

# INVENTORY MODEL OF TYPE $(s,S)$ WITH SUBEXPONENTIAL WEIBULL DISTRIBUTED DEMAND

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**Abstract:** In this study we consider a semi-Markovian inventory model of type  $(s,S)$  with subexponential Weibull distributed demand and uniform distributed interference of chance. By using a special asymptotic expansion proposed by Geluk and Frenk (2011) we derived an asymptotic expansion for the renewal function generated by the subexponential Weibull distributed demand random variables. Through this renewal function we obtained two term asymptotic expansion for the ergodic distribution function of the process which represents the model that we consider here. Moreover we proved weak convergence theorem for the ergodic distribution function and derived the limit distribution.

**Key words:** Inventory model of type  $(s,S)$ ; Heavy tailed distributions; Subexponential distributions; Renewal reward process; Asymptotic expansion

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## 1. Introduction

The events are called catastrophic or extreme which are highly improbable and unpredictable but have massive impacts. If the probability of extreme events become higher the use of heavy tailed distributions gives more realistic results when modeling uncertainty. Since such unusual events are encountered in many areas such as medical sciences, civil engineering applications, meteorology, financial risk management and inventory management, investigation of heavy-tailed distributions and the effects of heavy-tailed distributions has recently become a very important field of area. The aim of this study is to investigate inventory model of type  $(s,S)$  with heavy tailed Weibull distributed demands and to observe the impact of heavy tailed distributions on this model.

Inventory problems are one of the major application area of renewal-reward processes. As is known that, inventory management problems are an important issue related to the production

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policies of the companies. The main purpose of inventory management is to build and keep a convenient stock control model in order to balance supply and demand at minimum cost as well as to protect the company against demand uncertainty. Hence inventory problems has received a lot of attention among academics and has always been a popular field of study. In this paper we consider a spesific inventory model called "Inventory model of type (s,S)". The inventory model of type (s,S) has been comprehensively studied and some of their characteristics investigated see e.g. [7],[23],[8],[27]. In most of these studies some real world problems have been solved by using dynamic programming but analytic solutions could not be obtained. In order to estimate the behaviour of a system after a long running time it is important to investigate some stationary characteristics (ergodic moments, ergodic distribution function etc.) of the model which represents this system. In recent years many studies have been done to examine the stationary characteristics of an inventory model type (s,S) by using asymptotic tools. In publications Khaniyev and Aksop [16], and Khaniyev and Atalay [17] asymptotic expansion is obtained by assuming the interference of chance have generalized Beta distribution and triangular distribution respectively. Moreover Khaniyev et. al.[18] also considered this model with triangular distributed interference of chance and obtained asymptotic expansions for the ergodic moments. In all these studies it was assumed that the demand quantities  $\{\eta_n\}$ ,  $n \geq 1$  were light tailed with finite moments. The underlined feature of the present paper and the main difference from previous literature is that we consider subexponential demand distributions which is one of the most important subclass of heavy tailed distributions. The name of "subexponential" arises from one of their properties that their tails decrease more slowly than any exponential tail. This implies that large values can occur in a sample with non-negligible probability, and makes the subexponential distributions candidates for modeling situations where some extremely large values occur in a sample compared to the mean size of data [19]. In this study subexponential distributions are used to model the possibility of uncertain demands (very large or very small demands) relatively. The main purpose of this paper is to investigate the effect of subexponential distribution on the inventory model of type (s,S) by using asymptotic tools. Under the assumption of subexponential Weibull distributed demands a semi-Markovian inventory model of type (s,S) is considered. A stochastic process  $X(t)$  that express the evolution of the stock level in this model is constructed mathematically. Two term asymptotic expansion is obtained for the ergodic distribution of this process. Moreover weak convergence theorem is proved.

## 2. Preliminaries

Let us give the essential notations and explain this model before analyzing the main problem.

