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Some characterizations of color Hom-Poisson algebras

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Abstract

In this paper, we describe color Hom-Poisson structures in terms of a single bilinear operation. This enables us to explore color Hom-Poisson algebras in the realm of non-Hom-associative color algebras.

Keywords: Color Hom-Poisson algebras, color Hom-flexible algebras, admissible color Hom-Poisson algebras.

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1. Introduction

A Poisson algebra has simultaneously a Lie algebra structure and a commutative associative algebra structure, satisfying the Leibniz identity. These algebras firstly appeared in the work of Siméon-Denis Poisson two centuries ago when he was studying the three-body problem in celestial mechanics. Since then, Poisson algebras have shown to be connected to many areas of mathematics and physics. In mathematics, Poisson algebras play a fundamental role in Poisson geometry [23], quantum groups [7, 9] and deformation of commutative associative algebras [11]. In physics, Poisson algebras represent a major part of deformation quantization [16], Hamiltonian mechanics [4] and topological field theories [21]. Poisson-like structures are also used in the study of vertex operator algebras [10].

The first motivation to study nonassociative Hom-algebras comes from quasi-deformations of Lie algebras of vector fields, in particular q-deformations of Witt and Virasoro algebras [2, 6, 8, 14, 15]. Hom-Lie algebras were first introduced by Hartwig, Larsson and Silvestrov in order to describe q-deformations of Witt and Virasoro algebras using σ-derivations [13]. The corresponding associative type objects and non-commutative version, called Hom-associative algebras and Hom-Leibniz algebras respectively, were introduced by Makhlouf and Silvestrov in [17]. The notion of Hom-Poisson algebras appeared for the first time in [18] where it is shown that Hom-Poisson algebras play the same role in the deformation of commutative Hom-associative algebras as Poisson

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algebras do in the deformation of commutative associative algebras. They are further studied in [26] where the author proved that the polarisation of a given Hom-algebra is a Hom-Poisson algebra if and only if this Hom-algebra is an admissible Hom-Poisson algebra. The purpose of this paper is to study color Hom-Poisson algebras which are first introduced in [5]. For more informations on other color Hom-type algebras, the reader can refer to [1, 3, 5, 20, 27].

A description of the rest of this paper is as it follows.

In Section 2, we recall basic notions concerning color Hom-algebras. Color Hom-Poisson algebras [5] are defined without the ε-commutativity condition. Here we give the definition of these color Hom-algebras by adding this condition (Definition 2.5) and then Hom-Poissons algebras could be seen as color Hom-Poisson algebras with $G = \{0\}$. We then extend the notion of flexible algebras to the one of color Hom-flexible algebras (Definition 2.11). Theorem 2.8 as well as Theorem 2.12, produce a sequence of color Hom-Poisson and color Hom-flexible algebras respectively.

In Section 3, we define admissible color Hom-Poisson algebras (Definition 3.1) and then prove the main result of this paper (Theorem 3.10).

Throughout this paper, all graded vector spaces are assumed to be over a field $\mathbb{K}$ of characteristic 0.

2. Preliminaries and some results

Let G be an abelian group. A vector space V is said to be a G-graded if, there exists a family $(V_a)_{a\in G}$ of vector subspaces of V such that $V = \oplus_{a\in G} V_a$. An element $x \in V$ is said to be homogeneous of degree $a \in G$ if $x \in V_a$. We denote $\mathcal{H}(V)$ the set of all homogeneous elements in V. Let $V = \oplus_{a\in G} V_a$ and $V' = \oplus_{a\in G} V'_a$ be two G-graded vector spaces. A linear mapping $f : V \to V'$ is said to be homogeneous of degree $b \in G$ if $f(V_a) \subseteq V'_{a+b}$, $\forall a \in G$. If, f is homogeneous of degree zero i.e. $f(V_a) \subseteq V'_a$ holds for any $a \in G$, then f is said to be even. An algebra (A, μ) is said to be G-graded if its underlying vector space is G-graded i.e. $A = \oplus_{\in G} A_a$, and if furthermore $\mu(A_a, A_b) \subseteq A_{a+b}$, for all $a, b \in G$. Let A' be another G-graded algebra. A morphism $f : A \to A'$ of G-graded algebras is by definition an algebra morphism from A to A' which is, in addition an even mapping.

2.1. Definition. Let G be an abelian group. A mapping $\varepsilon : G \times G \longrightarrow \mathbb{K}^*$ is called a bicharacter on G if the following identities hold for all $a, b, c \in G$:
(i) $\varepsilon(a, b)\varepsilon(b, a) = 1$,
(ii) $\varepsilon(a + b, c) = \varepsilon(a, c)\varepsilon(b, c)$,
(iii) $\varepsilon(a, b + c) = \varepsilon(a, b)\varepsilon(a, c)$.

It is easy to see that $\varepsilon(0, a) = \varepsilon(a, 0) = 1$ and $\varepsilon(a, a) = \pm 1$ for all $a \in G$. In particular, for a fixed $a \in G$, the induced map $\varepsilon_a : G \longrightarrow \mathbb{K}^*$ defined by $\varepsilon_a(b) = \varepsilon(a, b)$ is a homomorphism of groups.

If x and y are two homogeneous elements of degree a and b respectively and ε is a bicharacter, then we shorten the notation by writing $\varepsilon(x, y)$ instead of $\varepsilon(a, b)$.

Unless stated, in the sequel all the graded spaces are over the same abelian group G and the bicharacter will be the same for all the structures. For the rest of this section, we give basic facts about color Hom-algebras [5],[22], [27] and prove some results concerning color Hom-Poisson and color Hom-flexible algebras.

