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On m-quasi class $\mathcal{A}(k^*)$ and absolute-(k^*, m)-paranormal operators

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Abstract

In this paper, we introduce a new class of operators, called m-quasi class $\mathcal{A}(k^*)$ operators, which is a superclass of hyponormal operators and a subclass of absolute-(k^*, m)-paranormal operators. We will show basic structural properties and some spectral properties of this class of operators. We show that if T is m-quasi class $\mathcal{A}(k^*)$, then $\sigma_{np}(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\}$, $\sigma_{na}(T) \setminus \{0\} = \sigma_a(T) \setminus \{0\}$ and $T - \mu$ has finite ascent for all $\mu \in \mathbb{C}$. Also, we consider the tensor product of m-quasi class $\mathcal{A}(k^*)$ operators.

Dedicated to the memory of Professor Takayuki Furuta with deep gratitude.

Keywords: m-quasi class $\mathcal{A}(k^*)$, absolute-(k^*, m)-paranormal.

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1. Introduction

Let H be an infinite dimensional complex Hilbert space and $L(H)$ be the set of all bounded operators on H. For $T \in L(H)$, we denote by $\ker T$ the null space and by $T(H)$ the range of T. The closure of a set M will be denoted by $\overline{M}$. An operator $T \in L(H)$ is said to be positive $T \geq 0$ if $\langle Tx, x\rangle \geq 0$ for all $x \in H$. We write $r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$ for the spectral radius. It is well known that $r(T) \leq \|T\|$. An operator T is called a normaloid operator if $r(T) = \|T\|$.

An operator T is said to be paranormal if $\|T^2x\| \geq \|Tx\|^2$ for every unit vector $x \in H$ ([6]). Also, T is said to be a $*$-paranormal operator if $\|T^2x\| \geq \|T^*x\|^2$ for every unit vector $x \in H$ ([4]).

In [7], Furuta, Ito and Yamazaki introduced a class $\mathcal{A}(k)$ operator T with $k > 0$ defined as

$$\left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} \geq |T|^2$$

(for $k = 1$ it defines the class $\mathcal{A}$ operator). The set of class $\mathcal{A}(1)$ operators includes log-hyponormal operators by Theorem 2 of [7] and paranormal operators by Theorem 1 of [7]. In [7], an absolute-k-paranormal operator T with $k > 0$ was introduced as

$$\left\||T|^kTx\right\| \geq \|Tx\|^{k+1}$$

for every unit vector $x \in H$. Every class $\mathcal{A}(k)$ operator with $k > 0$ is an absolute-k-paranormal operator by Theorem 2 of [7].

An operator T is said to be a class $\mathcal{A}(k^*)$ operator with $k > 0$ if

$$\left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} \geq |T^*|^2.$$

In case where $k = 1$ it defines class $\mathcal{A}^*$ operators. Every class $\mathcal{A}^*$ operator is a $*$-paranormal operator by Theorem 1.3 of [5].

In paper [13], an absolute-k^*-paranormal operator T with $k > 0$ was introduced as follows:

$$\||T|^kTx\| \geq \|T^*x\|^{k+1}$$

for every unit vector $x \in H$. Every class $\mathcal{A}(k^*)$ operator is an absolute-k^*-paranormal operator by Theorem 2.4 of [13].

1.1. Lemma. [12, Hölder-McCarthy's inequality] *Let T be a positive operator. Then the following inequalities hold for all $x \in H$:*

(1) $\langle T^rx, x\rangle \leq \langle Tx, x\rangle^r \|x\|^{2(1-r)}$ *for* $0 < r < 1$,
(2) $\langle T^rx, x\rangle \geq \langle Tx, x\rangle^r \|x\|^{2(1-r)}$ *for* $r \geq 1$.

1.2. Lemma. [9, Hansen's inequality] *If $A, B \in L(H)$ satisfy $A \geq 0$ and $\|B\| \leq 1$, then*

$$(B^*AB)^\delta \geq B^*A^\delta B \text{ for all } \delta \in (0, 1].$$

2. Definition and examples

2.1. Definition. Let $k > 0$ and m be a non-negative integer. An operator $T \in L(H)$ is said to be an m-quasi class $\mathcal{A}(k^*)$ operator (abbreviate $\mathcal{Q}(\mathcal{A}(k^*), m)$) if

$$T^{*m}\left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} T^m \geq T^{*m}|T^*|^2T^m.$$

1-quasi class $\mathcal{A}(k^*)$ operator is called a quasi class $\mathcal{A}(k)^*$ operator. 1-quasi class $\mathcal{A}(1^*)$ operator is called a quasi class $\mathcal{A}^*$ operator. 0-quasi class $\mathcal{A}(k^*)$ operator is called a class $\mathcal{A}(k^*)$operator and 0-quasi class $\mathcal{A}(1^*)$ operator is called a class $\mathcal{A}^*$ operator. If T is an m-quasi class $\mathcal{A}(k^*)$ operator, then T is an $(m+1)$-quasi class $\mathcal{A}(k^*)$ operator. The inverse is not true as it can be seen below.

2.2. Example. Consider a unilateral weighted shift operator as an infinite dimensional Hilbert space operator. Recall that given a bounded sequence of positive numbers $\alpha := \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, ...\}$ (called weights), a unilateral weighted shift W_α associated with weight α is defined by $W_\alpha e_n = \alpha_n e_{n+1}$ for all $n \geq 1$, where $\{e_n\}_{n=1}^{\infty}$ is the canonical orthonormal basis on $l_2(\mathbb{N})$, i.e.,

$$\mathbf{W}_\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ \alpha_1 & 0 & 0 & 0 & 0 & \dots \\ 0 & \alpha_2 & 0 & 0 & 0 & \dots \\ 0 & 0 & \alpha_3 & 0 & 0 & \dots \\ 0 & 0 & 0 & \alpha_4 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then W_α is an m-quasi class $\mathcal{A}(k^*)$ operator if and only if

$$\alpha_{m+l+1}^2 \alpha_{m+l+2}^{2k} \geq \alpha_{m+l}^{2(k+1)} \text{ for all } l \in \mathbb{N} \cup \{0\}.$$

If $\alpha_{m+1} \leq \alpha_{m+2} \leq \alpha_{m+3} \leq \alpha_{m+4} \leq ...$ and $\alpha_m > \alpha_{m+1}$, then W_α is an $(m+1)$-quasi class $\mathcal{A}(k^*)$ but it is not an m-quasi class $\mathcal{A}(k^*)$ operator. For example, if $1 = \alpha_1 = \alpha_2 = \cdots = \alpha_m$ and $2 = \alpha_{m+1} = \alpha_{m+2} = \cdots$, then W_α is an $(m+1)$-quasi class $\mathcal{A}(k^*)$ but W_α is not an m-quasi class $\mathcal{A}(k^*)$ operator.