### 2.1. The model

Consider a company want to create the optimal inventory policy. Assume that  $z$  is the initial stock level in a depot at time  $t = 0$ , hence  $X(0) = X_0 = z \in (s, S)$ ,  $0 < s < S < \infty$ . Here  $s$  is the stock control level and  $S$  is the maximum stock level. In addition suppose that  $\{\eta_n\}$ ,  $n \geq 1$  which describe the random amount of demands are coming to the system at random times  $T_1, T_2, \dots, T_n, \dots$ . Here  $T_n = \sum_{i=1}^n \xi_i$ , where  $\xi_n$ 's are interarrival times between two successive demands ( $n=1,2,3,\dots$ ). Hence the stock level  $X(t)$  decreases by  $\eta_1, \eta_2, \dots, \eta_n, \dots$  at random times  $T_1, T_2, \dots, T_n, \dots$  until the stock level  $X(t)$  falls below the control level  $s$ , at random time  $\tau_1$ . In this instance the stock level changes as follows:

$$X(T_1) \equiv X_1 = z - \eta_1, X(T_2) \equiv X_2 = z - (\eta_1 + \eta_2), \dots, X(T_n) \equiv X_n = z - \sum_{i=1}^n \eta_i$$

where,  $\eta_n$  represents the amount of  $n^{th}$  demand,  $n = 1, 2, 3, \dots$ .  $\tau_1$  is the first time, that the stock level falls below the control level  $s$ . After the stock level falls below  $s$ , it is immediately refilled up

to the level  $\zeta_1$ , and thus the first period is completed. Second period starts with a new initial level  $\zeta_1$  and, continues in a similar manner to the first period. Note that the random variables  $\eta_n, \zeta_n, \xi_n$  i.i.d here. This model is referred in the literature as "inventory model of type (s,S)" (see e.g. [17], [16])

## 2.2. Subexponential distributions and the renewal function for subexponential random variables

Subexponential distributions are a special subclass of heavy tailed distributions. The most widely used definition of heavy tailedness can be given as follows: A distribution  $F$  on  $\mathbb{R}$  is said to be (right) heavy tailed if

$$\int_{-\infty}^{\infty} e^{\lambda x} F(dx) = \infty \text{ for all } \lambda > 0.$$

Foss [13]. Although there are three important subclass of heavy tailed distributions (the long tailed distributions, the regularly varying distributions, and the subexponential distributions) most of the heavy-tailed distributions used in practice are subexponential for instance lognormal, Weibull with  $\bar{F}(x) = \exp(-x^\alpha)$ ,  $0 < \alpha < 1$  and generalized Pareto distributions. This subclass was introduced by Chistyakov [9].

A distribution  $F$  said to be subexponential ( $F \in S$ ) if for all  $n \geq 2$ , independent random variables  $X_1, X_2, \dots, X_n$  satisfies

$$P\{X_1 + X_2 + \dots + X_n > t\} \sim nP\{X_1 > t\} \text{ as } t \rightarrow \infty.$$

With a simple calculation it is easy to see that

$$P\{\max(X_1, X_2, \dots, X_n) > t\} \sim nP\{X_1 > t\} \text{ as } t \rightarrow \infty.$$

Subexponential distributions are a proper subset of long-tailed distributions but also possess many of their own properties makes them more useful in practice; for instance, the class of subexponential distributions is closed under maxima, minima, mixtures, convolutions and random translations [25]. For a comprehensive survey about heavy tailed and subexponential distributions see the books [3], [11],[5], [24],[26].

The main purpose of this study is to obtain an asymptotic expansion for the ergodic distribution of an inventory model of type (s,S) by using asymptotic tools. The ergodic distribution of this model is expressed through renewal functions. Hence in order to achieve the asymptotic expansion for the distribution function of the considered process, we need the asymptotic expansion of the renewal function generated by the demand random variables. The most important factor that distinguishes this study from similar studies in the literature is the use of special asymptotic expansion for renewal function  $U_n(x)$ , where by  $U_n(x)$  we mean renewal function generated by the demand random variables  $\{\eta_n\}$ ,  $n \geq 1$ .

Our study is inspired by a special asymptotic expansion obtained by Geluk and Frenk [14]. They provide Asymptotic Expansion (2.1) for renewal function generated by subexponential random variables with finite variance. According to the mentioned study for a large class of subexponential random variables the renewal function  $U$  satisfies

$$U(x) - \frac{x}{\mu} - \frac{\mu_2}{2\mu^2} = -\frac{1}{\mu} \int_x^\infty \bar{F}_I(s) ds + O(\bar{F}_I(x)) \text{ as } x \rightarrow \infty. \quad (2.1)$$

under some assumptions. Here  $\bar{F}_I(x)$  is integrated tail function and defined as;

$$\bar{F}_I(x) = \frac{1}{\mu} \int_0^x \bar{F}(y) dy, \quad x > 0. \quad (2.2)$$