2.2. Definition. (i) By a color Hom-algebra, we mean a quadruple $(A, \mu, \varepsilon, \alpha)$ consisting of a G-graded vector space A, an even bilinear map $\mu : A\times A \longrightarrow A$ i.e $\mu(A_a, A_b) \subseteq A_{a+b}$ for all $a, b \in G$, a bicharacter $\varepsilon : G \times G \longrightarrow \mathbb{K}^*$ and an even linear map $\alpha : A \longrightarrow A$.

A color Hom-algebra $(A, \mu, \varepsilon, , \alpha)$ is said to be multiplicative if $\alpha \circ \mu = \mu \circ \alpha^{\otimes 2}$ and ε-commutative if $\mu(x, y) = \varepsilon(x, y)\mu(y, x)$ for all $x, y \in \mathcal{H}(A)$.
(ii) A weak morphism $f : (A, \mu, \varepsilon, \alpha) \longrightarrow (A', \mu', \varepsilon, \alpha')$ of two color Hom-algebras is an even linear map $f : A \longrightarrow A'$ of the underlying G-graded vector spaces, satisfying $f \circ \mu = \mu' \circ f^{\otimes 2}$. If furthermore $f \circ \alpha = \alpha' \circ f$, then f is said to be a morphism.

For the rest of this paper, we will often write $\mu(x, y)$ as xy for homogeneous element x, y.

2.3. Definition. (i) A color Hom-associative algebra is a color Hom-algebra $(A, \mu, \varepsilon, \alpha)$ such that $as_A(x, y, z) = 0$ where as_A is the Hom-associator defined for all $x, y, z \in \mathcal{H}(A)$ by

$$as_A(x, y, z) = (xy)\alpha(z)) - \alpha(x)(yz) \tag{2.1}$$

(ii) A color Hom-Lie algebra is a color Hom-algebra $(A, \{,\}, \varepsilon, \alpha)$ such that

$$\{x, y\} = \varepsilon(x, y)\{y, x\} \text{ (} \varepsilon\text{-skew-symmetry)} \tag{2.2}$$

$$\oint \varepsilon(z, x)\{\alpha(x), \{y, z\}\} = 0 \text{ (} \varepsilon\text{-Hom-Jacobi identity)} \tag{2.3}$$

for all $x, y, z \in \mathcal{H}(A)$ where $\oint$ means the cyclic summation over x, y, z.

By the ε-skew-symmetry (2.2) of the color Hom-Lie bracket $\{,\}$, the ε-Hom-Jacobi identity (2.3) is equivalent to $J_A(x, y, z) = 0$ where

$$J_A(x, y, z) = \oint \varepsilon(z, x)\{\{x, y\}, \alpha(z)\} \tag{2.4}$$

for all $x, y, z \in \mathcal{H}(A)$, is called the color Hom-Jacobian of A.

2.4. Remark. A graded associative (resp. color Lie) algebra is a color Hom-associative (resp. color Hom-Lie) algebra with $\alpha = Id$.

2.5. Definition. A color Hom-Poisson algebra consists of a G-graded vector space A, two even bilinear maps $\mu, \{,\} : A^{\otimes 2} \longrightarrow A$, an even linear map $\alpha : A \longrightarrow A$ and a bicharacter ε such that
(1) $(A, \mu, \varepsilon, \alpha)$ is an ε-commutative color Hom-associative algebra,
(2) $(A, \{,\}, \varepsilon, \alpha)$ is a color Hom-Lie algebra,
(3) the color Hom-Leibniz identity

$$\{\alpha(x), \mu(y, z)\} = \mu(\{x, y\}, \alpha(z)) + \varepsilon(x, y)\mu(\alpha(y), \{x, z\}) \tag{2.5}$$

is satisfied for all $x, y, z \in \mathcal{H}(A)$.

By the ε-skew-symmetry of $\{,\}$, the color Hom-Leibniz identity is equivalent to

$$\{\mu(x, y), \alpha(z)\} = \mu(\alpha(x), \{y, z\}) + \varepsilon(y, z)\mu(\{x, z\}, \alpha(y)) \tag{2.6}$$

In a color Hom-Poisson algebra $(A, \mu, \{,\}, \varepsilon, \alpha)$, the operations μ and $\{,\}$ are called the color Hom-associative product and the color Hom-Poisson bracket, respectively.

2.6. Remark. In [5], color Hom-Poisson algebras are defined without the ε-commutativity condition in Definition 2.5. In this case, if $G = \mathbb{Z}_2$ we get the notion of Hom-Poisson superalgebras defined in [24]. According to our definition, we could see Hom-Poisson algebras [18] (resp. Hom-Poisson superalgebra [28]) as color Hom-Poisson algebras with $G = \{0\}$ (resp. $G = \mathbb{Z}_2$ and $\varepsilon(x, y) = (-1)^{xy}$ for all homogeneous elements x, y).

Here, we give an example of a color Hom-Poisson algebra for $G = \mathbb{Z}_2$ which could be seen in [28].

2.7. Example. There is a three-dimensional multiplicative color Hom-Poisson algebra $\mathcal{A} = (A = A_0 \oplus A_1, \cdot, \{,\}, \alpha)$, where $A_0 = \mathbb{C}e_1 \oplus \mathbb{C}e_2$, $A_1 = \mathbb{C}e_3$ and an algebra morphism α is defined by

$$\alpha(e_1) = xe_1,\ \alpha(e_2) = e_1 + e_2,\ \alpha(e_3) = ye_3,$$

with x and y fixed nonzero complex numbers. The defining non-zero relations are

$$e_1 \cdot e_2 = xe_1,\ e_2 \cdot e_2 = e_1 + e_2,\ e_3 \cdot e_2 = ye_3,\ \{e_1, e_2\} = x^2 e_1.$$

In fact these color Hom-Poisson algebras (Hom-Poisson superalgbras) are not Poisson superalgebras for $x \neq 1$, or $y \neq 1$.