It is well known that every $*$-paranormal operator is normaloid by Theorem 1.1 of [4]. But an m-quasi class $\mathcal{A}(k^*)$ operator with $m \geq 2$ need not be a normaloid operator: if $\alpha_1 > \alpha_2 = \alpha_3 = \cdots$, then

$$\|W_\alpha\| = \alpha_1 \text{ and } r(W_\alpha) = \lim_{n\to\infty} \|W_\alpha^n\|^{\frac{1}{n}} = \alpha_2.$$

Now, we show that m-quasi class $\mathcal{A}((k+1)^*)$ and $(m+1)$-quasi class $\mathcal{A}(k^*)$ operator are independent.

2.3. Example. An example of a 1-quasi class $\mathcal{A}(2^*)$ operator which is not a 2-quasi class $\mathcal{A}(1^*)$ operator.

Let W_α be a unilateral weighted shift operator with weighted sequence $\{\alpha_n : n \in \mathbb{N}\}$, given by the relation:

$$\alpha_n = \begin{cases} 1 & \text{if } n = 1 \\ \sqrt{2} & \text{if } n = 2 \\ 2 & \text{if } n = 3 \\ \sqrt[4]{3} & \text{if } n = 4 \\ 3 & \text{if } n \geq 5. \end{cases}$$

Simple calculations show that W_α is a 1-quasi class $\mathcal{A}(2^*)$ operator, but W_α is not a 2-quasi class $\mathcal{A}(1^*)$ operator.

2.4. Example. An example of a 2-quasi class $\mathcal{A}(1^*)$ operator which is not a 1-quasi class $\mathcal{A}(2^*)$ operator.

Let W_α be a unilateral weighted shift operator with weighted sequence $\{\alpha_n : n \in \mathbb{N}\}$, given by the relation:

$$\alpha_n = \begin{cases} \sqrt[3]{2} & \text{if } n = 1 \\ \frac{1}{\sqrt{2}} & \text{if } n = 2 \\ \sqrt{2} & \text{if } n = 3 \\ 2 & \text{if } n = 4 \\ 4 & \text{if } n \geq 5. \end{cases}$$

Simple calculations show that W_α is a 2-quasi class $\mathcal{A}(1^*)$ operator, but W_α is not a 1-quasi class $\mathcal{A}(2^*)$ operator.

Given a bounded sequence of complex numbers $\alpha := \{\alpha_n : n \in \mathbb{Z}\}$(called weights), let T_α be a bilateral weighted shift defined by $T_\alpha e_n = \alpha_n e_{n+1}$ for all $n \in \mathbb{Z}$ on $H = l_2(\mathbb{Z})$ with the canonical orthonormal basis $\{e_n : n \in \mathbb{Z}\}$. Based on the definition of the m-quasi class $\mathcal{A}(k^*)$ operators the following facts are valid:

2.5. Lemma. *Let T_α be a bilateral weighted shift operator defined as above with weights $\{\alpha_n : n \in \mathbb{Z}\}$. Then T_α is an m-quasi class $\mathcal{A}(k^*)$ operator if and only if*

$$|\alpha_{n+m}|^2 \cdot |\alpha_{n+m+1}|^{2k} \geq |\alpha_{n+m-1}|^{2(k+1)},$$

for all $n \in \mathbb{Z}$ and $m \in \mathbb{N} \cup \{0\}$.

A subspace M of H is said to be a nontrivial invariant subspace of T if $\{0\} \neq M \neq H$ and $T(M) \subseteq M$.

2.6. Theorem. *Let $T \in \mathcal{Q}(\mathcal{A}(k^*), m)$ with $0 < k \leq 1$ and T does not have a dense range. Then*

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \quad \text{on} \quad H = \overline{T^m(H)} \oplus \ker(T^{*m}),$$

where $A = T|_{\overline{T^m(H)}}$ is a class $\mathcal{A}(k^*)$ operator on $\overline{T^m(H)}$, $C^m = 0$ and $\sigma(T) = \sigma(A) \cup \{0\}$.

Proof. Since $\overline{T^m(H)} \subsetneqq H$ is an invariant subspace of T, T can be written in

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \quad \text{on} \quad H = \overline{T^m(H)} \oplus \ker(T^{*m}).$$

Let $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ be the orthogonal projection of H onto $\overline{T^m(H)}$. Then $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} = TP = PTP$. Since $T \in \mathcal{Q}(\mathcal{A}(k^*), m)$, we have

$$P\left(\left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} - |T^*|^2\right)P \geq O.$$

By Hansen's inequality, we have

$$\begin{aligned} \begin{pmatrix} |A^*|^2 & 0 \\ 0 & 0 \end{pmatrix} &\leq \begin{pmatrix} |A^*|^2 + |B^*|^2 & 0 \\ 0 & 0 \end{pmatrix} \\ &= P|T^*|^2P \leq P\left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}}P \\ &\leq \left(PT^*|T|^{2k}TP\right)^{\frac{1}{k+1}} = \left(PT^*P|T|^{2k}PTP\right)^{\frac{1}{k+1}}. \end{aligned}$$