Our experiences showed that it is difficult to work with the whole subexponential class when applying them in this kind of special expansions. Hence we select Weibull distribution as a representative distribution from the subexponential class and apply the asymptotic expansion (2.1) to our model through this representative. One of the main reasons we chose the Weibull distribution from subexponential class in this study is the distribution function exist in closed form. Hence we obtain explicit asymptotic expressions for the integrated tail function and asymptotic expansion for the ergodic distribution function of the process. (For example the distribution function of Lognormal distribution does not have closed form and it is hard to obtain a simple asymptotic expansion for ergodic distribution in this case). Moreover Weibull distribution have similar asymptotic behavior with many other more complicated distributions. The Weibull distribution with  $\bar{F}(x) = \exp(-x^\alpha)$  can be considered as a generalization of the exponential distribution. The shape parameter  $\alpha$  specify the tail's asymptotic behavior. For  $0 < \alpha < 1$  the distribution belongs to the subexponential family with a tail heavier than exponential one, while for  $\alpha > 1$  the distribution is characterized as hyperexponential with tail thinner than exponential. Many distributions can be assumed tail equivalent with Weibull for a specific value of the parameter  $\alpha$ . Hence considering this model with subexponential Weibull distributed demands can be regarded as a good beginning step to investigate this kind of inventory models with heavy tailed demand quantities.

### 2.3. Mathematical construction of the process $X(t)$

The mathematical construction of the considered system is given by Khaniyev (see [17],[16]) as follows:

Let  $(\Omega, \mathfrak{F}, P)$  be probability space and  $(\xi_n, \eta_n, \zeta_n)$  be a vector of i.i.d random variables defined on  $(\Omega, \mathfrak{F}, P)$ . Here  $\xi_n$  and  $\eta_n$  are positive valued random variables, the random variable  $\zeta_n$  takes values in the interval  $[s, S]$  and  $\xi_n, \eta_n$  and  $\zeta_n$  are also independent from each other. Let the distributions of  $\xi_n, \eta_n$  and  $\zeta_n$  be denoted by  $\Phi(t), F(x)$  and  $\pi(z)$  respectively and these distributions defined as:

$$\Phi(t) = P\{\xi_1 \leq t\}, F(x) = P\{\eta_1 \leq x\}, \pi(z) = P\{\zeta_1 \leq z\}.$$

As mentioned before, we assume here that the random variables  $\zeta_n$  which represents the initial level of stock after the  $n^{\text{th}}$  refillment have uniform distribution.

We also assume here  $\{\eta_n\}, n \geq 1$  have subexponential Weibull distribution i.e:

$$F(x) = P\{\eta_n \leq x\} = 1 - \exp(-x^\alpha), 0 < \alpha < 1.$$

Now we can construct the process with all this information above. As a first step we need to define the renewal sequences  $\{T_n\}$  and,  $\{S_n\}$  as:

$$T_0 = S_0 = 0, T_n = \sum_{i=1}^n \xi_i, S_n = \sum_{i=1}^n \eta_i, n \geq 1.$$

Now define a sequence of integer-valued random variables  $\{N_n\}, n \geq 0$  as follows:

$$N_0 = 0, N_1 = N(z - s) = \inf\{k \geq 1 : z - S_k \leq s\}, z \in [s, S].$$

$$N_{n+1} = \inf\{k \geq N_n + 1 : \zeta_n - (S_k - S_{N_n}) < s\}, n \geq 1.$$

Let  $\tau_0 = 0, \tau_n = T_{N_n} = \sum_{i=1}^{N_n} \xi_i, n \geq 1, \nu(t) = \max\{n \geq 0 : T_n \leq t\}, t \geq 0.$

Under these assumptions the desired stochastic process  $X(t)$  constructed as follows:

$$X(t) = \zeta_n - (\eta_{N_{n+1}} + \dots + \eta_{\nu(t)}) = \zeta_n - (S_{\nu(t)} - S_{N_n}), t \in [\tau_n, \tau_{n+1}), n \geq 0. \quad (2.3)$$

The process  $X(t)$  represents the amount of stock in the depot at time  $t > 0.$

### 2.4. Ergodicity of the process $X(t)$

The following proposition, by Khaniyev and Atalay [17] states that the process  $X(t)$  is ergodic under some weak assumptions.

