Hacettepe Journal of Mathematics and Statistics
Volume 47 (6) (2018), 1634 – 1651

Estimation of reliability in a multicomponent stress- strength model based on generalized linear failure rate distribution

M. KH. Hassan*† and M. I. Alohali‡

Abstract

In this paper, we consider the problem of estimation reliability in multicomponent stress-strength model, when the system consists of k-components have strength are given by independently and identically distributed random variables $X_1, ..., X_k$ each component experiencing a random stress governed by a random variable Y. The reliability such system is obtained when strength and stress variables are given by a generalized linear failure rate distribution. The system is regarded as alive only if at least s out of k $(s < k)$ strength exceed the stress. The multicomponent reliability of the system is given by $R_{s,k} = P[$ at least s of $X_1, ..., X_k$ exceed $Y]$. The maximum likelihood estimator (MLE) and Bayes estimator of $R_{s,k}$ are obtained. A simulation study is performed to compare the different estimators of $R_{s,k}$. Real data is used as a practical application of the proposed procedure.

Keywords: Generalized linear failure rate, stress-strength model, maximum likelihood method, Bayes estimator, simulation.

Mathematics Subject Classification (2010): 62N05

Received : 22/04/2016 *Accepted* : 25/11/2016 *Doi* : 10.15672/HJMS.2016.390

*Department of Mathematics, Faculty of Education, Ain Shams University, Cairo, Egypt, Department of Mathematics, College of Science, University of Dammam, Saudi Arabia Email: mkhassan@uod.edu.sa

†Corresponding Author.

‡Department of Mathematics, College of Science, University of Dammam, Saudi Arabia, Email: malohali@uod.edu.sa

1. Introduction

Sarhan and Kundu [24] introduced a three parameter generalized linear failure rate distribution (GLFRD) by exponentiating the linear failure rate distribution as was done for the exponentiated Weibull distribution by Mubholkar et al [15]. The generalized linear failure rate distribution has the following probability density function (p.d.f)

$$(1.1) \qquad f(x;a,b,\alpha) = \alpha(a+bx)e^{-(ax+\frac{b}{2}x^2)}(1-e^{-(ax+\frac{b}{2}x^2)})^{\alpha-1}, a,b,\alpha > 0, x > 0.$$

The corresponding cumulative distribution function is as follows

$$(1.2) \qquad F(x) = (1-e^{-(ax+\frac{b}{2}x^2)})^{\alpha}$$

where a, and b are scale parameters and α is the shape parameter.
The stress strength model $R = P[Y < X]$ has many applications in quality control, reliability, biostatistics and other fields see (Kotz et al [9]). The important application of $R = P[Y < X]$ in reliability analysis. In reliability context, the stress strength model describes the component which has a random strength variable X and the random stress variable Y. If the stress exceeds the strength then the system will fail. The estimation of the stress strength model when X and Y are random variables having a specified distribution discussed by many authors starting from Birnbaum [3]. Some of the recent work on the stress strength model can obtained by Kundu and Gupta [11] considered the estimator of R when stress and strength variates are generalized exponential distribution, Kundu and Gupta [12] considered the estimator of R when stress and strength variates are Weibull distribution, Raqab and and Kundu [21] considered the estimator of R when stress and strength variates are Burr type X distribution, Kundu and Raqab [13]considered the estimator of R when stress and strength variates are three parameter Weibull distribution, Krishnamoorthy et al [10] considered the estimator of R when stress and strength variates are two parameter exponential distribution, Shahsanaei and Daneshkhah [25] considered the estimator of R when stress and strength variates are linear failure rate distribution, Al-Mutairi et al [1] considered the estimator of R when stress and strength variates are Lindley distribution and Ghitany et al [7] considered the estimator of R when stress and strength variates are power Lindley distribution.
The main aim of this paper is to study multicomponent stress strength model based on generalized linear failure rate distribution. The estimation of reliability in multicomponent stress strength model $R_{s,k} = P[\text{ at least } s \text{ of } X_1, ..., X_k \text{ exceed } Y]$ discussed by many authors starting from Bhattacharyya and Jonson [2] and Pandey et al [17]. Some of the recent work on multicomponent stress strength model can be obtained by Eryilmaz [5], Pakdaman and Ahamed [16], Rao and Kantan [18], and Rao ([19], [20]).
This paper is organized as follows. In section (2), the system model $R_{s,k}$ is determined based on generalized linear failure rate distribution. In section (3), the maximum likelihood estimator (MLE) and asymptotic confidence interval of $R_{s,k}$ are obtained. In section (4), the Bayes estimator of $R_{s,k}$ by using Lindley approximation. In section (5), a simulation study is performed to compare the estimators of $R_{s,k}$. In section (6), Real data is used as a practical application of the proposed procedure. Finally, we introduce the conclusion of this paper in section (7).

2. Model description

Let X and Y be two independent random variables with generalized linear failure rate distribution with scale parameters (a_1, b_1) and (a_2, b_2) respectively, and shape parameter α, β respectively (i.e. $X \sim GLFRD(a_1, b_1, \alpha), andY \sim GLFRD(a_2, b_2, \beta)$). Suppose that $Y, X_1, ..., X_k$ are independent, $G(y)$ is the cumulative function of Y and $F(x)$ is the common cumulative function of $X_1, X_2, ..., X_k$. The reliability in multicomponent stress strength for $GLFRD$ is

$$R_{s,k} = P[\text{ at least } s \text{ the } (X_1, ..., X_k) \text{ exceed } Y]$$
$$= \sum_{i=s}^{k} \int_0^\infty \binom{k}{i} [1 - G(y)]^i [G(y)]^{k-i} dF(y)$$
$$= \sum_{i=s}^{k} \binom{k}{i} \int_0^\infty (1 - (1 - e^{-(a_2 y + \frac{b_2}{2} y^2)})^\beta)^i$$
$$(1 - e^{-(a_2 y + \frac{b_2}{2} y^2)})^{\beta(k-i)} \alpha (a_1 + b_1 y) e^{-(a_1 y + \frac{b_1}{2} y^2)} (1 - e^{-(a_1 y + \frac{b_1}{2} y^2)})^{\alpha - 1} dy$$

using the binomial theorem we get

$$R_{s,k} = \sum_{i=s}^{k} \sum_{j=0}^{i} (-)^j \binom{k}{i} \binom{i}{j} \alpha \int_0^\infty (a_1 + b_1 y) e^{-(a_1 y + \frac{b_1}{2} y^2)} (1 - e^{-(a_1 y + \frac{b_1}{2} y^2)})^{\alpha - 1} (1 - e^{-(a_2 y + \frac{b_2}{2} y^2)})^{\beta(k+j-i)} dy \tag{2.1}$$

If X and Y are two independent random variables with generalized linear failure rate with common scale parameters (i.e.$X \sim GLFRD(a, b, \alpha)$,
and $Y \sim GLFRD(a, b, \beta)$). The multicomponent stress strength is

$$R_{s,k} = \sum_{i=s}^{k} \sum_{j=0}^{i} (-)^j \binom{k}{i} \binom{i}{j} \frac{\alpha}{\alpha + \beta(k + j - i)} \tag{2.2}$$

3. The maximum likelihood estimator (MLE) of $R_{s,k}$

The main aim of this section is to obtain the maximum estimator of $R_{s,k}$ in three cases as follows

- Case (1): If $X \sim GLFRD(a_1, b_1, \alpha), and Y \sim GLFRD(a_2, b_2, \beta)$.
- Case (2): If $X \sim GLFRD(a, b, \alpha), and Y \sim GLFRD(a, b, \beta)$.
- Case (3): If $X \sim GLFRD(1, 2, \alpha), and Y \sim GLFRD(1, 2, \beta)$.