Also, by Hansen's inequality, we have $P|T|^{2k}P \leq (P|T|^2P)^k$ and

$$PT^*P|T|^{2k}PTP \leq PT^*(P|T|^2P)^kTP.$$

By Löwner–Heinz's inequality we have

$$\left(PT^*P|T|^{2k}PTP\right)^{\frac{1}{k+1}} \leq \left(PT^*(P|T|^2P)^kTP\right)^{\frac{1}{k+1}}.$$

So, we have

$$\begin{pmatrix} |A^*|^2 & 0 \\ 0 & 0 \end{pmatrix} \le P|T^*|^2 P$$
$$\le \left(PT^*P|T|^{2k}PTP\right)^{\frac{1}{k+1}} \le \left(PT^*(P|T|^2P)^k TP\right)^{\frac{1}{k+1}}$$
$$= \begin{pmatrix} A^*|A^*|^{2k}A & 0 \\ 0 & 0 \end{pmatrix}^{\frac{1}{k+1}} = \begin{pmatrix} (A^*|A^*|^{2k}A)^{\frac{1}{k+1}} & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence A is a class $\mathcal{A}(k^*)$ operator on $\overline{T^m(H)}$.

Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in H = \overline{T^m(H)} \oplus \ker(T^{*m})$. Then,

$$\langle C^m x_2, x_2 \rangle = \langle T^m(I-P)x, (I-P)x \rangle = \langle (I-P)x, T^{*m}(I-P)x \rangle = 0,$$

thus $C^m = 0$.

By Corollary 7 of [8], $\sigma(A) \cup \sigma(C) = \sigma(T) \cup \vartheta$ where ϑ is the union of the holes in $\sigma(T)$, which happen to be a subset of $\sigma(A) \cap \sigma(C)$. Since $\sigma(C) = \{0\}$, $\sigma(A) \cap \sigma(C)$ has no interior point. Therefore $\sigma(T) = \sigma(A) \cup \sigma(C) = \sigma(A) \cup \{0\}$. □

2.7. Theorem. *Let $T \in \mathcal{Q}(\mathcal{A}(k^*), m)$ with $0 < k \le 1$ and M be an invariant subspace of T. Then the restriction $T|_M$ of T to M is also a $\mathcal{Q}(\mathcal{A}(k^*), m)$ operator.*

Proof. We can represent T as

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \quad \text{on} \quad H = M \oplus M^\perp$$

where $A = T|_M$. Let $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ be the orthogonal projection of H onto M. Then we have

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} = TP = PTP.$$

Since T is an m-quasi class $\mathcal{A}(k^*)$ operator, we have

$$T^{*m}\left(\left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} - |T^*|^2\right)T^m \ge 0.$$

We remark

$$PT^{*m}|T^*|^2 T^m P = PT^{*m}P|T^*|^2 PT^m P = PT^{*m}PTT^*PT^m P$$
$$= \begin{pmatrix} A^{*m}|A^*|^2A^m + |B^*A^m|^2 & 0 \\ 0 & 0 \end{pmatrix} \ge \begin{pmatrix} A^{*m}|A^*|^2A^m & 0 \\ 0 & 0 \end{pmatrix}.$$

By Hansen's inequality, we have

$$\begin{aligned}PT^{*m}\left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}}T^mP &= PT^{*m}P\left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}}PT^mP\\ &\le PT^{*m}\left(PT^*|T|^{2k}TP\right)^{\frac{1}{k+1}}T^mP\\ &= PT^{*m}\left(PT^*P|T|^{2k}PTP\right)^{\frac{1}{k+1}}T^mP\\ &\le PT^{*m}\left(PT^*P(PT^*TP)^kPTP\right)^{\frac{1}{k+1}}T^mP\\ &= \begin{pmatrix}A^{*m} & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}A^*|A|^{2k}A & 0\\ 0 & 0\end{pmatrix}^{\frac{1}{k+1}}\begin{pmatrix}A^m & 0\\ 0 & 0\end{pmatrix}\\ &= \begin{pmatrix}A^{*m}(A^*|A|^{2k}A)^{\frac{1}{k+1}}A^m & 0\\ 0 & 0\end{pmatrix}\end{aligned}$$

Hence

$$\begin{aligned}\begin{pmatrix}A^{*m}(A^*|A|^{2k}A)^{\frac{1}{k+1}}A^m & 0\\ 0 & 0\end{pmatrix} &\ge PT^{*m}\left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}}T^mP\\ &\ge PT^{*m}|T^*|^2T^mP \ge \begin{pmatrix}A^{*m}|A^*|^2A^m & 0\\ 0 & 0\end{pmatrix}.\end{aligned}$$

Thus A is an m-quasi class $\mathcal{A}(k^*)$ operator on M. □

3. On absolute-(k^*, m)-paranormal operator

3.1. Definition. Let $k > 0$ and m be a non-negative integer. An operator $T \in L(H)$ is said to be an absolute-(k^*, m)-paranormal operator if

$$\||T^*|T^mx\|^{k+1} \le \||T|^kT^{m+1}x\|\|T^mx\|^k \quad \text{for} \quad x \in H.$$

An absolute-$(k^*, 0)$-paranormal operator is called an absolute-k^*-paranormal operator. If T is an absolute-(k^*, m)-paranormal operator, then we have T is an absolute-$(k^*, m+1)$-paranormal operator by taking $x = Tz$ in the definition.

3.2. Lemma. *For positive real numbers $a > 0$ and $b > 0$,*

$$\lambda a + \mu b \ge a^\lambda b^\mu$$

holds for $\lambda > 0$ and $\mu > 0$ such that $\lambda + \mu = 1$.