PROPOSITION 1. (Khaniyev and Atalay[17] Proposition 3.1) *Let the initial sequence of random variables  $\{(\xi_n, \eta_n, \zeta_n)\}$  satisfy the following supplementary conditions:*

1.  $0 < E(\xi_1) < \infty$ ,
2.  $0 < E(\eta_1) < \infty$ ,
3.  $\eta_1$  is a non-arithmetic random variable.

Then, the process  $X(t)$  is ergodic and, the following expression is correct with probability 1 for each measurable bounded function  $f(x)$ , ( $f : (s, S) \rightarrow \mathbb{R}$ ) .

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(X(u)) du = \frac{\int_s^S \int_s^S f(x) [U_\eta(z-s) - U_\eta(z-x)] d\pi(z) dx}{\int_s^S U_\eta(z-s) d\pi(z)}.$$

Here  $U_\eta(x) = \sum_{n=0}^{\infty} F^{*n}(x)$  is the renewal function generated by the sequence of random variables  $\{\eta_n\}$  and  $F^{*n}(x)$  is the  $n^{\text{th}}$  power convolution of the distribution function  $F(x)$  and defined as:

$$F^{*n}(x) = \int_{-\infty}^{\infty} F^{*(n-1)}(x-y) F(dy).$$

A direct result of this proposition is Corollary 1 below, obtained by choosing  $f$  to be indicator function in Proposition 1.

COROLLARY 1. *Assume the process  $X(t)$  be satisfied the conditions of Proposition 1. Then the ergodic distribution of the process  $X(t)$  is given as follows:*

$$Q_X(x) \equiv \lim_{t \rightarrow \infty} P\{X(t) \leq x\} = 1 - \frac{\int_x^S U_\eta(z-x) d\pi(z)}{\int_s^S U_\eta(z-s) d\pi(z)}; x \in [s, S].$$

### 3. Exact and Asymptotic Results for the Ergodic Distribution of the Process

In order to obtain an asymptotic expansion for the ergodic distribution of the process, we followed the same method with articles[17],[16] and defined a process  $Y(t)$  as a standardized version of the process  $X(t)$  as follows:

$$Y(t) = \frac{X(t) - s}{\beta}, \beta \equiv \frac{S - s}{2}.$$

Here  $Q_Y(v) = \lim_{t \rightarrow \infty} P\{Y(t) \leq v\}$ ,  $v \in [0, 2)$ .

PROPOSITION 2. *Under the conditions of Proposition 1 the ergodic distribution function  $Q_Y(v)$  of the process  $Y(t)$  is given as:*

$$Q_Y(v) \equiv 1 - \frac{\int_{\beta v}^{2\beta} U_\eta(x - \beta v) dx}{\int_0^{2\beta} U_\eta(x) dx}, v \in [0, 2).$$

PROOF. Recall that,  $Q_Y(v) = \lim_{t \rightarrow \infty} P\{Y(t) \leq v\}$ ,  $v \in [0, 2)$ . According to the definition of  $Y(t)$ :

$$Q_Y(v) = \lim_{t \rightarrow \infty} P\left\{\frac{X(t) - s}{\beta} \leq v\right\} = \lim_{t \rightarrow \infty} P\{X(t) \leq \beta v + s\}.$$

In this case;

$$Q_Y(v) = Q_X(s + \beta v) = 1 - \frac{\int_{s+\beta v}^{s+2\beta} U_\eta(z - s - \beta v) d\pi(z)}{\int_s^{s+2\beta} U_\eta(z - s) d\pi(z)}; v \in [0, 2).$$

We assumed that the random variables  $\zeta_n$  has uniform distribution. Hence the random variable  $\tilde{\zeta}_n = \zeta_n - s$  has the same distribution defined in  $[0, 2\beta)$ . Thus; we have:

$$Q_Y(v) \equiv 1 - \frac{\frac{1}{2\beta} \int_{\beta v}^{2\beta} U_\eta(x - \beta v) dx}{\frac{1}{2\beta} \int_0^{2\beta} U_\eta(x) dx}. \quad (3.1)$$

Now we can work on the asymptotic expansion for the ergodic distribution function of the process  $Y(t)$  when  $\beta \rightarrow \infty$ .