3.1. The maximum likelihood estimator of $R_{s,k}$ in case (1). Suppose that $X_1, ..., X_n$ and $Y_1, ..., Y_m$ are two random samples of size n and m respectively from GLFRD (a_1, b_1, α) and GLFRD (a_2, b_2, β) respectively. The log-likelihood function of the observed sample is

$$\begin{aligned} logL(a_1, b_1, a_2, b_2, \alpha, \beta) &= nLog(\alpha) + mLog(\beta) + \sum_{i=1}^{n} Log(a_1 + b_1 x_i) \\ &+ \sum_{j=1}^{m} Log(a_2 + b_2 y_j) + (\alpha - 1) \sum_{i=1}^{n} Log(1 - e^{-(a_1 x_i + \frac{b_1}{2} x_i^2)}) + \\ &(\beta - 1) \sum_{j=1}^{m} Log(1 - e^{-(a_2 y_j + \frac{b_2}{2} y_j^2)}) - \sum_{i=1}^{n} (a_1 x_i + \frac{b_1}{2} x_i^2) - \sum_{j=1}^{m} (a_2 y_j + \frac{b_2}{2} y_j^2) \end{aligned} \tag{3.1}$$

The maximum likelihood estimators of a_1, a_2, b_1, b_2, α and β denoted by $\hat{a_1}^{ML}$, $\hat{a_2}^{ML}$, $\hat{b_1}^{ML}$, $\hat{b_2}^{ML}$, $\hat{\alpha}^{ML}$ and $\hat{\beta}^{ML}$ respectively can be obtained as the solution of the following nonlinear equations

$$\hat{\alpha}^{ML} = \frac{-n}{\sum_{i=1}^{n} Log(1 - e^{-(a_1 x_i + \frac{b_1}{2} x_i^2)})} \tag{3.2}$$

$$\hat{\beta}^{ML} = \frac{-m}{\sum_{j=1}^{m} Log(1 - e^{-(a_2 y_j + \frac{b_2}{2} y_j^2)})} \tag{3.3}$$

$$(3.4)\qquad \frac{\partial logL}{\partial a_1}=\sum_{i=1}^{n}\frac{1}{(a_1+b_1x_i)}+(\alpha-1)\sum_{i=1}^{n}\frac{x_ie^{-(a_1x_i+\frac{b_1}{2}x_i^2)}}{(1-e^{-(a_1x_i+\frac{b_1}{2}x_i^2)})}-\sum_{i=1}^{n}x_i$$

$$(3.5)\qquad \frac{\partial logL}{\partial a_2}=\sum_{j=1}^{m}\frac{1}{(a_2+b_2y_j)}+(\beta-1)\sum_{j=1}^{m}\frac{y_je^{-(a_2y_j+\frac{b_2}{2}y_j^2)}}{(1-e^{-(a_2y_j+\frac{b_2}{2}y_j^2)})}-\sum_{j=1}^{m}y_j$$

$$(3.6)\qquad \frac{\partial logL}{\partial b_1}=\sum_{i=1}^{n}\frac{x_i}{(a_1+b_1x_i)}+\frac{(\alpha-1)}{2}\sum_{i=1}^{n}\frac{x_i^2e^{-(a_1x_i+\frac{b_1}{2}x_i^2)}}{(1-e^{-(a_1x_i+\frac{b_1}{2}x_i^2)})}-\sum_{i=1}^{n}\frac{x_i^2}{2}$$

$$(3.7)\qquad \frac{\partial logL}{\partial b_2}=\sum_{j=1}^{m}\frac{y_j}{(a_2+b_2y_j)}+(\beta-1)\sum_{j=1}^{m}\frac{y_j^2e^{-(a_2y_j+\frac{b_2}{2}y_j^2)}}{(1-e^{-(a_2y_j+\frac{b_2}{2}y_j^2)})}-\sum_{j=1}^{m}\frac{y_j^2}{2}$$

Due to the invariance property the maximum likelihood estimator of $R_{s,k}$ denoted by $\hat{R}_{s,k}^{ML}$ can obtained by replacing a_1, a_2, b_1, b_2, α and β by their maximum likelihood estimators. Therefore the maximum likelihood of $R_{s,k}$ is obtained by

$$(3.8)\qquad \begin{aligned}\hat{R}_{s,k}^{ML} &= \sum_{i=s}^{k}\sum_{j=0}^{i}(-)^j\left(\begin{array}{c}k\\i\end{array}\right)\left(\begin{array}{c}i\\j\end{array}\right)\hat{\alpha}^{ML}\int_0^\infty(\hat{a_1}^{ML}\\ &+\hat{b_1}^{ML}y)e^{-(\hat{a_1}^{ML}y+\frac{\hat{b_1}^{ML}}{2}y^2)}(1-e^{-(\hat{a_1}^{ML}y+\frac{\hat{b_1}^{ML}}{2}y^2)})^{\hat{\alpha}^{ML}-1}\\ &(1-e^{-(\hat{a_2}^{ML}y+\frac{\hat{b_2}^{ML}}{2}y^2)})^{\hat{\beta}^{ML}(k+j-i)}dy\end{aligned}$$

Since the exact distribution of $\hat{R}_{s,k}^{ML}$ does not exist, then the asymptotic distribution and confidence interval of $R_{s,k}$ is important to study. Using the delta method, the maximum likelihood estimator of $R_{s,k}$ denoted by $\hat{R}_{s,k}^{ML}$ has asymptotic normal distribution with mean $R_{s,k}$ and asymptotic variance $AV((\hat{R}_{s,k}^{ML})$, where

$$(3.9)\qquad AV(\hat{R}_{s,k}^{ML})=\sum_{i=1}^{6}\sum_{j=1}^{6}\frac{\partial R_{s,k}}{\partial\theta_i}\frac{\partial R_{s,k}}{\partial\theta_j}I^{-1}(\underline{\theta})$$

where, $\underline{\theta}=(a_1,a_2,b_1,b_2,\alpha,\beta)$ and $I(\underline{\theta})$ is the asymptotic symmetric expected Fisher information matrix. Therefore, an asymptotic $100(1-\eta)\%$ confidence interval for $R_{s,k}$can obtain as $\hat{R}_{s,k}^{ML}\mp Z_{\frac{\eta}{2}}\sqrt{AV(\hat{R}_{s,k}^{ML})}$ where $Z_{\frac{\eta}{2}}$is the upper $\frac{\eta}{2}$-quantile of standard normal distribution.