The following theorem produces a sequence of color Hom-Poisson algebras. It says again that the category of color Hom-Poisson algebras is closed by weak morphisms.

2.8. Theorem. *Let $\mathcal{A} = (A, \mu, \{,\}, \varepsilon, \alpha)$ be a color Hom-Poisson algebra and $\beta : A \longrightarrow A$ a weak morphim. Then for each $n \in \mathbb{N}$, $\mathcal{A}_{\beta^n} = (A, \mu_{\beta^n} = \beta^n \circ \mu, \{,\}_{\beta^n} = \beta^n \circ \{,\}, \varepsilon, \beta^n \circ \alpha)$ is a color Hom-Poisson algebra. Moreover, if $\mathcal{A}$ is multiplicative and β is a morphism of $\mathcal{A}$, then $\mathcal{A}_{\beta^n}$ is also multiplicative.*

Proof. First we note that the ε-commutativity and the ε-skew-symmetry of μ_{β^n} and $\{,\}_{\beta^n}$ follow from the one of μ and $\{,\}$ respectively. Next, it is straightforward to check that

$$as_{\mathcal{A}_{\beta^n}} = \beta^{2n} \circ as_{\mathcal{A}} \text{ and } J_{\mathcal{A}_{\beta^n}} = \beta^{2n} \circ J_{\mathcal{A}} \tag{2.7}$$

Since $(A, \mu, \varepsilon, \alpha)$ is an ε-commutative color Hom-associative algebra and $(A, \{,\}, \varepsilon, \alpha)$ is a color Hom-Lie algebra, we deduce by (2.7) that $(A, \mu_{\beta^n}, \varepsilon, \beta^n \circ \alpha)$ is an ε-commutative color Hom-associative algebra and $(A, \{,\}_{\beta^n}, \varepsilon, \beta^n \circ \alpha)$ is a color Hom-Lie algebra.
Now, writting for readability the composition law "$\circ$" as juxtaposition, (2.5) is proved as it follows

$$\begin{aligned}
\{\beta^n\alpha(x), \mu_{\beta^n}(y,z)\}_{\beta^n} &= \beta^n\{\beta^n\alpha(x)), \beta^n(yz)\}) \\
&= \beta^{2n}\{\alpha(x), yz\} \\
&= \beta^{2n}(\mu(\{x,y\}, \alpha(z)) + \varepsilon(x,y)\mu(\alpha(y), \{x,z\})) \\
&\quad \text{(by (2.5) in } \mathcal{A} \text{)} \\
&= \beta^n\mu(\beta^n\{x,y\}, \beta^n\alpha(z)) + \varepsilon(x,y)\beta^n\mu(\beta^n\alpha(y), \beta^n\{x,z\}) \\
&= \mu_{\beta^n}(\{x,y\}_{\beta^n}, \beta^n\alpha(z)) + \varepsilon(x,y)\mu_{\beta^n}(\beta^n\alpha(y), \{x,z\}_{\beta^n})
\end{aligned}$$

Finally, observing that the conditions $\beta\circ\alpha = \alpha\circ\beta$ and $\beta\circ\mu = \mu\circ\beta^{\otimes 2}$ implie $\beta^n\circ\alpha = \alpha\circ\beta^n$ and $\beta^n \circ \mu = \mu \circ (\beta^n)^{\otimes 2}$ respectively, we have for all $x, y \in \mathcal{H}(A)$: $\beta^n\alpha\mu_{\beta^n}(x,y) = \beta^n\alpha\beta^n\mu(x,y) = \beta^n\alpha\mu(\beta^n(x), \beta^n(y)) = \beta^n\mu(\alpha\beta^n(x), \alpha\beta^n(y)) = \beta^n\mu(\beta^n\alpha(x), \beta^n\alpha(y)) = \mu_{\beta^n}(\beta^n\alpha(x), \beta^n\alpha(y))$. Similarly, we prove that
$\beta^n\alpha\{x,y\}_{\beta^n} = \{\beta^n\alpha(x), \beta^n\alpha(y)\}_{\beta^n}$. Therefore if $\mathcal{A}$ is multiplicative and β is a morphism of $\mathcal{A}$, then $\mathcal{A}_{\beta^n}$ is also multiplicative. □

If we drop the ε-commutativity condition in Definition 2.5 and set $\beta = \alpha$, (resp. $n = 1$ and $\alpha = Id$) in Theorem 2.8, we get some results in [5].

2.9. Example. From the multiplicative color Hom-Poisson algebra $\mathcal{A}$ in Example 2.7, we get the familly of multiplicative color Hom-Poisson algebras $(\mathcal{A}_{\alpha^n})_{n\in\mathbb{N}}$ where for each

$n \in \mathbb{N}$,

$\mathcal{A}_{\alpha^n} = (A, \cdot_{\alpha^n}, \{,\}_{\alpha^n}, \alpha^{n+1})$ with the following non-zero products :
$e_1 \cdot_{\alpha^n} e_2 = x^n e_1,\ e_2 \cdot_{\alpha^n} e_2 = (1 + x + x^2 + \cdots + x^{n-1})e_1 + e_2,\ e_3 \cdot_{\alpha^n} e_2 = y^n e_3,$
$\{e_1, e_2\}_{\alpha^n} = x^{n+2}e_1$ and the morphism defined by: $\alpha^{n+1}(e_1) = x^{n+1}e_1,$
$\alpha^{n+1}(e_2) = (1 + x + x^2 + \cdots + x^n)e_1 + e_2,\ \alpha^{n+1}(e_3) = y^{n+1}e_3.$

The next result shows that, color Hom-Novikov Poisson algebras can be gotten from ε-commutative color Hom-associative algebras.