LEMMA 1. (*Geluk and Frenk (2011) Theorem 1*) Let  $\eta_1, \eta_2, \dots, \eta_n$  be independent and identically distributed (i.i.d) non-negative random variables with common distribution function  $F$  with unbounded support and  $E(\eta_1^2) < \infty$ . For a large class of heavy-tailed random variables with finite variance the renewal function  $U$  satisfies:

$$U(x) - \frac{x}{\mu} - \frac{\mu_2}{2\mu^2} \sim -\frac{1}{\mu x} \int_x^\infty \int_s^\infty (1 - F(u)) du ds; x \rightarrow \infty. \quad (3.2)$$

Equivalently;

$$U(x) - \frac{x}{\mu} - \frac{\mu_2}{2\mu^2} = -\frac{1}{\mu} \int_x^\infty \bar{F}_I(t) dt + O(\bar{F}_I(x)) \text{ as } x \rightarrow \infty \quad (3.3)$$

$\bar{F}_I(x)$  is defined in (2.2).

In order to attain the asymptotic expression of the ergodic distribution function  $Q_Y(v)$  defined by (3.2) we need to find the asymptotic expansion of the renewal function  $U_\eta(x)$  generated by the subexponential Weibull distributed random variables. The following lemma shows the asymptotic expression of the integrated tail function:

LEMMA 2. In addition to the conditions of Lemma 1 let  $\eta_1$  be subexponential Weibull distributed random variable with parameters  $\lambda = 1$  and,  $0 < \alpha < 1$ , i.e:

$$F(x) = P\{\eta_1 \leq x\} = 1 - \exp\{-x^\alpha\}.$$

Then:

$$\int_x^\infty \bar{F}_I(t) dt = \frac{1}{\mu_1 \alpha^2} O(x^{2-2\alpha} \exp(-x^\alpha)).$$

Where  $\mu_k = E(\eta_1^k)$ ,  $0 < \alpha < 1$ .

PROOF. It is straightforward to see

$$\bar{F}_I(t) = \frac{1}{\mu_1} \int_t^\infty \bar{F}(y) dy$$

Since we consider the demand quantity  $\eta_1$  with subexponential Weibull distributed random variable with parameters  $\lambda = 1$  and,  $0 < \alpha < 1$ , then:

$$F(x) = 1 - \exp\{-x^\alpha\}.$$

In this case

$$\begin{aligned} \overline{F}_I(t) &= \frac{1}{\mu_1} \int_t^\infty \exp\{-y^\alpha\} dy \\ &= \frac{1}{\mu_1 \alpha} \int_{t^\alpha}^\infty \exp\{-u\} u^{\frac{1}{\alpha}-1} du \\ &= \frac{1}{\mu_1 \alpha} \Gamma\left(t^\alpha, \frac{1}{\alpha}\right). \end{aligned} \tag{3.4}$$

Hence,

$$\int_x^\infty \overline{F}_I(t) dt = \frac{1}{\mu_1 \alpha} \int_x^\infty \Gamma\left(t^\alpha, \frac{1}{\alpha}\right) dt. \tag{3.5}$$

Here  $\Gamma(z, a)$  is upper incomplete Gamma function defined as:

$$\Gamma(z, a) = \int_z^\infty x^{a-1} \exp\{-x\} dx.$$

Incomplete Gamma function admits the following asymptotic expansion for large  $z$  (see e.g. Abramowitz[1]).

$$\begin{aligned} \Gamma(z, a) &= z^{a-1} e^{-z} \left( 1 + \frac{a-1}{z} + \frac{(a-1)(a-2)}{z^2} + \dots \right) \\ &= z^{a-1} e^{-z} \left( 1 + O\left(\frac{1}{z}\right) \right). \end{aligned} \tag{3.6}$$

In order to compute integral (3.5) we used the asymptotic expansion of incomplete Gamma function (3.6) and obtained:

$$\Gamma\left(t^\alpha, \frac{1}{\alpha}\right) = (t)^{1-\alpha} \exp\{-t^\alpha\} g(t).$$