3.2. The maximum likelihood estimator of $R_{s,k}$ in case (2). Suppose that $X_1,...,X_n$ and $Y_1,...,Y_m$ are two random samples of size n and m respectively from GLFRD (a,b,α) and GLFRD (a,b,β) respectively. The log-likelihood function of the observed sample is

$$(3.10)\qquad \begin{aligned}logL(a,b,\alpha,\beta) &= nLog(\alpha)+mLog(\beta)+\sum_{i=1}^{n}Log(a+x_i)+\sum_{j=1}^{m}Log(a+by_j)\\ &+(\alpha-1)\sum_{i=1}^{n}Log(1-e^{-(ax_i+\frac{b}{2}x_i^2)})+(\beta-1)\sum_{j=1}^{m}Log(1-e^{-(ay_j+\frac{b}{2}y_j^2)})\\ &-\sum_{i=1}^{n}(ax_i+\frac{b}{2}x_i^2)-\sum_{j=1}^{m}(ay_j+\frac{b}{2}y_j^2)\end{aligned}$$

The maximum likelihood estimators of a, b, α and β denoted by $\hat{a}^{ML}$, $\hat{b}^{ML}$, $\hat{\alpha}^{ML}$ and $\hat{\beta}^{ML}$ respectively can be obtained as the solution of the following nonlinear equations

$$(3.11)\qquad \hat{\alpha}^{ML}=\frac{-n}{\sum_{i=1}^{n}Log(1-e^{-(ax_i+\frac{b}{2}x_i^2)})}$$

$$\hat{\beta}^{ML} = \frac{-m}{\sum_{j=1}^{m} Log(1 - e^{-(ay_j + \frac{b}{2}y_j^2)})} \tag{3.12}$$

$$\begin{aligned} \frac{\partial logL}{\partial a} &= \sum_{i=1}^{n} \frac{1}{(a+bx_i)} + (\alpha-1)\sum_{i=1}^{n} \frac{x_i e^{-(ax_i+\frac{b}{2}x_i^2)}}{(1-e^{-(ax_i+\frac{b}{2}x_i^2)})} - \sum_{i=1}^{n} x_i \\ &+ \sum_{j=1}^{m} \frac{1}{(a+by_j)} + (\beta-1)\sum_{j=1}^{m} \frac{y_j e^{-(ay_j+\frac{b}{2}y_j^2)}}{(1-e^{-(ay_j+\frac{b}{2}y_j^2)})} - \sum_{j=1}^{m} y_j \end{aligned} \tag{3.13}$$

$$\begin{aligned} \frac{\partial logL}{\partial b} &= \sum_{i=1}^{n} \frac{x_i}{(a+bx_i)} + \frac{(\alpha-1)}{2}\sum_{i=1}^{n} \frac{x_i^2 e^{-(ax_i+\frac{b}{2}x_i^2)}}{(1-e^{-(ax_i+\frac{b}{2}x_i^2)})} - \sum_{i=1}^{n} \frac{x_i^2}{2} \\ &\sum_{j=1}^{m} \frac{y_j}{(a+by_j)} + (\beta-1)\sum_{j=1}^{m} \frac{y_j^2 e^{-(ay_j+\frac{b}{2}y_j^2)}}{(1-e^{-(ay_j+\frac{b}{2}y_j^2)})} - \sum_{j=1}^{m} \frac{y_j^2}{2} \end{aligned} \tag{3.14}$$

Due to the invariance property the maximum likelihood estimator of $R_{s,k}$ denoted by $\hat{R}_{s,k}^{ML}$ can obtained by replacing a, b, α and β by their maximum likelihood estimators. Therefore the maximum likelihood of $R_{s,k}$ is obtained by

$$\hat{R}_{s,k}^{ML} = \sum_{i=s}^{k}\sum_{j=0}^{i}(-)^j \begin{pmatrix} k \\ i \end{pmatrix} \begin{pmatrix} i \\ j \end{pmatrix} \frac{\hat{\alpha}^{ML}}{\hat{\alpha}^{ML} + (k+j-i)\hat{\beta}^{ML}} \tag{3.15}$$

Since the exact distribution of $\hat{R}_{s,k}^{ML}$ does not exist, then the asymptotic distribution and confidence interval of $R_{s,k}$ is important to study. Using the delta method, the maximum likelihood estimator of $R_{s,k}$ denoted by $\hat{R}_{s,k}^{ML}$ has asymptotic normal distribution with mean $R_{s,k}$ and asymptotic variance $AV((\hat{R}_{s,k}^{ML})$, where

$$AV(\hat{R}_{s,k}^{ML}) = \sum_{i=1}^{4}\sum_{j=1}^{4} \frac{\partial R_{s,k}}{\partial \theta_i}\frac{\partial R_{s,k}}{\partial \theta_j} I^{-1}(\underline{\theta}) \tag{3.16}$$

where, $\underline{\theta} = (a, b, \alpha, \beta)$ and $I(\underline{\theta})$ is the asymptotic symmetric expected Fisher information matrix. Therefore, an asymptotic $100(1-\eta)\%$ confidence interval for $R_{s,k}$can obtain as $\hat{R}_{s,k}^{ML} \mp Z_{\frac{\eta}{2}}\sqrt{AV(\hat{R}_{s,k}^{ML})}$ where $Z_{\frac{\eta}{2}}$is the upper $\frac{\eta}{2}$-quantile of standard normal distribution.

3.3. The maximum likelihood estimator of $R_{s,k}$ in case (3). Suppose that $X_1, ..., X_n$ and $Y_1, ..., Y_m$ are two random samples of size n and m respectively from GLFRD $(1, 2, \alpha)$ and GLFRD $(1, 2, \beta)$ respectively. The log-likelihood function of the observed sample is

$$\begin{aligned} &logL(\alpha, \beta) = nLog(\alpha) + mLog(\beta) + \sum_{i=1}^{n} Log(1+2x_i) + \\ &\sum_{j=1}^{m} Log(1+2y_j) + (\alpha-1)\sum_{i=1}^{n} Log(1-e^{-(1x_i+x_i^2)}) + \\ &(\beta-1)\sum_{j=1}^{m} Log(1-e^{-(y_j+y_j^2)}) - \sum_{i=1}^{n}(x_i+x_i^2) - \sum_{j=1}^{m}(y_j+y_j^2) \end{aligned} \tag{3.17}$$

The maximum likelihood estimators of α and β denoted by $\hat{\alpha}^{ML}$ and $\hat{\beta}^{ML}$ respectively can be obtained as the solution of the following nonlinear equations

$$\hat{\alpha}^{ML} = \frac{-n}{\sum_{i=1}^{n} Log(1-e^{-(x_i+x_i^2)})} \tag{3.18}$$

$$\hat{\beta}^{ML} = \frac{-m}{\sum_{j=1}^{m} Log(1-e^{-(y_j+y_j^2)})} \tag{3.19}$$