3.3. Theorem. *Let $k > 0$ and m be a non-negative integer. Then an operator $T \in L(H)$ is an absolute-(k^*, m)-paranormal operator if and only if*

$$T^{*(m+1)}|T|^{2k}T^{m+1} - (k+1)\lambda^kT^{*m}|T^*|^2T^m + k\lambda^{k+1}T^{*m}T^m \ge 0 \quad \textit{for all} \quad \lambda > 0.$$

Proof. Suppose T is an absolute-(k^*, m)-paranormal operator. Then

$$\||T^*|T^mx\| \le \||T|^kT^{m+1}x\|^{\frac{1}{k+1}}\|T^mx\|^{\frac{k}{k+1}}. \tag{3.1}$$

Using Lemma 3.2, we have

$$\begin{aligned}\left\langle T^{*m}|T^*|^2T^mx, x\right\rangle &\le \left\langle T^{*(m+1)}|T|^{2k}T^{m+1}x, x\right\rangle^{\frac{1}{k+1}}\langle T^{*m}T^mx, x\rangle^{\frac{k}{k+1}}\\ &= \left\{\frac{1}{\lambda^k}\left\langle T^{*(m+1)}|T|^{2k}T^{m+1}x, x\right\rangle\right\}^{\frac{1}{k+1}}\{\lambda\langle T^{*m}T^mx, x\rangle\}^{\frac{k}{k+1}}\\ &\le \frac{1}{k+1}\frac{1}{\lambda^k}\left\langle T^{*(m+1)}|T|^{2k}T^{m+1}x, x\right\rangle + \frac{k}{k+1}\lambda\langle T^{*m}T^mx, x\rangle\end{aligned}$$

for all $x \in H$ and $\lambda > 0$. Hence

$$T^{*(m+1)}|T|^{2k}T^{m+1} - (k+1)\lambda^k T^{*m}|T^*|^2 T^m + k\lambda^{k+1}T^{*m}T^m \geq 0. \tag{3.2}$$

Conversely, we assume (3.2). If $T^m x = 0$, then (3.1) is trivial. Hence we may assume $T^m x \neq 0$. If $\langle T^{*(m+1)}|T|^{2k}T^{m+1}x, x\rangle > 0$, put

$$\lambda = \left(\frac{\langle T^{*(m+1)}|T|^{2k}T^{m+1}x, x\rangle}{\langle T^m x, T^m x\rangle}\right)^{\frac{1}{k+1}} > 0$$

in (3.3), i.e.,

$$\langle T^{*(m+1)}|T|^{2k}T^{m+1}x, x\rangle - (k+1)\lambda^k \langle T^{*m}|T^*|^2 T^m x, x\rangle + k\lambda^{k+1}\langle T^{*m}T^m x, x\rangle \geq 0. \tag{3.3}$$

Then we have (3.1). If $\langle T^{*(m+1)}|T|^{2k}T^{m+1}x, x\rangle = 0$, we have

$$0 - (k+1)\langle T^{*m}|T^*|^2 T^m x, x\rangle + k\lambda\langle T^{*m}T^m x, x\rangle \geq 0 \quad \text{for all} \quad \lambda > 0$$

by (3.3). By letting $\lambda \to +0$, we have $\langle T^{*m}|T^*|^2 T^m x, x\rangle = 0$ and we gain (3.1).

□

3.4. Theorem. *If $T \in \mathcal{Q}(\mathcal{A}(k^*), m)$, then T is an absolute-(k^*, m)-paranormal operator. The converse is not true.*

Proof. Suppose $T \in \mathcal{Q}(\mathcal{A}(k^*), m)$. From Hölder-McCarthy's inequality, we have

$$\begin{aligned} \||T^*|T^m x\|^2 &= \langle T^{*m}|T^*|^2 T^m x, x\rangle \\ &\leq \left\langle T^{*m}\left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} T^m x, x\right\rangle \\ &\leq \left\langle T^{*m}(T^*|T|^{2k}T)T^m x, x\right\rangle^{\frac{1}{k+1}} \|T^m x\|^{\frac{2k}{k+1}} \\ &= \||T|^k T^{m+1}x\|^{\frac{2}{k+1}} \|T^m x\|^{\frac{2k}{k+1}}. \end{aligned}$$

Hence T is an absolute-(k^*, m)-paranormal operator. To prove that the converse is not true we will consider a following example. □

3.5. Lemma. *Let $H = \oplus_{n=1}^{\infty} H_n$ where $H_n = \mathbb{C}^2$. Let $A_j \in B(H_j)$ and define $T \in B(H)$ as*

$$T = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ A_1 & 0 & 0 & \cdots \\ 0 & A_2 & 0 & \cdots \\ 0 & 0 & A_3 & \\ \vdots & \vdots & & \ddots \end{pmatrix}.$$

Let $k > 0$ and m be a non-negative integer. Then the following assertions hold:

(1) *T is an m-quasi class $\mathcal{A}(k^*)$ operator if and only if*

$$\begin{aligned} &A_j^* A_{j+1}^* \cdots A_{j+m-1}^* \left(A_{j+m}^*|A_{j+m+1}|^{2k}A_{j+m}\right)^{\frac{1}{k+1}} A_{j+m-1}\cdots A_{j+1}A_j \\ &\geq A_j^* A_{j+1}^* \cdots A_{j+m-1}^*|A_m^*|^2 A_{j+m-1}\cdots A_{j+1}A_j \quad \textit{for} \quad j = 1, 2, \cdots. \end{aligned} \tag{3.4}$$

(2) *T is an absolute-(k^*, m)-paranormal operator if and only if*

$$A_j^* \cdots A_{j+m-1}^*\left(A_{j+m}^*|A_{j+m+1}|^{2k}A_{j+m} - (k+1)\lambda^k|A_{j+m-1}^*|^2 + k\lambda^{k+1}\right)A_{j+m-1}\cdots A_j \geq 0 \tag{3.5}$$

for $j = 1, 2, \cdots$.

