{lm-})^2}\right],\\ T_{n,a}^{(2)}=&\frac{1}{2n}\sqrt{\frac{\pi}{a}}\sum_{j,k=1}^{n}\left[\exp\left(-\frac{y_{jk-}^2}{4a}\right)+\exp\left(-\frac{y_{jk+}^2}{4a}\right)\right]\\ &-\frac{1}{n^2}\sqrt{\frac{\pi}{a}}\sum_{j,k=1}^{n}\sum_{l=1}^{n}\left[\exp\left(-\frac{(y_{jk-}-y_l)^2}{4a}\right)+\exp\left(-\frac{(y_{jk-}+y_l)^2}{4a}\right)\right]\\ &+\frac{1}{2n^3}\sqrt{\frac{\pi}{a}}\sum_{j,k=1}^{n}\sum_{l=1}^{n}\left[\exp\left(-\frac{(y_{jk-}-y_{lm-})^2}{4a}\right)+\exp\left(-\frac{(y_{jk-}+y_{lm-})^2}{4a}\right)\right],\end{aligned}$$

where $y_{jk-}=y_j-y_k, y_{jk+}=y_j+y_k$ and $a>0$. We have taken $a=2.5$, following the recommendation of Henze and Meintanis (2005).

(13) The test statistics proposed by Alizadeh and Arghami (2011) which are based on the Shannon entropy of transformed data:

$$T_1==-\frac{1}{n'}\sum_{i=1}^{n'}\ln\left[\frac{n'}{2m}(w_{i+m:n}-w_{i-m:n})\right],$$

$$T_2 = = -\frac{1}{n'}\sum_{i=1}^{n'} \ln\left[\frac{n'}{2m}(t_{i+m:n} - t_{i-m:n})\right] + \frac{2}{n'}\sum_{i=1}^{n'} \ln(1+t_i),$$
$$T_3 = = -\frac{1}{n'}\sum_{i=1}^{n'} \ln\left[\frac{n'}{2m}(z_{i+m:n} - z_{i-m:n})\right] + \ln 2,$$

where $n' = n(n-1), w_{ij} = \frac{x_i}{x_i+x_j}, t_{ij} = \frac{x_i}{x_j}$ and $z_{ij} = \frac{x_i - x_j}{x_i+x_j}$ for $i \neq j = 1, ..., n$.

For power comparisons, we consider the same alternatives listed in Henze and Meintanis (2005) and Alizadeh and Arghami (2011) and their choices of parameters:

- the Weibull distribution with density $\theta x^{\theta-1}\exp(-x^\theta)$, denoted by $W(\theta)$,
- the gamma distribution with density $\Gamma(\theta)^{-1}x^{\theta-1}\exp(-x)$, denoted by $\Gamma(\theta)$,
- the lognormal distribution with density $(\theta x)^{-1}(2\pi)^{-1/2}\exp(-(\ln x)^2/(2\theta^2))$, denoted by $LN(\theta)$,
- the half-normal distribution with density $\Gamma(2/\pi)^{1/2}\exp(-x^2/2)$, denoted by HN,
- the uniform distribution with density 1, $0 \leq x \leq 1$, denoted by $U(0,1)$,
- the modified extreme value distribution with distribution function $1-\exp(\theta^{-1}(1-e^x))$, denoted by $EV(\theta)$,
- the linear increasing failure rate law with density $(1+\theta x)\exp(-x-\theta x^2/2)$, denoted by $LF(\theta)$,
- Dhillon's (1981) law with distribution function $1-\exp(-(\ln(x+1))^{\theta+1})$, denoted by $DL(\theta)$,
- Chen's (2000) distribution with distribution function $1-\exp(2(1-e^{x^\theta}))$, denoted by $CH(\theta)$.

These distributions comprise of widely used alternatives to the exponential model and include densities f with decreasing hazard rates, increasing hazard rates as well as models with non-monotone hazard functions.

Tables 8 and 9 shows the estimated powers of the tests TV, TE, TA and TZ and those of the competing tests, at the significance level $\alpha = 0.05$, based on the results of 10000 simulations of sample size 20. The maximum power are given in bold type.

For these alternatives, we have taken $m = 50$ for $n = 20$ $(n' = 190)$, based on simulation results.

It is evident from Tables 8 and 9 that, the test statistic TZ has the greatest power for almost all alternatives among the test statistics based on entropy and among other test statistics $T_{n,a}^{(1)}$ has the best performance in terms of the powers.