Due to the invariance property the maximum likelihood estimator of $R_{s,k}$ denoted by $\hat{R}_{s,k}^{ML}$ can obtained by replacing α and β by their maximum likelihood estimators. Therefore the maximum likelihood of $R_{s,k}$ is obtained by

$$\hat{R}_{s,k}^{ML} = \sum_{i=s}^{k} \sum_{j=0}^{i} (-)^j \begin{pmatrix} k \\ i \end{pmatrix} \begin{pmatrix} i \\ j \end{pmatrix} \frac{\hat{\alpha}^{ML}}{\hat{\alpha}^{ML} + (k+j-i)\hat{\beta}^{ML}} \tag{3.20}$$

To get the distribution of $\hat{R}_{s,k}^{ML}$ in this case, using the transformation rule we find $-Log(1-e^{-(X+X^2)})$ has an exponential distribution with mean $\frac{1}{\alpha}$, hence $-2\alpha \sum_{i=1}^{n} Log(1-e^{-(X_i+X_i^2)})$ has chi-square distribution with degrees of freedom equals $2n$ similar, $-Log(1-e^{-(Y+Y^2)})$ has an exponential distribution with mean $\frac{1}{\beta}$, hence $-2\beta \sum_{j=1}^{m} Log(1-e^{-(Y_j+Y_j^2)})$ has chi-square distribution with degrees of freedom equals $2m$.
Let

$$\begin{aligned} W &= \tfrac{\hat{\alpha}^{ML}}{\hat{\alpha}^{ML}+(k+j-i)\hat{\beta}^{ML}} \\ &= \tfrac{1}{1+(k+j-i)\frac{\hat{\beta}^{ML}}{\hat{\alpha}}{}^{ML}} \\ &= \tfrac{1}{1+(k+j-i)\frac{\beta}{\alpha}\left(\frac{m\alpha \sum_{i=1}^{n} Log(1-e^{-(X_i+X_i^2)})}{n\beta \sum_{j=1}^{m} Log(1-e^{-(Y_j+Y_j^2)})}\right)} \end{aligned}$$

Using the relation between chi-square and f-distribution we get, $W = \frac{1}{1+(k+j-i)F\frac{\beta}{\alpha}}$, where F is the random variable has f-distribution with degrees of freedom equal $2n$ and $2m$. Now, we want to get the distribution of W, using transformation rule we get,

$$f(w) = \tfrac{1}{w^2\beta(n,m)}\left(\tfrac{n\alpha}{m(k+j-i)\beta}\right)^n \tfrac{\left(\frac{1-w}{w}\right)^{n-1}}{\left(1+\frac{n\alpha}{m\beta(k+j-i)}\left(\frac{1-w}{w}\right)\right)^{-(n+m)}}$$

Now to obtain the expectation of $\hat{R}_{s,k}^{ML}$,

$$\begin{aligned} E(\hat{R}_{s,k}^{ML}) &= \textstyle\sum_{i=s}^{k} \sum_{j=0}^{i} (-)^j \begin{pmatrix} k \\ i \end{pmatrix} \begin{pmatrix} i \\ j \end{pmatrix} E(W) \\ &= \textstyle\sum_{i=s}^{k} \sum_{j=0}^{i} (-)^j \begin{pmatrix} k \\ i \end{pmatrix} \begin{pmatrix} i \\ j \end{pmatrix} \frac{m}{m-1},\ m > 1 \end{aligned}$$

We can not obtain an analytical form of mean square error, so we will using numerical calculation to get it.

4. Bayes estimator of $R_{s,k}$

To find the Bayesian estimators of unknown parameters a_1, a_2, b_1, b_2, α , and β and stress-strength reliability model $R_{s,k}$which denoted by$\hat{R}_{s,k}^{B}$. We consider a non-informative and an informative gamma priors for unknown parameters a_1, a_2, b_1, b_2, α , and β see Jeffrey [8]. Let$X_1, ..., X_n$ be a random sample is distributed as $GLFRD(a_1, b_1, \alpha)$ and $Y_1, ..., Y_m$ be a random sample is distributed as as $GLFRD(a_2, b_2, \beta)$. We assume a_1, a_2, b_1, b_2, α , and β have gamma prior distributions of the following forms

$$\pi(\theta_i) = \frac{\lambda_i^{\gamma_i}}{\Gamma(\gamma_i)} \theta_i^{\gamma_i - 1} e^{-\theta_i \lambda_i}, \theta_i > 0, \lambda_i > 0, \gamma_i > 0 \tag{4.1}$$

where, $\theta_i = (\theta_1, ..., \theta_6) = (a_1, a_2, b_1, b_2, \alpha, \beta)$, $i = 1, ..., 6$.
The joint posterior distribution of unknown parameters a_1, a_2, b_1, b_2, α , and β is defined

by

$$\pi(a_1, a_2, b_1, b_2, \alpha, \beta \,\Big|\, data) = kL(a_1, a_2, b_1, b_2, \alpha, \beta \,\Big|\, data) \prod_{i=1}^{6} \pi(\theta_i)$$

where, $k = 1/\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty L(a_1, a_2, b_1, b_2, \alpha, \beta \,|data) \prod_{i=1}^{6} \pi(\theta_i) d\theta_i$,

and

$L(a_1, a_2, b_1, b_2, \alpha, \beta \,\Big|data) = \alpha^n \prod_{i=1}^{n} (a_1 + b_1 x_i)(1 - e^{-(a_1 x_i + \frac{b_1}{2} x_i^2)})^{\alpha-1}$
$e^{-\sum_{i=1}^{n}(a_1 x_i + \frac{b_1}{2} x_i^2)} \beta^m \prod_{j=1}^{m} (a_2 + b_2 y_j)(1 - e^{-(a_2 y_j + \frac{b_2}{2} y_j^2)})^{\beta-1} e^{-\sum_{j=1}^{m}(a_2 y_j + \frac{b_2}{2} y_j^2)}$

The Bayes estimator of $R_{s,k}$ under square error loss function (SELF) as

$$\hat{R}^B_{s,k} = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty R_{s,k} \pi(a_1, a_2, b_1, b_2, \alpha, \beta \,\Big|\, data) \prod_{i=1}^{6} d\theta_i \tag{4.2}$$

We have not closed form for $\hat{R}^B_{s,k}$, hence numerical computations are needed.
The Bayes estimator of $R_{s,k}$ under square error loss function (SELF) when X and Y have the same scale parameters (a, b) and different shape parameters α and β respectively is obtained as

$$\hat{R}^B_{s,k} = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty R_{s,k} \pi(a, b, \alpha, \beta \,\Big|\, data) \prod_{i=1}^{4} d\theta_i \tag{4.3}$$