3.6. Example. Examples of m-quasi class $\mathcal{A}(k^*)$ operators and an absolute-(k^*, m)-paranormal operators.

Consider $T = T(m,c)$ with $0 < c < \sqrt{3}/4 = 0.433\cdots$ as in Lemma 3.5 where $0 < A_1 = A_2 = \cdots = A_m = \begin{pmatrix} \frac{3}{4} & c \\ c & \frac{1}{4} \end{pmatrix}$ and $0 < \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} = A_{m+1} = A_{m+2} = \cdots$. Since every A_j is invertible, (3.4) means

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \geq \begin{pmatrix} \frac{3}{4} & c \\ c & \frac{1}{4} \end{pmatrix}.$$

Since

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} - \begin{pmatrix} \frac{3}{4} & c \\ c & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & -c \\ -c & \frac{1}{4} \end{pmatrix}$$

and

$$\begin{vmatrix} \frac{1}{4} & -c \\ -c & \frac{1}{4} \end{vmatrix} = \frac{1}{16} - c^2,$$

we have that $T(m,c)$ is an m-quasi class $\mathcal{A}(k^*)$ operator if $0 < c \leq 0.25$ and $T(m,c)$ is not an m-quasi class $\mathcal{A}(k^*)$ operator if $0.25 < c < \sqrt{3}/4$. Also, $T(m,c)$ is an $(m+1)$-quasi class $\mathcal{A}(k^*)$ operator for all $0 < c < \sqrt{3}/4$. On the otherhand (3.5) means

$$\begin{pmatrix} 1 - \frac{3}{4}(k+1)\lambda^k + k\lambda^{k+1} & -(k+1)\lambda^k c \\ -(k+1)\lambda^k c & \left(\frac{1}{2}\right)^{k+1} - \frac{1}{4}(k+1)\lambda^k + k\lambda^{k+1} \end{pmatrix} \geq 0 \text{ for all } \lambda > 0. \tag{3.6}$$

Since

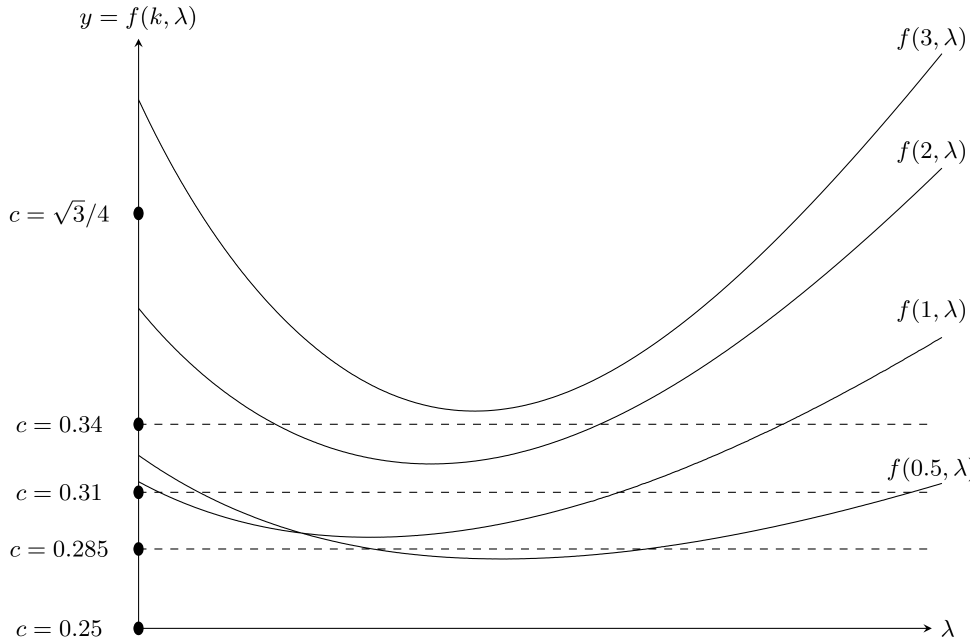
$$\begin{aligned} 1 - \frac{3}{4}(k+1)\lambda^k + k\lambda^{k+1} &> 0, \\ \left(\frac{1}{2}\right)^{k+1} - \frac{1}{4}(k+1)\lambda^k + k\lambda^{k+1} &> 0 \quad \text{for all} \quad \lambda > 0, \end{aligned}$$

the inequality (3.6) means

$$\begin{vmatrix} 1 - \frac{3}{4}(k+1)\lambda^k + k\lambda^{k+1} & -(k+1)\lambda^k c \\ -(k+1)\lambda^k c & \left(\frac{1}{2}\right)^{k+1} - \frac{1}{4}(k+1)\lambda^k + k\lambda^{k+1} \end{vmatrix} \geq 0 \text{ for all } \lambda > 0, \tag{3.7}$$

or equivalently,

$$\begin{aligned} f(k,\lambda) := &\left(\frac{1}{(k+1)\lambda^k} - \frac{3}{4} + \frac{k\lambda}{k+1}\right)^{\frac{1}{2}} \left(\frac{1}{(k+1)2^{k+1}\lambda^k} - \frac{1}{4} + \frac{k\lambda}{k+1}\right)^{\frac{1}{2}} \\ &\geq c \quad \text{for all } \lambda > 0. \end{aligned} \tag{3.8}$$



The above is graph of $y = f(0.5, \lambda), f(1, \lambda), f(2, \lambda), f(3, \lambda)$. Hence $T(m, 0.285)$ is an absolute-$(1^*, m)$-paranormal operator, but $T(m, 0.285)$ is not an absolute-$(0.5^*, m)$-paranormal operator. Also, $T(m, 0.31)$ is an absolute-$(2^*, m)$-paranormal operator, but $T(m, 0.31)$ is not an absolute-$(1^*, m)$-paranormal operator, and $T(m, 0.34)$ is an absolute-$(3^*, m)$-paranormal operator, but $T(m, 0.34)$ is not an absolute-$(2^*, m)$-paranormal operator.