In general, we can not say a single test perform the best for all alternatives. We observe that the proposed test statistics based on Renyi entropy perform very well compared with the other tests for Weibull(0.8), gamma(2), lognormal(0.8) and Dhillon, $DL(\theta)$ alternatives. However, the test of Henze and Meintanis (2005), $T_{n,a}^{(1)}$, has the greatest powers for almost all other alternatives.

TABLE 8. Power comparison of various exponentiality tests for $n = 20$ with $\alpha = 0.05$.

Altern.	KS	CV	W	AD	EP	$\overline{KS}$	$\overline{CV}$	S	CO	BH	HE	$T^{(1)}_{n,a}$	$T^{(2)}_{n,a}$
$W(0.8)$	18	20	14	**28**	24	14	22	**28**	24	24	24	4	10
$W(1.4)$	28	35	28	31	36	35	35	35	37	36	37	**45**	42
$\Gamma(0.4)$	70	74	63	90	76	62	75	76	**91**	77	79	33	50
$\Gamma(1)$	5	5	5	5	5	5	5	5	5	5	5	5	5
$\Gamma(2)$	40	47	41	45	48	46	47	46	54	48	48	**55**	50
$LN(0.8)$	32	35	**37**	34	25	28	27	24	33	26	29	27	22
$LN(1.5)$	58	63	49	63	**67**	55	66	**67**	60	**67**	66	18	38
HN	18	21	18	17	21	24	22	21	19	21	21	**33**	31
U	51	68	61	64	66	72	70	70	50	65	62	**86**	85
$CH(0.5)$	56	62	51	79	63	47	61	63	**80**	63	63	23	37
$CH(1)$	12	14	12	11	15	18	16	15	13	14	14	**25**	23
$CH(1.5)$	69	81	72	77	84	79	83	84	81	84	84	**91**	**91**
$LF(2)$	24	29	23	24	28	32	30	29	28	28	28	**42**	39
$LF(4)$	35	41	35	36	42	44	43	42	37	42	42	**56**	54
$EV(0.5)$	12	15	12	12	15	18	16	15	13	14	15	**25**	23
$EV(1.5)$	34	42	36	36	45	48	47	46	37	44	43	**63**	61
$DL(1)$	19	23	22	21	20	22	21	19	25	21	20	**25**	21
$DL(1.5)$	58	67	60	64	64	62	63	62	**72**	64	64	67	61

TABLE 9. Power comparison of exponentiality tests based on entropy for $n = 20$ with $\alpha = 0.05$.

Altern.	KL	T_1	T_2	T_3	TV	TE	TA	TZ
$W(0.8)$	**28**	16	13	16	16	14	12	6
$W(1.4)$	29	35	36	35	44	39	42	**45**
$\Gamma(0.4)$	**88**	86	82	85	83	80	79	68
$\Gamma(1)$	5	5	5	5	5	5	5	5
$\Gamma(2)$	44	55	55	55	60	63	64	**67**
$LN(0.8)$	35	51	50	50	56	60	61	**64**
$LN(1.5)$	**66**	10	7	9	17	15	13	13
HN	16	18	18	17	20	21	22	**23**
U	**61**	52	55	51	50	48	47	44
$CH(0.5)$	**77**	73	67	70	70	66	64	51
$CH(1)$	11	13	14	13	14	14	15	**16**
$CH(1.5)$	76	72	74	72	**77**	**77**	**77**	**77**
$LF(2)$	23	23	25	24	26	27	27	**28**
$LF(4)$	34	34	36	35	37	38	38	**39**
$EV(0.5)$	11	13	14	13	14	14	15	**16**
$EV(1.5)$	**37**	34	35	34	36	**37**	**37**	**37**
$DL(1)$	21	31	31	31	36	39	40	**44**
$DL(1.5)$	63	74	75	74	80	82	83	**84**

4. Conclusion

In this paper, we proposed three modified estimators of Renyi entropy and compared them with the existing estimator based on their RMSEs. It is observed that one of these estimators has better performance than other estimators especially for small sample sizes.

Also we introduced a goodness of fit test for exponentiality based on Renyi distance. At first we use transformations of data which turn the test of exponentiality into one of uniformity and then we use a corresponding test based on estimators of Renyi entropy. We can see that the proposed tests performs well for some alternatives.

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