The Bayes estimator of $R_{s,k}$ under square error loss function (SELF) when X and Y have the same scale parameters $(1, 2)$ and different shape parameters α and β respectively is obtained as

$$\hat{R}^B_{s,k} = \int_0^\infty \int_0^\infty R_{s,k} \pi(\alpha, \beta \,|data) d\alpha d\beta \tag{4.4}$$

where,

$$\pi(\alpha, \beta \,\Big|\, data) = \frac{(\lambda_5 - T_1)^{n+\nu_5} (\lambda_6 - T_2)^{m+\nu_6} \alpha^{n+\nu_5-1} \beta^{m+\nu_6-1}}{\Gamma(n+\nu_5)\Gamma(m+\nu_6)} e^{-\alpha(\lambda_5 - T_1) - \beta(\lambda_6 - T_2)} \tag{4.5}$$

$$T_1 = \sum_{i=1}^{n} log(1 - e^{-(x_i + x_i^2}), T_2 = \sum_{j=1}^{m} log(1 - e^{-(y_j + y_j^2})) \tag{4.6}$$

Hence from (2.2) and (4.5) in (4.4) and using binomial theorem we get the Bayes estimator as

$$\hat{R}^B_{s,k} = \sum_{l=0}^{\infty} \sum_{i=s}^{k} \sum_{j=0}^{i} (-)^j \begin{pmatrix} k \\ i \end{pmatrix} \begin{pmatrix} i \\ j \end{pmatrix} \begin{pmatrix} -1 \\ l \end{pmatrix} (k+j-i)^l (\frac{\lambda_5 - T_1}{\lambda_6 - T_2})^l \frac{\Gamma(n+\nu_5 - l)\Gamma(m+\nu_6 + l)}{\Gamma(n+\nu_5)\Gamma(m+\nu_6)} \tag{4.7}$$

We have not a closed form for $\hat{R}^B_{s,k}$, hence numerical computations are needed. Therefore we can use approximate approaches to find Bayes estimator such as Lindley approximation see Lindley [14] and Markov Chain Monte Carlo (MCMC) method for more details about MCMC and the related methodologies, one can refer to Gentle [6], Chen et al. [4] and Robert and Casella [22] but this method is not suitable in our study, so in the next subsection we present only Lindley approximation.

4.1. Lindley approximate. Lindley [14] considered that the Bayes estimator of any parametric function $f(\underline{\theta})$ under square error loss function (SELF) is defined as the ratio of integral as follows

$$f^B(\underline{\theta}) = \frac{\int_{\underline{\theta}} f(\underline{\theta}) e^{logL(\underline{\theta})+\rho(\underline{\theta})} d\underline{\theta}}{\int_{\underline{\theta}} e^{logL(\underline{\theta})+\rho(\underline{\theta})} d\underline{\theta}} \tag{4.8}$$

where $f^B(\underline{\theta})$ the Bayes estimator of parametric function $f(\underline{\theta})$,$\underline{\theta} = (\theta_1 ... \theta_p)$ and $\rho(\underline{\theta})$ is the log of joint prior of $\underline{\theta}$.
For large sample and under some conditions Lindley approximate equation (4.8) to

$$\begin{aligned} f^B(\underline{\theta}) &= \hat{f}(\underline{\theta}) + \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}(\hat{f}_{ij}(\underline{\theta}) \\ &+ 2\hat{f}_i(\underline{\theta})\hat{\rho}_j(\underline{\theta}))\sigma_{ij} + \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{l=1}^{p} L_{ijk}\hat{f}_l(\underline{\theta})\sigma_{ij}\sigma_{kl} \end{aligned} \tag{4.9}$$

where $\hat{()}$ means we replace $\underline{\theta}$ by its maximum estimator, $\hat{f}_{ij}(\underline{\theta}) = \frac{\partial^2 f(\hat{\underline{\theta}})}{\partial\theta_i \partial\theta_j}$,
$\hat{\rho}_i(\underline{\theta}) = \frac{\partial\rho(\hat{\underline{\theta}})}{\partial\theta_i}$, $\hat{L_{ijk}} = \frac{\partial^3 logL(\underline{\theta})}{\partial\theta_i\partial\theta_j\partial\theta_k}$ and σ_{ij} elements of inverse Fisher information matrix. In our study we replace the parametric function $f(\underline{\theta})$ by $R_{s,k}$ in (4.8)and (4.9). In (4.9) the value of p as

- Case (1): $p = 6$, if $X \sim GLFRD(a_1, b_1, \alpha), andY \sim GLFRD(a_2, b_2, \beta)$.
- Case (2): $p = 4$, if $X \sim GLFRD(a, b, \alpha), andY \sim GLFRD(a, b, \beta)$.
- Case (3): $p = 2$, if $X \sim GLFRD(1, 2, \alpha), andY \sim GLFRD(1, 2, \beta)$.

5. Simulation study

In this section, we present some results based on Monte Carlo simulation method to compare the performance of different estimators of $R_{s,k}$. First, we perform the simulation study when the scale parameters of random variables X and Y are known and equal (case 3). Second,we perform the simulation study when the scale parameters of random variables X and Y are unknown and equal (case 2) also the scale parameters are unknown and not equal (case 1).

5.1. Maximum likelihood estimator of $R_{s,k}$ when the scale parameters of random variables X and Y are known and equal (case 3).

In this subsection, we find and study $R_{s,k}$ when $(s, k) = (1, 3)$ and $(2, 4)$ respectively, also we perform our simulation study when $X \sim GLFRD(1, 2, \alpha), andY \sim GLFRD(1, 2, \beta)$. Now to study the behavior of $\hat{R}^{ML}_{s,k}$ we use the following algorithm.

Algorithm

(1) For given values of $(\alpha, \beta) = (1, 2), (1, 1.5), (1, 0.5)$ compute $R_{1,3}$ and $R_{2,4}$ from (2.2).
(2) Using Monte Carlo simulation method to generate 1000 random sample of sizes 5(5)(20) from (1.1).
(3) Compute $\hat{\alpha}^{ML}$ and $\hat{\beta}^{ML}$ from (3.18) and (3.19) respectively.
(4) Compute $\hat{R}^{ML}_{1,3}$ and $\hat{R}^{ML}_{2,4}$ from (3.20).
(5) Compute the average bias, average mean square and asymptotic 95% confidence interval of $R_{1,3}$ and $R_{2,4}$. Where, $Bias = \frac{1}{N}\sum_{i=1}^{N}(\hat{R}^{ML}_{s,k} - R_{s,k})$, $(s, k) =$