4. Spectral properties

A complex number λ is said to be in the point spectrum $\sigma_p(T)$ of T if there is a nonzero $x \in H$ such that $(T - \mu)x = 0$. If in addition, $(T - \mu)^* x = 0$, then μ is said to be in the normal point spectrum $\sigma_{np}(T)$ of T. Clearly $\sigma_{np}(T) \subseteq \sigma_p(T)$. In general $\sigma_{np}(T) \neq \sigma_p(T)$. A complex number μ is said to be in the approximate point spectrum $\sigma_a(T)$ of T if there is a sequence $\{x_n\}_{n=1}^{\infty} \subset H$ of unit vectors satisfying $(T - \mu)x_n \to 0$ as $n \to \infty$. If in addition $(T - \mu)^* x_n \to 0$ as $n \to \infty$, then μ is said to be in the normal approximate point spectrum $\sigma_{na}(T)$ of an operator T. Clearly $\sigma_{na}(T) \subseteq \sigma_a(T)$. In general $\sigma_{na}(T) \neq \sigma_a(T)$. Let $\alpha(T) = \dim \ker(T)$ and $\beta(T) = \dim Ker(T^*)$.

4.1. Theorem. *Let $0 < k \leq 1$ and m be a non-negative integer. If $T \in \mathcal{Q}(\mathcal{A}(k^*), m)$ and $(T - \mu)x = 0$ with $\mu \neq 0$, then $(T - \mu)^* x = 0$.*

Proof. We may assume that $x \neq 0$. Let M be a span of $\{x\}$. Then M is an invariant subspace of T. Let

$$T = \begin{pmatrix} \mu & B \\ 0 & C \end{pmatrix} \quad \text{on} \quad H = M \oplus M^{\perp}.$$

From the Theorem 2.7 we have

$$\begin{aligned}\begin{pmatrix}|\mu|^{2m}(|\mu|^2+|B^*|^2) & 0\\ 0 & 0\end{pmatrix} &= PT^{*m}|T^*|^2T^mP\\ &\le PT^{*m}\left(PT^*P|T|^{2k}PTP\right)^{\frac{1}{k+1}}T^mP\\ &\le PT^{*m}\left(PT^*P(P|T|^2P)^kPTP\right)^{\frac{1}{k+1}}T^mP\\ &= \begin{pmatrix}|\mu|^{2+2m} & 0\\ 0 & 0\end{pmatrix}.\end{aligned}$$

Hence $B=0$. Thus

$$(T-\mu)^*x = \begin{pmatrix}0 & 0\\ 0 & C-\mu\end{pmatrix}^*\begin{pmatrix}x\\ 0\end{pmatrix} = 0.$$

□

4.2. Corollary. *If T is an m-quasi class $\mathcal{A}(k^*)$ operator with $0<k\le 1$, then $\sigma_{np}(T)\setminus\{0\}=\sigma_p(T)\setminus\{0\}$.*

4.3. Corollary. *If T is an m-quasi class $\mathcal{A}(k^*)$ operator with $0<k\le 1$, then $\alpha(T-\mu)\le\beta(T-\mu)$ for all $\mu\neq 0$.*

4.4. Theorem. *Let $0<k\le 1$ and m be a non-negative integer. If $T\in\mathcal{Q}(\mathcal{A}(k^*),m)$ and γ,δ are nonzero numbers such that $\gamma\neq\delta$, then $\ker(T-\gamma)\perp\ker(T-\delta)$.*

Proof. Let $x\in\ker(T-\gamma)$ and $y\in\ker(T-\delta)$. Then $Tx=\gamma x$ and $Ty=\delta y$. Therefore

$$\gamma\langle x,y\rangle=\langle Tx,y\rangle=\langle x,T^*y\rangle=\langle x,\overline{\delta}y\rangle=\delta\langle x,y\rangle,$$

then $\langle x,y\rangle=0$. Therefore, $\ker(T-\gamma)\perp\ker(T-\delta)$. □

4.5. Theorem. *Let $0<k\le 1$ and m be a non-negative integer. If $T\in\mathcal{Q}(\mathcal{A}(k^*),m)$ and $(T-\mu)x_n\to 0$ with $\mu\neq 0$ and $\|x_n\|=1$, then $(T-\mu)^*x_n\to 0$.*

Proof. By the assumption $(T-\mu)x_n\to 0$, from

$$T^l=(T-\mu+\mu)^l=\sum_{i=1}^{l}\binom{l}{i}\mu^{l-i}(T-\mu)^i+\mu^l,\ \text{for } l\in\mathbb{N},$$

we have $(T^l-\mu^l)x_n\to 0$. By

$$|\|T^lx_n\|-|\mu|^l|\le\|(T^l-\mu^l)x_n\|,$$

we have

(4.1) $\quad \|T^lx_n\|\to|\mu|^l.$

Moreover

(4.2) $\quad |\|T^*\mu^mx_n\|-\|T^*(T^m-\mu^m)x_n\||\le\|T^*T^mx_n\|.$

Since T is an m-quasi class $\mathcal{A}(k^*)$ operator, we get

$$\begin{aligned}\|T^*T^mx\|^2=\||T^*|T^mx\|^2 &\le \||T|^kT^{m+1}x\|^{\frac{2}{k+1}}\|T^mx\|^{\frac{2k}{k+1}}\\ &=\langle|T|^{2k}T^{m+1}x,T^{m+1}x\rangle^{\frac{1}{k+1}}\|T^mx\|^{\frac{2k}{k+1}}\\ &\le\langle|T|^2T^{m+1}x,T^{m+1}x\rangle^{\frac{k}{k+1}}\|T^{m+1}x\|^{\frac{2(1-k)}{k+1}}\|T^mx\|^{\frac{2k}{k+1}}\\ &=\|T^{m+2}x\|^{\frac{2k}{k+1}}\|T^{m+1}x\|^{\frac{2(1-k)}{k+1}}\|T^mx\|^{\frac{2k}{k+1}}\end{aligned}$$

by Hölder-McCarthy's inequality. Hence

$$\|T^*T^m x\| \le \|T^{m+2}x\|^{\frac{k}{k+1}}\|T^{m+1}x\|^{\frac{1-k}{k+1}}\|T^m x\|^{\frac{k}{k+1}}. \tag{4.3}$$