Here  $g(t)$  is a bounded function hence there exists a positive constant  $D$  such that,

$$\sup_{t \geq 0} |g(t)| \equiv D < \infty$$

So

$$\begin{aligned} \left| \int_x^\infty \Gamma\left(t^\alpha, \frac{1}{\alpha}\right) dt \right| &= \left| \frac{1}{\mu_1^2 \alpha} \int_x^\infty t^{1-\alpha} \exp(-t^\alpha) g(t) dt \right| \\ &\leq \frac{1}{\mu_1^2 \alpha} \int_x^\infty t^{1-\alpha} \exp(-t^\alpha) |g(t)| dt \\ &\leq \frac{D}{\mu_1^2 \alpha} \int_x^\infty t^{1-\alpha} \exp(-t^\alpha) dt \\ &= \frac{D}{\mu_1^2 \alpha^2} \int_{x^\alpha}^\infty u^{\frac{2}{\alpha}-2} \exp(-u) du \\ &= \frac{D}{\mu_1^2 \alpha^2} \Gamma\left(x^\alpha, \frac{2}{\alpha} - 1\right). \end{aligned} \tag{3.7}$$

Again by using the asymptotic expansion (3.6) we obtain the following result:

$$\frac{D}{\mu_1^2 \alpha^2} \Gamma\left(x^\alpha, \frac{2}{\alpha} - 1\right) = \frac{D}{\mu_1^2 \alpha^2} \left[ x^{2-2\alpha} \exp(-x^\alpha) \left( 1 + O\left(\frac{1}{x^\alpha}\right) \right) \right].$$

Where  $\mu_k = E(\eta_1^k)$ ,  $0 < \alpha < 1$ .

COROLLARY 2. Let  $\eta_1, \eta_2, \eta_3, \dots$  be (i.i.d) random variables and the conditions of Lemma 2 be satisfied. Then the renewal function  $U_\eta(x)$  which is generated by subexponential Weibull distributed random variables satisfies the following asymptotic expansion:

$$U_\eta(x) = \frac{x}{\mu_1} + \frac{\mu_2}{2\mu_1^2} + O(x^{2-2\alpha} \exp(-x^\alpha)).$$

where  $\mu_k = E(\eta_1^k)$ ,  $0 < \alpha < 1$ .

PROOF. By using the results of Lemma 1 and Lemma 2 we obtain:

$$U_\eta(x) = \frac{x}{\mu_1} + \frac{\mu_2}{2\mu_1^2} + O(x^{2-2\alpha} \exp(-x^\alpha)) + O(\overline{F_I}(x)).$$

and,

$$O(\overline{F_I}(x)) = O(x^{1-\alpha} \exp(-x^\alpha)).$$

Proof follows by comparing the last two asymptotic terms.

LEMMA 3. For all measurable bounded functions  $h: \mathbb{R} \rightarrow \mathbb{R}$ , the following limit holds:

$$\lim_{\beta \rightarrow \infty} \int_0^{2\beta - \beta v} \exp(-t^\alpha) t^{2-2\alpha} h(t) dt = O(1), \quad (3.8)$$

where  $\beta \equiv \frac{(S-s)}{2}$

PROOF. Since  $h(x)$  is given as a bounded function, there exists an  $K > 0$  such that  $\sup_{x \geq 0} |h(x)| \equiv K < \infty$  holds. Thus for  $v \in [0, 2)$ , and  $0 < \alpha < 1$  we have :

$$\begin{aligned} \left| \int_0^{2\beta - \beta v} \exp(-t^\alpha) t^{2-2\alpha} h(t) dt \right| &= \left| \int_0^\infty \exp(-t^\alpha) t^{2-2\alpha} h(t) dt - \int_{2\beta - \beta v}^\infty \exp(-t^\alpha) t^{2-2\alpha} h(t) dt \right| \\ &\leq \left| \int_0^\infty \exp(-t^\alpha) t^{2-2\alpha} h(t) dt \right| + \left| \int_{2\beta - \beta v}^\infty \exp(-t^\alpha) t^{2-2\alpha} h(t) dt \right| \\ &\leq \int_0^\infty \exp(-t^\alpha) t^{2-2\alpha} |h(t)| dt + \int_{2\beta - \beta v}^\infty \exp(-t^\alpha) t^{2-2\alpha} |h(t)| dt \\ &\leq K \left\{ \int_0^\infty \exp(-t^\alpha) t^{2-2\alpha} dt + \int_{2\beta - \beta v}^\infty \exp(-t^\alpha) t^{2-2\alpha} dt \right\} \\ &= \frac{K}{\alpha} \left\{ \int_0^\infty \exp(-u) u^{\frac{3}{\alpha}-3} du + \int_{(2\beta - \beta v)^\alpha}^\infty \exp(-u) u^{\frac{3}{\alpha}-3} du \right\} \\ &= \frac{K}{\alpha} \left\{ \Gamma\left(\frac{3}{\alpha} - 2\right) + \Gamma\left((2\beta - \beta v)^\alpha, \frac{3}{\alpha} - 2\right) \right\}. \quad (3.9) \end{aligned}$$