(1,3) and (2,4), $MSE = \frac{1}{N}\sum_{i=1}^{N}(\hat{R}_{s,k}^{ML} - R_{s,k})^2$,$(s,k) = (1,3)$ and $(2,4)$, asymptotic 95% confidence interval of $R_{s,k}$ (ACI) is given by

$$\hat{R}_{s,k}^{ML} \mp Z_{0.95}\sqrt{AV(\hat{R}_{s,k}^{ML})}$$

where,

$$AV(\hat{R}_{s,k}^{ML}) = V(\hat{\alpha})(\frac{\partial R_{s,k}}{\partial \alpha})^2 + V(\hat{\beta})(\frac{\partial R_{s,k}}{\partial \beta})^2,$$

$$V(\hat{\alpha}) = (E[-\frac{logL(\alpha,\beta)}{\partial\alpha\partial\alpha}])^{-1} = \frac{\alpha^2}{n}, \qquad V(\hat{\beta}) = (E[-\frac{logL(\beta,\beta)}{\partial\alpha\partial\beta}])^{-1} = \frac{\beta^2}{m}$$

$\frac{\partial R_{1,3}}{\partial\alpha} = \frac{-3\beta}{(\alpha+3\beta)^2}, \qquad \frac{\partial R_{1,3}}{\partial\beta} = \frac{3\alpha}{(\alpha+3\beta)^2},$

$\frac{\partial R_{2,4}}{\partial\alpha} = \frac{-12\beta^2(2\alpha+7\beta)}{(\alpha+3\beta)^2(\alpha+4\beta)^2}$ and $\frac{\partial R_{2,4}}{\partial\beta} = 12\alpha(\frac{1}{(\alpha+3\beta)^2-(\alpha+4\beta)^2})$.

5.2. Bayes estimator of $R_{s,k}$ when the scale parameters of random variables X and Y are known and equal (case 3). In this subsection, we use Lindley approximation to find Bayes estimator of $R_{s,k}$ when $X \sim GLFRD(1,2,\alpha), andY \sim GLFRD(1,2,\beta)$. Note that to avoid the difficulty of computations, we find and study $R_{s,k}$ when $(s,k) = (1,3)$ and $(2,4)$. Now to find $\hat{R}_{s,k}^{B}$ we use the following algorithm.

Algorithm

(1) Find $\hat{\alpha}^{ML}$, $\hat{\beta}^{ML}$ and $\hat{R_{s,k}}^{ML}$.

(2) Find $\hat{R_{s,b}}^{B} = \hat{R_{s,k}}^{ML} + \frac{1}{2}[\sigma_{11}(R_{11}+2R_1\rho_1)+\sigma_{22}(R_{22}+2R_2\rho_2)] + \frac{1}{2}\,[L_{111}R_1\sigma_{11}^2 + L_{222}R_2\sigma_{22}^2]$

where,

$(s,k) = (1,3)$ and $(2,4)$, $\sigma_{11} = \frac{(\hat{\alpha}^{ML})^2}{n}$, $\sigma_{22} = \frac{(\hat{\beta}^{ML})^2}{m}$,

$R_1 = \frac{\partial R_{s,k}}{\partial\alpha} = \sum_{i=s}^{k}\sum_{j=0}^{i}(-)^j \binom{k}{i}\binom{i}{j}\frac{(k+j-i)\beta}{(\alpha+(k+j-i)\beta)^2},$

$R_2 = \frac{\partial R_{s,k}}{\partial\beta} = \sum_{i=s}^{k}\sum_{j=0}^{i}(-)^j \binom{k}{i}\binom{i}{j}\frac{-(k+j-i)\alpha}{(\alpha+(k+j-i)\beta)^2},$

$R_{11} = \frac{\partial^2 R_{s,k}}{\partial\alpha\partial\alpha} = \sum_{i=s}^{k}\sum_{j=0}^{i}(-)^j \binom{k}{i}\binom{i}{j}\frac{-2(k+j-i)\beta}{(\alpha+(k+j-i)\beta)^3},$

$R_{22} = \frac{\partial^2 R_{s,k}}{\partial\beta\partial\beta} = \sum_{i=s}^{k}\sum_{j=0}^{i}(-)^j \binom{k}{i}\binom{i}{j}\frac{2(k+j-i)\alpha}{(\alpha+(k+j-i)\beta)^3},$

$L_{111} = \frac{\partial^3 LogL}{\partial\alpha^3} = \frac{2n}{\alpha^3}$, $L_{222} = \frac{\partial^3 LogL}{\partial\beta^3} = \frac{2m}{\beta^3}$,

$\sigma_{11}^2 = (\frac{(\hat{\alpha}^{ML})^2}{n})^2$, $\sigma_{22}^2 = (\frac{(\hat{\beta}^{ML})^2}{m})^2$

$\frac{\partial\rho}{\partial\alpha} = -\frac{(2+4\beta+\beta^2)(2+6\alpha+6\alpha^2+\alpha^3)}{\alpha^2(1+\alpha)^2\beta(1+\beta)\sqrt{(2+4\alpha+\alpha^2)(2+4\beta+\beta^2)}}$,

$\frac{\partial\rho}{\partial\beta} = -\frac{(2+4\alpha+\alpha^2)(2+6\beta+6\beta^2+\beta^3)}{\beta^2(1+\beta)^2\alpha(1+\alpha)\sqrt{(2+4\beta+\beta^2)(2+4\alpha+\alpha^2)}}$

(3) Repeat 2 $N = 10^4$ times.

(4) Compute $Bias = \frac{1}{N}\sum_{i=1}^{N}(\hat{R}_{s,k}^{B} - R_{s,k})$, $MSE = \frac{1}{N}\sum_{i=1}^{N}(\hat{R}_{s,k}^{B} - R_{s,k})^2$. Note we use sample of sizes (5)(5)(20) from equation (1.1).

5.3. Maximum likelihood estimator of $R_{s,k}$ when the scale parameters of random variables X and Y are unknown and equal (case 2) also the scale parameters are unknown and not equal (case 1).

In this subsection, we find and study $R_{s,k}$ when $(s,k) = (1,3)$ and $(2,4)$ respectively, also we perform our simulation study when $X \sim GLFRD(a_1,b_1,\alpha), andY \sim$

$GLFRD(a_2, b_2, \beta)$. Now to study the behavior of $\hat{R}_{s,k}^{ML}$ we use the following algorithm.

Algorithm

(1) For given values of $(a_1, b_1, a_2, b_2, \alpha, \beta)$ compute $R_{1,3}$ and $R_{2,4}$ from (2.1) or(2.2).
(2) Using Monte Carlo simulation method to generate 1000 random sample of sizes (15),(25)(50) from (1.1).
(3) Compute $\hat{a_1}^{ML}$, $\hat{a_2}^{ML}$, $\hat{b_1}^{ML}$, $\hat{b_2}^{ML}$, $\hat{\alpha}^{ML}$ and $\hat{\beta}^{ML}$ from subsection (3.1) or (3.2).
(4) Compute $\hat{R}_{1,3}^{ML}$ and $\hat{R}_{2,4}^{ML}$ from (3.10) or (3.15).
(5) Compute the average bias, average mean square and asymptotic 95% confidence interval of $R_{1,3}$ and $R_{2,4}$. Where, $Bias = \frac{1}{N}\sum_{i=1}^{N}(\hat{R}_{s,k}^{ML} - R_{s,k})$, $(s, k) = (1, 3)$ and $(2, 4)$, $MSE = \frac{1}{N}\sum_{i=1}^{N}(\hat{R}_{s,k}^{ML} - R_{s,k})^2$,
$(s, k) = (1, 3)$ and $(2, 4)$.