2.10. Proposition. *Let $(A, \mu, \varepsilon, \alpha)$ be an ε-commutative color Hom-associative algebra. Then*

$$A^- = (A, \mu, \{,\}, \varepsilon, \alpha)$$

is a color Hom-Poisson algebra where $\{x, y\} = \mu(x, y) - \varepsilon(x, y)\mu(y, x)$ for all $x, y \in \mathcal{H}(A)$.

Proof. It is proved in [27] (Proposition 3. 13) that $(A, \{,\}, \varepsilon, \alpha)$ is a color Hom-Lie algebra. To check the color Hom-Leibniz identity (2.5) for A^-, we write μ as juxtaposition and compute as follows:

$$\begin{aligned}
&\mu(\{x, y\}, \alpha(z)) + \varepsilon(x, y)\mu(\alpha(y), \{x, z\}) - \{\alpha(x), \mu(y, z)\}\\
&= (xy)\alpha(z) - \varepsilon(x, y)(yx)\alpha(z) + \varepsilon(x, y)\alpha(y)(xz) - \varepsilon(x, y)\varepsilon(x, z)\alpha(y)(zx)\\
&-\alpha(x)(yz) + \varepsilon(x, y)\varepsilon(x, z)(yz)\alpha(x)\\
&= as_A(x, y, z) - \varepsilon(x, y)as_A(z, x, z) + \varepsilon(x, y)\varepsilon(x, z)as_A(y, z, x)
\end{aligned}$$

Since $as_A = 0$, we conclude that A^- satisfies the color Hom-Leibniz identity. □

The following definition will be useful in Section 3.

2.11. Definition. A color Hom-flexible algebra is a color Hom-algebra $(A, \mu, \varepsilon, \alpha)$ that satisfies the ε-Hom-flexible (color Hom-flexible) identity i.e for all $x, y, x \in \mathcal{H}(A)$

$$as_A(x, y, z) = -\varepsilon(x, y)\varepsilon(x, z)\varepsilon(y, z)as_A(z, y, x) \tag{2.8}$$

It follows that when $G = \mathbb{Z}_2$ and $\varepsilon(x, y) = (-1)^{xy}$ (resp. $G = \{0\}$) in Definition 2.11, we recover the notion of Hom-flexible superalgebra [1] (resp. Hom-flexible algebra [25]).

As for color Hom-Poisson algebras, we get the following:

2.12. Theorem. *Let $\mathcal{A} = (A, \mu, \varepsilon, \alpha)$ be a color Hom-flexible algebra and $\beta : A \longrightarrow A$ a weak morphim. Then for each $n \in \mathbb{N}$, $\mathcal{A}_{\beta^n} = (A, \mu_{\beta^n} = \beta^n \circ \mu, \varepsilon, \beta^n \circ \alpha)$ is a color Hom-flexible algebra. Moreover, if $\mathcal{A}$ is multiplicative and β is a morphism of $\mathcal{A}$, then $\mathcal{A}_{\beta^n}$ is also multiplicative.*

Proof. The proof follows from (2.7) and the proof of Theorem 2.8. □

3. Characterizations

In [12] and [19], it is shown that Poisson algebras can be described using only one operation of its two binary operations via the polarization-depolarization process. This enables to explore Poisson algebras in the realm of non-associative algebras. The similar is done for Hom-Poisson algebras [26]. The purpose of this section is to extend this alternative description of Poisson algebras or Hom-Poisson algebras to color Hom-Poisson algebras. Let's first define the notion of an admissible color Hom-Poisson algebras.

3.1. Definition. Let $(A, \mu, \varepsilon, \alpha)$ be a color Hom-algebra. Then A is called an admissible color Hom-Poisson algebra if it satisfies

$$\begin{aligned} as_A(x,y,z) &= \frac{1}{3}\{\varepsilon(y,z)(xz)\alpha(y) - \varepsilon(x,z)\varepsilon(y,z)(zx)\alpha(y) \\ &\quad +\varepsilon(x,y)\varepsilon(x,z)(yz)\alpha(x) - \varepsilon(x,y)(yx)\alpha(z)\} \end{aligned} \tag{3.1}$$

for all $x, y, z \in \mathcal{H}(A)$, where as_A is the Hom-associator (2.1) of A.

An admissible color Hom-Poisson algebra with $G = \{0\}$ is exactly an admissible Hom-Poisson algebra as defined in [26]. If furthermore $\alpha = Id$, we get the notion of an admissible Poisson algebra [12]. To compare color Hom-Poisson algebras and admissible color Hom-Poisson algebras, we need the following function, which generalizes a similar function in [19, 26].

3.2. Definition. Let $(A, \mu, \varepsilon, \alpha)$ be a color Hom-algebra. Define the quintuple

$$P(A) = (A, *, \{,\}, \varepsilon, \alpha) \tag{3.2}$$

called the polarization of A, where $x*y = \frac{1}{2}(xy+\varepsilon(x,y)yx)$ and $\{x,y\} = \frac{1}{2}(xy-\varepsilon(x,y)yx)$ for all $x, y \in \mathcal{H}(A)$. We call P the polarization function.

The main result is to prove that admissible color Hom-Poisson algebras, and only these color Hom-algebras, give rise to color Hom-Poisson algebras via polarization. It is the color Hom-version of [19, Example 2]. To do that, we need some useful ingredients.

3.3. Definition. Let $(A, *, \{,\}, \varepsilon, \alpha)$ be a quintuple in which A is a graded vector space, $*, \{,\} : A \longrightarrow A$ are linear even maps, $\alpha : A \longrightarrow A$ an even linear map and ε a bicharacter. Define the color Hom-algebra

$$P^-(A) = (A, \mu = * + \{,\}, \varepsilon, \alpha) \tag{3.3}$$

called the depolarization of A. We call P^- the depolarization function.