Then it follows from (4.1),(4.2) and (4.3) that

$$\limsup_{n\to\infty}\|T^*x_n\| \le |\mu|.$$

Since

$$\begin{aligned}\|(T-\mu)^*x_n\|^2 &= \|T^*x_n\|^2 - 2\mathrm{Re}\langle T^*x_n, \overline{\mu}x_n\rangle + |\mu|^2\|x_n\|^2\\ &= \|T^*x_n\|^2 - 2\mathrm{Re}\langle x_n, \overline{\mu}Tx_n\rangle + |\mu|^2\|x_n\|^2,\end{aligned}$$

we have

$$\limsup_{n\to\infty}\|(T-\mu)^*x_n\|^2 \le |\mu|^2 - 2|\mu|^2 + |\mu|^2 = 0.$$

This implies $(T-\mu)^*x_n \to 0$. □

4.6. Corollary. *If $T \in \mathcal{Q}(\mathcal{A}(k^*), m)$ with $0 < k \le 1$, then $\sigma_{na}(T)\setminus\{0\} = \sigma_a(T)\setminus\{0\}$.*

4.7. Lemma. [2, Corollary 2] *Let $T = U|T|$ be the polar decomposition of T, $\mu = |\mu|e^{i\theta} \neq 0$ and $\{x_n\}$ a sequence of vectors. Then the following assertions are equivalent:*

(1) *$(T-\mu)x_n \to 0$ and $(T^*-\overline{\mu})x_n \to 0$, as $n\to\infty$,*
(2) *$(|T|-|\mu|)x_n \to 0$ and $(U-e^{i\theta})x_n \to 0$, as $n\to\infty$,*
(3) *$(|T^*|-|\mu|)x_n \to 0$ and $(U^*-e^{-i\theta})x_n \to 0$, as $n\to\infty$.*

4.8. Corollary. *If $T \in \mathcal{Q}(\mathcal{A}(k^*), m)$ with $0 < k \le 1$ and $\mu \in \sigma_a(T)\setminus\{0\}$ then $|\mu| \in \sigma_a(|T|)\cap\sigma_a(|T^*|)$.*

Proof. If $\mu \in \sigma_a(T)\setminus\{0\}$, then by Theorem 4.5, there exists a sequence of unit vectors $\{x_n\}$ such that $(T-\mu)x_n \to 0$ and $(T-\mu)^*x_n \to 0$, as $n\to\infty$. Hence we have $|\mu| \in \sigma_a(|T|)\cap\sigma_a(|T^*|)$ by Lemma 4.7 □

4.9. Corollary. *Let $T \in \mathcal{Q}(\mathcal{A}(k^*), m)$ with $0 < k \le 1$ and $T = U|T|$ be the polar decomposition of T. If $\mu = |\mu|e^{i\theta} \neq 0$ and $\mu \in \sigma_a(T)$, then $e^{i\theta} \in \sigma_{na}(U)$.*

Proof. Let $\mu \in \sigma_a(T)$. From Corollary 4.6, $\mu \in \sigma_{na}(T)$. Then, there exists a sequence of unit vectors $\{x_n\}$ such that $(T-\mu)x_n \to 0$ and $(T-\mu)^*x_n \to 0$, as $n\to\infty$. From Lemma 4.7 we have $(U-e^{i\theta})x_n \to 0$ and $(U^*-e^{-i\theta})x_n \to 0$, as $n\to\infty$. Thus $e^{i\theta} \in \sigma_{na}(U)$. □

An operator T on a complex Banach space X has the single-valued extension property, abbreviated SVEP, if, for every open set $U \subset \mathbb{C}$, the only analytic solution $f: U \to X$ of the equation $(T-\lambda)f(\lambda) = 0$ for all $\lambda \in U$ is the zero function on U.

4.10. Corollary. *If $T \in \mathcal{Q}(\mathcal{A}(k^*), m)$ with $0 < k \le 1$, then $\ker(T-\mu) = \ker(T-\mu)^2$ if $\mu \neq 0$ and $\ker(T^m) = \ker(T^{m+1})$.*

Proof. Let $\mu \neq 0$. Since $\ker(T-\mu) \subset \ker(T-\mu)^2$ is clear, we prove $\ker(T-\mu)^2 \subset \ker(T-\mu)$. Let $x \in \ker(T-\mu)^2$. Since $(T-\mu)(T-\mu)x = (T-\mu)^2x = 0$, we have $(T-\mu)^*(T-\mu)x = 0$ by Corollary 4.1. Hence,

$$\|(T-\mu)x\|^2 = \langle (T-\mu)^*(T-\mu)x, x\rangle = 0,$$

so we have $(T-\mu)x = 0$. Hence $x \in \ker(T-\mu)$.

Let $x \in \ker(T^{m+1})$. Then

$$\||T^*|T^m x\|^2 \le \||T^k|T^{m+1}x\|^{\frac{2}{1+k}}\|T^m x\|^{\frac{2k}{k+1}} = 0.$$

Hence $|T^*|T^m x = 0$. Then

$$\|T^m x\|^2 = \langle T^*T^m x, T^{m-1}x\rangle = \langle U^*|T^*|T^m x, T^{m-1}x\rangle = 0.$$

Thus $x \in \ker(T^m)$. □

4.11. Corollary. *If $T \in \mathcal{Q}(\mathcal{A}(k^*), m)$ with $0 < k \le 1$, then T has SVEP.*

Proof. The proof is obvious from Theorem 2.39 of [1]. □

5. Tensor product for $\mathcal{Q}(\mathcal{A}(k^*), m)$

Let H and K denote Hilbert spaces. For given non zero operators $T \in L(H)$ and $S \in B(K)$, $T \otimes S$ denotes the tensor product on the product space $H \otimes K$. It is known that the normaloid property is invariant under tensor products by [14], and there exist paranormal operators T and S such that $T \otimes S$ is not paranormal by [3], and $T \otimes S$ is normal if and only if T and S are normal by [15]. These results were extended to the class $\mathcal{A}$ operators, class $A(k)$ operators, and $*$-class $\mathcal{A}$ operators by [10] [11] and [5]. In this section, we prove an analogues result for $\mathcal{Q}(\mathcal{A}(k^*), m)$ operators.