5.4. Bayes estimator of $R_{s,k}$ when the scale parameters of random variables X and Y are unknown and equal (case 2) also the scale parameters are unknown and not equal (case 1).

In this subsection, we use Lindley approximation to find Bayes estimator of $R_{s,k}$ when $X \sim GLFRD(a_1, b_1, \alpha)$, and $Y \sim GLFRD(a_2, b_2, \beta)$. Note that to avoid the difficulty of computations, we find and study $R_{s,k}$ when $(s, k) = (1, 3)$ and $(2, 4)$. Now to find $\hat{R}_{s,k}^{B}$ we use the following algorithm.

Algorithm

(1) Find $\hat{a_1}^{ML}$, $\hat{a_2}^{ML}$, $\hat{b_1}^{ML}$, $\hat{b_2}^{ML}$, $\hat{\alpha}^{ML}$, $\hat{\beta}^{ML}$ and $\hat{R_{s,k}}^{ML}$.
(2) Find $\hat{R_{s,b}}^{B}$ from (4.9).
(3) Repeat 2 $N = 10^4$ times.
(4) Compute $Bias = \frac{1}{N}\sum_{i=1}^{N}(\hat{R}_{s,k}^{B} - R_{s,k})$, $MSE = \frac{1}{N}\sum_{i=1}^{N}(\hat{R}_{s,k}^{B} - R_{s,k})^2$. Note we use sample of sizes $(15), (25), (50)$ from equation (1.1).

Table 1: Estimate of $R_{s,k}$ using $X \sim GLFRD(1,2,1)$, and $Y \sim GLFRD(1,2,0.5)$

(s,k)	$R_{s,k}$	(n,m)	$\hat{R}^{ML}_{s,k}$	$\hat{R}^{B}_{s,k}$	$95\%ACI$
(1,3)	0.6	(5,5)	0.6088	0.6038	(0.313571,0.904029)
		(5,10)	0.6829	0.7225	(0.450434,0.915366)
		(5,15)	0.5706	0.7057	(0.322616,0.818584)
		(5,20)	0.5031	0.7204	(0.258110,0.748090)
		(10,10)	0.6626	0.6564	(0.450043,0.875157)
		(10,15)	0.5479	0.5797	(0.0.34969,0.74610)
		(10,20)	0.4801	0.5484	(0.290623,0.669577)
		(15,15)	0.4111	0.4161	(0.248855,0.733450)
		(15,20)	0.3473	0.3637	(0.206791,0.878090)
		(20,20)	0.4390	0.4436	(0.284055,0.939450)
(2,4)	0.4	(5,5)	0.4108	0.4220	(0.115571,0.706029)
		(5,10)	0.5065	0.5700	(0.274034,0.738966)
		(5,15)	0.3647	0.5364	(0.116716,0.612684)
		(5,20)	0.2891	0.5339	(0.044110,0.534090)
		(10,10)	0.4794	0.4786	(0.266843,0.691957)
		(10,15)	0.3384	0.3830	(0.140195,0.536605)
		(10,20)	0.2649	0.3431	(0.075423,0.454377)
		(15,15)	0.1982	0.2098	(0.035954,0.604450)
		(15,20)	0.1441	0.1628	(0.003591,0.646090)
		(20,20)	0.2241	0.2339	(0.069154,0.790450)

Table 2: Estimate of $R_{s,k}$ using $X \sim GLFRD(1,2,1)$, and $Y \sim GLFRD(1,2,1.5)$

(s,k)	$R_{s,k}$	(n,m)	$\hat{R}^{ML}_{s,k}$	$\hat{R}^{B}_{s,k}$	$95\%ACI$
(1,3)	0.8182	(5,5)	0.8379	0.8189	(0.680556,0.995244)
		(5,10)	0.8381	0.8323	(0.701836,0.974364)
		(5,15)	0.7962	0.8049	(0.667729,0.924671)
		(5,20)	0.7398	0.7749	(0.615409,0.864191)
		(10,10)	0.8252	0.8156	(0.713941,0.936459)
		(10,15)	0.7808	0.7776	(0.679235,0.882365)
		(10,20)	0.7216	0.7293	(0.625247,0.879530)
		(15,15)	0.6724	0.6675	(0.581558,0.963242)
		(15,20)	0.5990	0.5989	(0.520328,0.977672)
		(20,20)	0.6872	0.6834	(0.608528,0.965872)
(2,4)	0.7013	(5,5)	0.7317	0.7069	(0.574356,0.889044)
		(5,10)	0.7321	0.7265	(0.595836,0.868364)
		(5,15)	0.6679	0.6856	(0.539429,0.796371)
		(5,20)	0.5853	0.6418	(0.622800,0.801300)
		(10,10)	0.7121	0.7000	(0.460909,0.709691)
		(10,15)	0.6450	0.6435	(0.600841,0.823359)
		(10,20)	0.5597	0.5747	(0.543435,0.746565)
		(15,15)	0.4925	0.4906	(0.463347,0.756053)
		(15,20)	0.3988	0.4038	(0.320128,0.777472)
		(20,20)	0.5123	0.5104	(0.433628,0.790972)

Table 3: Estimate of $R_{s,k}$ using $X \sim GLFRD(1,2,1),$ and $Y \sim GLFRD(1,2,3)$

(s,k)	$R_{s,k}$	(n,m)	$\hat{R}_{s,k}^{ML}$	$\hat{R}_{s,k}^{B}$	$95\% ACI$
(1,3)	0.9	(5,5)	0.8797	0.8632	(0.748513,1,000000)
		(5,10)	0.9383	0.9297	(0.824689,1,000000)
		(5,15)	0.9305	0.9221	(0.823386,1,000000)
		(5,20)	0.8878	0.8801	(0.784088,0.991512)
		(10,10)	0.9328	0.9273	(0.840037,1,000000)
		(10,15)	0.9243	0.9190	(0.839619,1,000000)
		(10,20)	0.8783	0.8729	(0.797965,0.958635)
		(15,15)	0.8756	0.8702	(0.799859,0.951341)
		(15,20)	0.8062	0.8002	(0.740607,0.971793)
		(20,20)	0.8595	0.8551	(0.793907,0.925093)
(2,4)	0.8307	(5,5)	0.7978	0.7741	(0.666613,0.928987)
		(5,10)	0.8942	0.8805	(0.780589,1,000000)
		(5,15)	0.8812	0.8678	(0.774086,0.988314)
		(5,20)	0.81106	0.8015	(0.707348,0.914772)
		(10,10)	0.8849	0.8763	(0.792137,0.977663)
		(10,15)	0.8709	0.8625	(0.786219,0.955581)
		(10,20)	0.7957	0.7880	(0.715365,0.876035)
		(15,15)	0.7913	0.7836	(0.715559,0.867041)
		(15,20)	0.8062	0.6759	(0.617507,0.848693)
		(20,20)	0.7656	0.7596	(0.700007,0.831193)