The following observation says that admissible color Hom-Poisson algebras are color Hom-flexible algebras. It is the color Hom-version of [12, Proposition 4].

3.4. Lemma. *Every admissible color Hom-Poisson algebra $(A, \mu, \varepsilon, \alpha)$ is a color Hom-flexible algebra.*

Proof. The color Hom-flexibility identity (2.8) is proved using (3.1) as it follows:

$$\begin{aligned} as_A(z,y,x) &= \frac{1}{3}\{\varepsilon(y,x)(zx)\alpha(y) - \varepsilon(z,x)\varepsilon(y,x)(xz)\alpha(y) \\ &\quad +\varepsilon(z,y)\varepsilon(z,x)(yx)\alpha(z) - \varepsilon(z,y)(yz)\alpha(x)\} \\ &= -\frac{1}{3}\varepsilon(y,x)\varepsilon(z,x)\varepsilon(z,y)\{\varepsilon(y,z)(xz)\alpha(y) - \varepsilon(x,z)\varepsilon(y,z)(zx)\alpha(y) \\ &\quad +\varepsilon(y,x)\varepsilon(z,x)(yz)\alpha(x) - \varepsilon(x,y)(yx)\alpha(z)\} \\ &= -\varepsilon(y,x)\varepsilon(z,x)\varepsilon(z,y)as_A(x,y,z) \end{aligned}$$

i.e. $as_A(x,y,z) = -\varepsilon(x,y)\varepsilon(x,z)\varepsilon(y,z)as_A(z,y,x)$ for all $x, y, z \in \mathcal{H}(A)$ and then we get (2.8). □

For a given color Hom-algebra A, the color cyclic sum S_A of the Hom-associator is defined by:

$$\begin{aligned} S_A(x,y,z) &:= as_A(x,y,z) + \varepsilon(y,z)\varepsilon(x,z)as_A(z,x,y) + \\ &\quad \varepsilon(x,y)\varepsilon(x,z)as_A(y,z,x) \end{aligned} \tag{3.4}$$

for all $x, y, z \in \mathcal{H}(A)$.

Next we observe that in an admissible color Hom-Poisson algebra the color cyclic sum of the Hom-associator is identically zero.

3.5. Lemma. *Let* $(A, \mu, \varepsilon, \alpha)$ *be an admissible color Hom-Poisson algebra. Then* $S_A(x, y, z) = 0$ *for all* $x, y, z \in \mathcal{H}(A)$.

Proof. Using the defining identity (3.1), we have for all $x, y, z \in \mathcal{H}(A)$:

$$\begin{aligned}
as_A(x,y,z) &= \frac{1}{3}\{\varepsilon(y,z)(xz)\alpha(y) - \varepsilon(x,z)\varepsilon(y,z)(zx)\alpha(y) \\
&\quad +\varepsilon(x,y)\varepsilon(x,z)(yz)\alpha(x) - \varepsilon(x,y)(yx)\alpha(z)\} \\
&= -\frac{1}{3}\{\varepsilon(y,z)\varepsilon(x,z)\varepsilon(x,y)(zy)\alpha(x) - \varepsilon(x,z)\varepsilon(x,y)(yz)\alpha(x) \\
&\quad +(xy)\alpha(z) - \varepsilon(y,z)(xz)\alpha(y)\} + \frac{1}{3}\{(xy)\alpha(z) - \varepsilon(x,y)(yx)\alpha(z) \\
&\quad +\varepsilon(y,z)\varepsilon(x,z)\varepsilon(x,y)(zy)\alpha(x) - \varepsilon(y,z)\varepsilon(x,z)(zx)\alpha(y)\} \\
&= -\frac{1}{3}\varepsilon(y,z)\varepsilon(x,z)\{\varepsilon(x,y)(zy)\alpha(x) - \varepsilon(z,y)\varepsilon(x,y)(yz)\alpha(x) \\
&\quad +\varepsilon(z,x)\varepsilon(z,y)(xy)\alpha(z) - \varepsilon(z,x)(xz)\alpha(y)\} \\
&\quad +\frac{1}{3}\varepsilon(y,z)\{\varepsilon(z,y)(xy)\alpha(z) - \varepsilon(x,y)\varepsilon(z,y)(yx)\alpha(z) \\
&\quad +\varepsilon(x,z)\varepsilon(x,y)(zy)\alpha(x) - \varepsilon(x,z)(zx)\alpha(y)\} \\
&= -\varepsilon(y,z)\varepsilon(x,z)as_A(z,x,y) + \varepsilon(y,z)as_A(x,z,y) \\
&= -\varepsilon(y,z)\varepsilon(x,z)as_A(z,x,y) - \varepsilon(x,y)\varepsilon(x,z)as_A(y,z,x) \\
&\quad \text{(by Lemma 3.4)}
\end{aligned}$$

Therefore, we conclude that $S_A = 0$. □

Next we show that the polarization of an admissible color Hom-Poisson algebra is ε-commutative Hom-associative.