Let $T \in L(H)$ and $S \in L(K)$ be non zero operators. Then $(T \otimes S)^*(T \otimes S) = T^*T \otimes S^*S$ holds. By the uniqueness of positive square roots, we have $|T \otimes S|^r = |T|^r \otimes |S|^r$ for any positive rational number r. From the density of the rationales in the real, we obtain $|T \otimes S|^p = |T|^p \otimes |S|^p$ for any positive real number p.

5.1. Theorem. *Let $0 < k$ and m be a non-negative integer. If (1) $T, S \in \mathcal{Q}(\mathcal{A}(k^*), m)$ or (2) $T^m = 0$ or $S^m = 0$ holds, then $T \otimes S \in \mathcal{Q}(\mathcal{A}(k^*), m)$.*

Proof. By simple calculation we have:

$$
\begin{aligned}
&(T \otimes S)^{*m} \left(\left((T \otimes S)^* |(T \otimes S)|^{2k} (T \otimes S) \right)^{\frac{1}{k+1}} - |(T \otimes S)^*|^2 \right) (T \otimes S)^m \\
&= T^{*m} \left(\left(T^* |T|^{2k} T \right)^{\frac{1}{k+1}} - |T^*|^2 \right) T^m \otimes S^{*m} \left(S^* |S|^{2k} S \right)^{\frac{1}{k+1}} S^m \\
&\qquad + T^{*m} |T^*|^2 T^m \otimes S^{*m} \left(\left(S^* |S|^{2k} S \right)^{\frac{1}{k+1}} - |S^*|^2 \right) S^m.
\end{aligned}
$$

Hence, if either (1) or (2), then $T \otimes S \in \mathcal{Q}(\mathcal{A}(k^*), m)$.

□

5.2. Theorem. *Let m be a non-negative integer and $T \in L(H)$ and $S \in L(K)$ be non-zero operators. If $T \otimes S \in \mathcal{Q}(\mathcal{A}(1^*), m)$, then (1) $T, S \in \mathcal{Q}(\mathcal{A}(1^*), m)$ or (2) $T^{m+1} = 0$ or $S^{m+1} = 0$ holds.*

Proof. Suppose $T \otimes S \in \mathcal{Q}(\mathcal{A}(1^*), m)$. Then we get

$$
\begin{aligned}
&\left\langle T^{*m} \left(\left(T^* |T|^2 T \right)^{\frac{1}{2}} - |T^*|^2 \right) T^m x, x \right\rangle \left\langle S^{*m} \left(S^* |S|^2 S \right)^{\frac{1}{2}} S^m y, y \right\rangle \\
&\quad + \left\langle T^{*m} |T^*|^2 T^m x, x \right\rangle \left\langle S^{*m} \left(\left(S^* |S|^2 S \right)^{\frac{1}{2}} - |S^*|^2 \right) S^m y, y \right\rangle \ge 0
\end{aligned}
$$

for $x \in H$, $y \in K$.

Assume $T \notin \mathcal{Q}(\mathcal{A}(1^*), m)$. Then there exists $x_0 \in H$ such that:

$$\left\langle T^{*m} \left(\left(T^* |T|^2 T \right)^{\frac{1}{2}} - |T^*|^2 \right) T^m x_0, x_0 \right\rangle := \alpha < 0$$

and

$$\left\langle T^{*m} |T^*|^2 T^m x_0, x_0 \right\rangle := \beta > 0.$$

From the above relation, we have

$$(\alpha + \beta) \left\langle S^{*m} \left(S^* |S|^2 S \right)^{\frac{1}{2}} S^m y, y \right\rangle \ge \beta \left\langle S^{*m} |S^*|^2 S^m y, y \right\rangle.$$

Thus $S \in \mathcal{Q}(\mathcal{A}(1^*), m)$ because $\alpha + \beta < \beta$ and $0 < \beta$.

Since

$$\left\langle S^{*m}|S^*|^2 S^m y, y\right\rangle = \left\langle |S^*|^2 S^m y, S^m y\right\rangle = \langle S^* S^m y, S^* S^m y\rangle = \|S^* S^m y\|^2$$

and using Holder McCarthy's inequality, we get

$$\begin{aligned}\left\langle S^{*m}\left(S^*|S|^2 S\right)^{\frac{1}{2}} S^m y, y\right\rangle &= \left\langle \left(S^*|S|^2 S\right)^{\frac{1}{2}} S^m y, S^m y\right\rangle \\ &\leq \left\langle (S^*|S|^2 S) S^m y, S^m y\right\rangle^{\frac{1}{2}} \|S^m y\| \\ &= \||S| S^{m+1} y\| \|S^m y\|.\end{aligned}$$

Then

$$(\alpha + \beta)\||S|S^{m+1}y\|\|S^m y\| \geq \beta \|S^* S^m y\|^2.$$

Since $S \in \mathcal{Q}(\mathcal{A}(1^*), m)$, S has decomposition of the form

$$S = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \text{ on } H = \overline{S^m(H)} \oplus \ker(S^{*m})$$

where $A = S|_{\overline{S^m(H)}}$ is a $\mathcal{A}(1^*)$ operator by Theorem 2.6. Then we have

$$(\alpha + \beta)\|A^2 z\|\|z\| = (\alpha + \beta)\||A|Az\|\|z\| \geq \beta \|S^* z\|^2 \geq \beta \|A^* z\|^2 \ ,$$

for all $z \in \overline{S^m(H)}$. Since $A \in \mathcal{A}(1^*)$, A is normaloid by Theorem 1.1 of [4]. By taking supremum on both sides of the above inequality, we have

$$(\alpha + \beta)\|A\|^2 \geq \beta \|A^*\|^2 = \beta \|A\|^2.$$

This implies $A = 0$. Then we have

$$S^{m+1} = \begin{pmatrix} 0 & BC^m \\ 0 & C^{m+1} \end{pmatrix} = 0.$$

A similar argument shows that if $S \notin \mathcal{Q}(\mathcal{A}(1^*), m)$, then $T^{m+1} = 0$. Hence the proof is completed. $\square$

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