Note that for  $0 < \alpha < 1$ ,  $\Gamma\left(\frac{3}{\alpha} - 2\right)$  is finite and, has a positive value. Hence  $\Gamma\left(\frac{3}{\alpha} - 2\right) = O(1)$ . Moreover we obtain by using (9) (asymptotic expansion of incomplete Gamma function):

$$\Gamma\left((2\beta - \beta v)^\alpha, \frac{3}{\alpha} - 2\right) = O\left((2\beta - \beta v)^{3-3\alpha} \exp(-(2\beta - \beta v)^\alpha)\right).$$

Where  $\beta \equiv \frac{(S-s)}{2}$ ,  $v \in [0, 2)$ , and  $0 < \alpha < 1$ .

This completes the proof.

LEMMA 4. For all  $v \in [0, 2)$  the following equation holds as  $\beta \rightarrow \infty$ :

$$J(v) \equiv \frac{1}{2\beta} \int_{\beta v}^{2\beta} U_\eta(x - \beta v) dx = \frac{1}{\mu_1} \frac{\beta(2-v)^2}{4} + \frac{\mu_2}{2\mu_1^2} \frac{(2-v)}{2} + O\left(\frac{1}{\beta}\right). \quad (3.10)$$

where  $\mu_k = E(\eta_1^k)$ ,  $k = 1, 2, \dots$



PROOF. Let  $h : \mathbb{R} \rightarrow \mathbb{R}$  be a measurable bounded function. Then, by using the results of Lemma 3 we obtain:

$$\begin{aligned}
 J(v) &\equiv \frac{1}{2\beta} \int_{\beta v}^{2\beta} U_{\eta}(x - \beta v) dx = \frac{1}{2\beta} \int_0^{2\beta - \beta v} U_{\eta}(t) dt \\
 &= \frac{1}{2\beta} \int_0^{2\beta - \beta v} \left\{ \frac{t}{\mu_1} + \frac{\mu_2}{2\mu_1^2} + \exp(-t^{\alpha}) t^{2-2\alpha} h(t) \right\} dt \\
 &= \frac{1}{2\beta} \left\{ \frac{(2\beta - \beta v)^2}{2\mu_1} + \frac{\mu_2}{2\mu_1^2} (2\beta - \beta v) + O(1) \right\} \\
 &= \frac{1}{\mu_1} \frac{\beta(2-v)^2}{4} + \frac{\mu_2}{2\mu_1^2} \frac{(2-v)}{2} + O\left(\frac{1}{\beta}\right). \tag{3.11}
 \end{aligned}$$

By using Lemma 4 the following result can be obtained as  $v \rightarrow 0$ .

COROLLARY 3. The following expansion holds as  $\beta \rightarrow \infty$ :

$$J(0) \equiv \frac{1}{2\beta} \int_0^{2\beta} U_{\eta}(x) dx = \frac{\beta}{\mu_1} + \frac{\mu_2}{2\mu_1^2} + O\left(\frac{1}{\beta}\right). \tag{3.12}$$

The following theorem is the main result of this study.

THEOREM 1. Let the conditions of Proposition 2 be satisfied, then for all  $v \in [0, 2)$  the following asymptotic expansion can be written for the ergodic distribution function  $Q_Y(v)$  of the process  $Y(t)$  as  $\beta \equiv \frac{(S-s)}{2} \rightarrow \infty$

$$Q_Y(v) = F(v) + \frac{\mu_2}{2\mu_1\beta} G(v) + O\left(\frac{1}{\beta^2}\right). \tag{3.13}$$

where

$$F(v) = \frac{4v - v^2}{4}, \tag{3.14}$$

$$G(v) = \frac{v^2 - 2v - 2}{4},$$

and  $v \in [0, 2)$ .