3.6. Lemma. *Let* $(A, \mu, \varepsilon, \alpha)$ *be an admissible color Hom-Poisson algebra. Then*

$$(A, *, \varepsilon, \alpha) \tag{3.5}$$

is an ε*-commutative Hom-associative color Hom-algebra.*

Proof. It is obvious that $*$ is ε-commutative. To show that $as_{P(A)} = 0$, pick $x, y, z \in \mathcal{H}(A)$ and write μ using juxtatposition of homogeneous elements.
Expanding $as_{P(A)}$ in terms of μ, we have:

$$\begin{aligned}
as_{P(A)} &= (x*y)*\alpha(z) - \alpha(x)*(y*z) \\
&= \frac{1}{2}\{(xy + \varepsilon(x,y)yx)*\alpha(z) - \alpha(x)*(yz + \varepsilon(y,z)zy)\} \\
&= \frac{1}{4}\{(xy)\alpha(z) + \varepsilon(x,y)(yx)\alpha(z) + \varepsilon(x,z)\varepsilon(y,z)\alpha(z)(xy) \\
&\quad +\varepsilon(x,z)\varepsilon(y,z)\varepsilon(x,y)\alpha(z)(yx) - \alpha(x)(yz) - \varepsilon(y,z)\alpha(x)(zy) \\
&\quad -\varepsilon(x,y)\varepsilon(x,z)(yz)\alpha(x) - \varepsilon(x,y)\varepsilon(x,z)\varepsilon(y,z)(zy)\alpha(x)\}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4}\{as_A(x,y,z) - \varepsilon(x,y)\varepsilon(x,z)\varepsilon(y,z)as_A(z,y,x) + \varepsilon(x,y)(yx)\alpha(z)\\
&\quad -\varepsilon(x,y)\varepsilon(x,z)(yz)\alpha(x) - \varepsilon(x,z)\varepsilon(y,z)as_A(z,x,y)\\
&\quad +\varepsilon(x,z)\varepsilon(y,z)(zx)\alpha(y) + \varepsilon(y,z)as_A(x,z,y) - \varepsilon(y,z)(xz)\alpha(y)\}\\
&= \frac{1}{4}\{as_A(x,y,z) - \varepsilon(x,y)\varepsilon(x,z)\varepsilon(y,z)as_A(z,y,x) - [\varepsilon(y,z)(xz)\alpha(y)\\
&\quad -\varepsilon(x,z)\varepsilon(y,z)(zx)\alpha(y) + \varepsilon(x,y)\varepsilon(x,z)(yz)\alpha(x) - \varepsilon(x,y)(yx)\alpha(z)]\\
&\quad -\varepsilon(x,z)\varepsilon(y,z)as_A(z,x,y) + \varepsilon(y,z)as_A(x,z,y)\}\text{(rearranging terms)}\\
&= \frac{1}{4}\{as_A(x,y,z) - \varepsilon(x,y)\varepsilon(x,z)\varepsilon(y,z)as_A(z,y,x) - 3as_A(x,y,z)\\
&\quad -\varepsilon(x,z)\varepsilon(y,z)as_A(z,x,y) + \varepsilon(y,z)as_A(x,z,y)\}\text{(by (3.1))}\\
&= \frac{1}{4}\{as_A(x,y,z) + as_A(x,y,z) - 3as_A(x,y,z)\\
&\quad -\varepsilon(x,z)\varepsilon(y,z)as_A(z,x,y) - \varepsilon(x,y)\varepsilon(x,z)as_A(y,z,x)\}\\
&\quad \text{(by Lemma 3.4)}\\
&= \frac{1}{4}\{as_A(x,y,z) + as_A(x,y,z) - 3as_A(x,y,z) + as_A(x,y,z)\}\\
&\quad \text{(by Lemma 3.5)}
\end{aligned}$$

and thus $as_{P(A)} = 0$. ☐

Now we observe that the polarization of an admissible color Hom-Poisson algebra is a color Hom-Lie algebra.

3.7. Lemma. *Let $(A, \mu, \varepsilon, \alpha)$ be a color Hom-algebra. Then*

$$4J_{P(A)}(x,y,z) = \varepsilon(z,x)S_A(x,y,z) - \varepsilon(x,y)\varepsilon(z,x)S_A(y,x,z) \tag{3.6}$$

for all $x, y, z \in \mathcal{H}(A)$ where $J_{P(A)}$ is the Hom-Jacobian (2.4) of the polarisation of A see (3.2). Moreover, if A is an admissible color Hom-Poisson algebra, then

$$(A, \{,\}, \varepsilon, \alpha)$$

is a color Hom-Lie algebra where $\{x,y\} = \mu(x,y) - \varepsilon(x,y)\mu(y,x)$ for all $x, y \in \mathcal{H}(A)$.

Proof. To show this relation, pick $x, y, z \in \mathcal{H}(A)$ and write μ using juxtatposition of homogeneous elements. Expanding $J_{P(A)}$ in terms of μ, we have:

$$\begin{aligned}
4J_{P(A)}(x,y,z) &= \oint \varepsilon(z,x)\{\{x,y\},\alpha(z)\}\\
&= \varepsilon(z,x)(xy)\alpha(z) - \varepsilon(y,z)\alpha(z)(xy) - \varepsilon(z,x)\varepsilon(x,y)(yx)\alpha(z)\\
&\quad +\varepsilon(x,y)\varepsilon(y,z)\alpha(z)(yx) + \varepsilon(y,z)(zx)\alpha(y) - \varepsilon(x,y)\alpha(y)(zx)\\
&\quad -\varepsilon(y,z)\varepsilon(z,x)(xz)\alpha(y) + \varepsilon(z,x)\varepsilon(x,y)\alpha(y)(xz)\\
&\quad +\varepsilon(x,y)(yz)\alpha(x) - \varepsilon(z,x)\alpha(x)(yz) - \varepsilon(x,y)\varepsilon(y,z)(zy)\alpha(x)\\
&\quad +\varepsilon(y,z)\varepsilon(z,x)\alpha(x)(zy)\\
&= \varepsilon(z,x)as_A(x,y,z) + \varepsilon(y,z)as_A(z,x,y)\\
&\quad -\varepsilon(z,x)\varepsilon(x,y)as_A(y,x,z) - \varepsilon(x,y)\varepsilon(y,z)as_A(z,y,x)\\
&\quad +\varepsilon(x,y)as_A(y,z,x) - \varepsilon(y,z)\varepsilon(z,x)as_A(x,z,y)\\
&= \varepsilon(z,x)S_A(x,y,z) - \varepsilon(x,y)\varepsilon(z,x)S_A(y,x,z) \text{ (by (3.4))}
\end{aligned}$$

and then the desired relation holds. ☐

If A is an admissible color Hom-Poisson algebra, then by Lemma 3.5, it follows that $J_{P(A)} = 0$ and therefore $(A, \{,\}, \varepsilon, \alpha)$ is a color Hom-Lie algebra.

The following result says that the polarization of an admissible color Hom-Poisson algebra satisfies the color Hom-Leibniz identity (2.5).

3.8. Lemma. *Let $(A, \mu, \varepsilon, \alpha)$ be a color Hom-algebra. Then the polarization $P(A)$ satisfies*

$$\begin{aligned} & 4(\{\alpha(x), y * z\} - \{x, y\} * \alpha(z) - \varepsilon(x, y)\alpha(y) * \{x, z\}) \\ = \; & -as_A(x, y, z) - \varepsilon(x, y)\varepsilon(x, z)as_A(y, z, x) - \varepsilon(y, z)as_A(x, z, y) \\ & -\varepsilon(x, y)\varepsilon(x, z)\varepsilon(y, z)as_A(z, y, x) + \varepsilon(x, y)as_A(y, x, z) \\ & +\varepsilon(y, z)\varepsilon(x, z)as_A(z, x, y) \end{aligned}$$

for all $x, y, z \in \mathcal{H}(A)$. Moreover, if A is an admissible color Hom-Poisson algebra, then the polarization $P(A)$ satisfies the color Hom-Leibniz identity.

Proof. To prove this relation, pick $x, y, z \in \mathcal{H}(A)$ and write μ using juxtatposition of homogeneous elements. Expanding the left-hand side in terms of μ, we have:

$$\begin{aligned} & 4(\{\alpha(x), y * z\} - \{x, y\} * \alpha(z) - \varepsilon(x, y)\alpha(y) * \{x, z\}) \\ = \; & \alpha(x)(yz) - \varepsilon(x, y)\varepsilon(x, z)(yz)\alpha(x) + \varepsilon(y, z)\alpha(x)(zy) \\ & -\varepsilon(y, z)\varepsilon(x, z)\varepsilon(x, y)(zy)\alpha(x) - (xy)\alpha(z) + \varepsilon(x, y)(yx)\alpha(z) \\ & -\varepsilon(x, z)\varepsilon(y, z)\alpha(z)(xy) + \varepsilon(x, y)\varepsilon(x, z)\varepsilon(y, z)\alpha(z)(yx) - \varepsilon(x, y)\alpha(y)(xz) \\ & +\varepsilon(x, y)\varepsilon(x, z)\alpha(y)(zx) - \varepsilon(y, z)(xz)\alpha(y) + \varepsilon(y, z)\varepsilon(x, z)(zx)\alpha(y) \\ = \; & -as_A(x, y, z) - \varepsilon(x, y)\varepsilon(x, z)as_A(y, z, x) - \varepsilon(y, z)as_A(x, z, y) \\ & -\varepsilon(x, y)\varepsilon(x, z)\varepsilon(y, z)as_A(z, y, x) + \varepsilon(x, y)as_A(y, x, z) \\ & +\varepsilon(y, z)\varepsilon(x, z)as_A(z, x, y) \end{aligned}$$

For the second assertion, suppose that A is an admissible color Hom-Poisson algebra. Then the color Hom-flexibility (Lemma 3.4) implies that the right-hand side of (3.7) is 0. We conclude that

$$\{\alpha(x), y * z\} = \{x, y\} * \alpha(z) + \varepsilon(x, y)\alpha(y) * \{x, z\}$$

which is the color Hom-Leibniz identity in the polarization $P(A)$. □

Next we show that only admissible color Hom-Poisson algebras can give rise to color Hom-Poisson algebras via polarization.

3.9. Lemma. *Let $(A, \mu, \varepsilon, \alpha)$ be a color Hom-algebra such that the polarization $P(A)$ is a color Hom-Poisson algebra. Then A is an admissible color Hom-Poisson algebra.*

Proof. We need to prove the identity (3.1). Pick $x, y, z \in \mathcal{H}(A)$. We will express the Hom-associator as_A in several different forms and compare them.

On the one hand, the color Hom-Jacobi identity $J_{P(A)} = 0$ and (3.6) imply that

$$\begin{aligned} as_A(x, y, z) \; = \; & -\varepsilon(y, z)\varepsilon(x, z)as_A(z, x, y) - \varepsilon(x, y)\varepsilon(x, z)as_A(y, z, x) \\ & +\varepsilon(x, y)as_A(y, x, z) + \varepsilon(x, y)\varepsilon(x, z)\varepsilon(y, z)as_A(z, y, x) \\ & +\varepsilon(y, z)as_A(x, z, y) \; (\text{ by } (3.4)\ ) \end{aligned} \tag{3.7}$$

Moreover, the color Hom-Leibniz identity in $P(A)$ and (3.7) imply that

$$\begin{aligned} as_A(x,y,z) &= -\varepsilon(x,y)\varepsilon(x,z)as_A(y,z,x) - \varepsilon(y,z)as_A(x,z,y) \\ &\quad -\varepsilon(x,y)\varepsilon(x,z)\varepsilon(y,z)as_A(z,y,x) + \varepsilon(x,y)as_A(y,x,z) \\ &\quad +\varepsilon(y,z)\varepsilon(x,z)as_A(z,x,y) \end{aligned} \tag{3.8}$$

Adding (3.7) and (3.8) and dividing the result by 2, we obtain

$$as_A(x,y,z) = \varepsilon(x,y)as_A(y,x,z) - \varepsilon(x,y)\varepsilon(x,z)as_A(y,z,x) \tag{3.9}$$

which we will use in a moment.