PROOF. Using Lemma 4 and, Corollary 3, as  $\beta \rightarrow \infty$  we have:

$$\begin{aligned}
 Q_Y(v) &= 1 - \frac{J(v)}{J(0)} \\
 &= \left[ \frac{\beta}{\mu_1} \frac{4v - v^2}{4} + \frac{\mu_2}{2\mu_1^2} \frac{v - 1}{2} + O\left(\frac{1}{\beta}\right) \right] \\
 &\quad \times \left[ \frac{\beta}{\mu_1} + \frac{\mu_2}{2\mu_1^2} + O\left(\frac{1}{\beta}\right) \right]^{-1} \\
 &= \left\{ \frac{4v - v^2}{4} + \frac{\mu_2}{2\mu_1\beta} \frac{v - 1}{2} + O\left(\frac{1}{\beta^2}\right) \right\} \left\{ 1 - \frac{\mu_2}{2\mu_1\beta} + O\left(\frac{1}{\beta^2}\right) \right\} \\
 &= \frac{4v - v^2}{4} + \frac{\mu_2}{2\mu_1\beta} \left\{ \left(\frac{v - 1}{2}\right) - \left(\frac{4v - v^2}{4}\right) \right\} + O\left(\frac{1}{\beta^2}\right) \\
 &= F(v) + \frac{\mu_2}{2\mu_1\beta} G(v) + O\left(\frac{1}{\beta^2}\right). \tag{3.15}
 \end{aligned}$$

The following weak convergence theorem is straightforward result from Theorem 1.

**THEOREM 2.** Assume that the conditions of Theorem 1 are satisfied. Then the ergodic distribution ( $Q_Y(v)$ ) of  $Y(t)$  converges to  $F(v)$  as  $\beta \rightarrow \infty$  i.e:

$$Q_Y(v) \rightarrow F(v)$$

where

$$F(v) = \frac{4v - v^2}{4}$$

**PROOF.** Since  $v \in [0, 2)$  then we have

$$|G(v)| = \left| \frac{v^2 - 2v - 2}{4} \right| < \frac{1}{2} < \infty.$$

Moreover according to the conditions of Theorem 1 and, Proposition 1 we have  $\mu_2 \equiv E(\eta_1^2) < \infty$  and,  $\mu_1 \equiv E(\eta_1) > 0$  respectively. Therefore,  $\left| \frac{\mu_2}{2\mu_1} G(v) \right| < \frac{\mu_2}{4\mu_1} < \infty$ . Hence  $\frac{\mu_2}{2\mu_1\beta} G(v) \rightarrow 0$  as  $\beta \rightarrow \infty$ . Hence  $Q_Y(v) \rightarrow F(v)$  as  $\beta \rightarrow \infty$ . Which completes the proof. Note that  $F(v) = \frac{4v - v^2}{4}$  is limit distribution in the interval  $[0, 2)$ .

#### 4. Conclusions

In this study an inventory model of type (s,S) with subexponential demands is represented with a semi-Markov model called a renewal- reward process. A stochastic process which express this model is constructed mathematically. The limit distribution is obtained for the ergodic distribution function of this system. We consider the demand quantities  $\eta_n$ ,  $n = 1, 2, \dots$  have Weibull distribution with parameters  $\lambda = 1$  and,  $0 < \alpha < 1$  which belongs to the subexponential subclass of heavy tailed distributions in this case. Two term asymptotic expansion for the ergodic distribution is obtained when  $\beta \equiv \frac{S-s}{2} \rightarrow \infty$ . In order to obtain the limit distribution weak convergence theorem is proved. Our basic idea was to examine the semi-Markovian (s, S) inventory systems with heavy tailed distributions which was considered only with light tailed demand quantities previously. Some data analysis studies regarding inventory control models have shown that the random variables which represents the demand quantities tend to heavy-tailed distribution especially when sudden unexpected fluctuations happen in the demand quantities. Thus investigating the stock control models with heavy tailed demand quantities theoretically and analyze the effects of these demand quantities on asymptotic results is an important field of study. As a result of the literature research we have done so far, we have pointed out that there is no comprehensive and theoretical study about this field in the literature. The aim of this study is to fill this gap in the literature. This study might be considered as a beginning step to investigation an inventory model of type (s,S) with heavy tailed demands. The results are new and important to expand the application area of this model. As is known that there are some different asymptotic expansions for the renewal functions generated from the different subclasses of heavy tailed distributions. Hence this approach can also be applied to another subclasses of heavy tailed distributions by using different asymptotic expansions. For instance studying an inventory model of type (s,S) with the random variables which are a member of regularly varying heavy tailed distributions might be another suggestion for future research. In addition if exists finite moments of this process can be calculated by using this approach and simulation methods can be applied to concrete models.

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