On the other hand, since $\mu = \{,\} + *$, we can expand the Hom-associator as_A in terms of $\{,\}$ and $*$ as follows:

$$\begin{aligned} as_A(x,y,z) &= \mu(\mu(x,y),\alpha(z)) - \mu(\alpha(x),\mu(y,z)) \\ &= \{\{x,y\},\alpha(z)\} + \{x*y,\alpha(z)\} + \{x,y\}*\alpha(z) + (x*y)*\alpha(z) \\ &\quad -\{\alpha(x),\{y,z\}\} - \{\alpha(x),y*z\} - \alpha(x)*\{y,z\} - \alpha(x)*(y*z) \end{aligned} \tag{3.10}$$

Since the polarization $P(A)$ is assumed to be a color Hom-Poisson algebra, we have:

$$0 = as_{P(A)}(x,y,z) = (x*y)*\alpha(z) - \alpha(x)*(y*z) \tag{3.11}$$

$$\begin{aligned} 0 &= \{x,z\}*\alpha(y) - \varepsilon(x,y)\varepsilon(z,y)\alpha(y)*\{x,z\} \text{ (by } \varepsilon\text{-commutativity)} \\ &= \{x*y,\alpha(z)\} - \alpha(x)*\{y,z\} - \{\alpha(x),y*z\} + \{x,y\}*\alpha(z) \\ &\quad \text{(by (2.5) and (2.6))} \end{aligned} \tag{3.12}$$

$$\{\{x,z\},\alpha(y)\} = \varepsilon(z,y)\{\{x,y\},\alpha(z)\} - \varepsilon(z,y)\{\alpha(x),\{y,z\}\} \tag{3.13}$$

by (2.3) and the ε-skew-symmetry of $\{,\}$.

Using the identities (3.11) in (3.13), we obtain from (3.10):

$$\begin{aligned} 4as_A(x,y,z) &= 4\varepsilon(y,z)\{\{x,z\},\alpha(y)\} \\ &= \varepsilon(y,z)(xz)\alpha(y) - \varepsilon(y,z)\varepsilon(x,z)(zx)\alpha(y) - \varepsilon(x,y)\alpha(y)(xz) \\ &\quad +\varepsilon(x,y)\varepsilon(x,z)\alpha(y)(zx) \\ &= \varepsilon(y,z)(xz)\alpha(y) - \varepsilon(y,z)\varepsilon(x,z)(zx)\alpha(y) + \varepsilon(x,y)as_A(y,x,z) \\ &\quad -\varepsilon(x,y)(yx)\alpha(z) - \varepsilon(x,y)\varepsilon(x,z)as_A(y,z,x) \\ &\quad +\varepsilon(x,y)\varepsilon(x,z)(yz)\alpha(x) \\ &= \varepsilon(y,z)(xz)\alpha(y) - \varepsilon(y,z)\varepsilon(x,z)(zx)\alpha(y) + \varepsilon(x,y)\varepsilon(x,z)(yz)\alpha(x) \\ &\quad -\varepsilon(x,y)(yx)\alpha(z) + as_A(x,y,z) \text{ (by (3.9))} \end{aligned}$$

Finally, subtracting $as_A(x,y,z)$ in the above calculation and dividing the result by 3, we obtain the desired identity (3.1). □

Now the main result of this section is the following

3.10. Theorem. *Let $(A,\mu,\varepsilon,\alpha)$ be a color Hom-algebra. Then the polarization $P(A)$ is a color Hom-Poisson algebra if and only if A is an admissible color Hom-Poisson algebra.*

Proof. If A is an admissible color Hom-Poisson algebra, then Lemmas 3.6, 3.7, and 3.8 imply that the polarization $P(A)$ is a color Hom-Poisson algebra. The converse is Lemma 3.9. □

3.11. Corollary. *The polarization and the depolarization functions*

$$P : \{\textit{admissible color Hom-Poisson algebras}\} \rightleftarrows \{\textit{color Hom-Poisson algebras}\} : P^-$$

preserve multiplicativity and are the inverses of each other.

Proof. If $(A, \mu, \varepsilon, \alpha)$ is an admissible color Hom-Poisson algebra, then $P(A)$ is a color Hom-Poisson algebra by Theorem 3.10. Furthermore we have for all $x, y \in \mathcal{H}(A)$:

$$\begin{aligned} \{x, y\} + x * y &= \frac{1}{2}(\mu(x, y) - \varepsilon(x, y)\mu(y, x)) + \frac{1}{2}(\mu(x, y) + \varepsilon(x, y)\mu(y, x)) \\ &= \mu(x, y) \end{aligned}$$

i.e. $P^-(P(A)) = A$.
Conversely, suppose that $(A, \{,\}, *, \varepsilon, \alpha)$ is a color Hom-Poisson algebra. To show that $P^-(A)$ is an admissible color Hom-Poisson algebra, note by the ε-skew-symmetry of $\{,\}$ and the ε-commutativity of $*$ that for all $x, y \in \mathcal{H}(A)$,

$$\begin{aligned} \frac{1}{2}[(\{x, y\} + x * y) - \varepsilon(x, y)(\{y, x\} + y * x)] &= \{x, y\} \\ \frac{1}{2}[(\{x, y\} + x * y) + \varepsilon(x, y)(\{y, x\} + y * x)] &= x * y \end{aligned}$$

i.e. $P(P^-(A)) = A$, which is a color Hom-Poisson algebra. It follows from Theorem 3.10 that $P^-(A)$ is an admissible color Hom-Poisson algebra. Since P^-P and PP^- are both identity functions, P and P^- are the inverses of each other.
The fact that P and P^- preserve multiplicativity is straightforward. □

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