Table 4: Bias and MSE using $X \sim GLFRD(1,2,1),$ and $Y \sim GLFRD(1,2,0.5)$

		MLE		Bayes	
(s,k)	(n,m)	Bias	MSE	Bias	MSE
(1,3)	(5,5)	0.0088	0.0000	0.0038	0.000000
	(5,10)	0.0829	0.0068	0.1225	0.015006
	(5,15)	-0.0294	0.0008	0.1057	0.011170
	(5,20)	-0.0969	0.0093	0.1204	0.014490
	(10,10)	0.0626	0.0039	0.0564	0.003100
	(10,15)	-0.0521	0.0027	-0.0203	0.000400
	(10,20)	-0.1199	0.0143	-0.0516	0.002660
	(15,15)	-0.1889	0.0354	-0.1839	0.033800
	(15,20)	0.0626	0.0039	-0.2363	0.055800
	(20,20)	-0.1610	0.0259	-0.1564	0.024400
(2,4)	(5,5)	0.0108	0.0001	0.0220	0.000400
	(5,10)	0.1065	0.0113	0.1700	0.028900
	(5,15)	-0.0353	0.0012	0.1364	0.018600
	(5,20)	-0.1109	0.0122	0.1339	0.017920
	(10,10)	0.0794	0.0063	0.0786	0.006100
	(10,15)	-0.0616	0.0037	-0.0170	0.000200
	(10,20)	-0.1351	0.0182	-0.0569	0.003200
	(15,15)	-0.2018	0.0407	-0.1902	0.036100
	(15,20)	-0.2559	0.0654	-0.2372	0.056260
	(20,20)	-0.1759	0.0309	-0.1661	0.027500

Table 5: Bias and MSE using $X \sim GLFRD(1,2,1),$ and $Y \sim GLFRD(1,2,1.5)$

		MLE		Bayes	
(s,k)	(n,m)	Bias	MSE	Bias	MSE
(1,3)	(5,5)	0.0197	0.0003	0.0007	0.0000
	(5,10)	0.0199	0.0003	0.0141	0.0001
	(5,15)	-0.0220	0.0004	-0.0133	0.0001
	(5,20)	-0.0784	0.0061	-0.0433	0.0018
	(10,10)	0.0007	0.0000	-0.0026	0.0000
	(10,15)	-0.0374	0.0013	-0.0406	0.0016
	(10,20)	-0.0966	0.00933	-0.0889	0.0079
	(15,15)	-0.1458	0.0212	-0.1507	0.0227
	(15,20)	-0.2192	0.0480	-0.2193	0.0481
	(20,20)	-0.1310	0.0171	-0.1348	0.0181
(2,4)	(5,5)	0.0304	0.0009	0.0056	0.0000
	(5,10)	0.0309	0.0009	0.0252	0.0006
	(5,15)	-0.0333	0.0011	-0.0157	0.0002
	(5,20)	-0.1159	0.0134	-0.0595	0.0035
	(10,10)	0.0108	0.0001	-0.0013	0.0001
	(10,15)	-0.0562	0.0031	-0.0578	0.0033
	(10,20)	-0.1415	0.0200	-0.1266	0.0160
	(15,15)	-0.2088	0.0435	-0.2107	0.0443
	(15,20)	-0.3024	0.0914	-0.2975	0.0885
	(20,20)	-0.1890	0.0357	-0.1909	0.0364

Table 6: Bias and MSE using $X \sim GLFRD(1,2,1),$ and $Y \sim GLFRD(1,2,3)$

		MLE		Bayes	
(s,k)	(n,m)	Bias	MSE	Bias	MSE
(1,3)	(5,5)	-0.0203	0.0004	-0.0368	0.0013
	(5,10)	0.0383	0.0014	0.0297	0.0008
	(5,15)	0.0305	0.0009	0.0221	0.0004
	(5,20)	-0.0122	0.0001	-0.0199	0.0003
	(10,10)	0.0328	0.0010	0.0273	0.0007
	(10,15)	0.0243	0.0005	0.019	0.0003
	(10,20)	-0.0217	0.0004	-0.0271	0.0007
	(15,15)	-0.0244	0.0005	-0.0298	0.0008
	(15,20)	-0.0938	0.0087	-0.0998	0.0099
	(20,20)	-0.0405	0.0016	-0.0449	0.0020
(2,4)	(5,5)	-0.0329	0.0010	-0.0566	0.0032
	(5,10)	0.0635	0.0040	0.0498	0.0024
	(5,15)	0.0505	0.0025	0.0371	0.0013
	(5,20)	-0.0196	0.0003	-0.0292	0.0008
	(10,10)	0.0542	0.0029	0.0456	0.0021
	(10,15)	0.0402	0.0016	0.0318	0.0010
	(10,20)	-0.0350	0.0012	-0.0427	0.0018
	(15,15)	-0.0394	0.0015	-0.0471	0.0022
	(15,20)	-0.1476	0.0217	-0.1548	0.0239
	(20,20)	-0.0651	0.0042	-0.0711	0.0050

Table 7: Estimate of $R_{s,k}$ using $X \sim GLFRD(a,b,\alpha),$ and $Y \sim GLFRD(a,b,\beta)$

(s,k)	$R_{s,k}$	(n,m)	$\hat{R}^{ML}_{s,k}$	$\hat{R}^{B}_{s,k}$	$95\%ACI$
(1,3)	0.75	(15,15)	0.7475	0.7412	(0.61240,0.82582)
		(25,25)	0.7520	0.7538	(0.64860,0.85530)
		(50,50)	0.7533	0.7514	(0.68040,0.82610)
		(15,25)	0.7471	0.7469	(0.62617,0.86800)
		(15,50)	0.7389	0.7393	(0.64620,0.83150)
(2,4)	0.6	(15,15)	0.5964	0.5907	(0.46130,0.73140)
		(25,25)	0.6030	0.6073	(0.49960,0.70630)
		(50,50)	0.6048	0.6030	(0.55760,0.70330)
		(15,25)	0.5959	0.5982	(0.47490,0.71680)
		(15,50)	0.5842	0.5874	(0.49150,0.67680)

Table 8: Bias and MSE using $X \sim GLFRD(a,b,\alpha),$ and $Y \sim GLFRD(a,b,\beta)$

		MLE		Bayes	
(s,k)	(n,m)	Bias	MSE	Bias	MSE
(1,3)	(15,15)	-0.0025	0.0000	0.0088	0.0000
	(25,25)	0.0020	0.0000	0.0038	0.0150
	(50,50)	-0.0033	0.0000	0.0000	0.0000
	(15,25)	-0.0029	0.0000	-0.0031	0.0000
	(15,50)	-0.0114	0.0001	-0.0107	0.0001
(2,4)	(15,15)	-0.0036	0.0000	-0.0093	0.0000
	(25,25)	0.0030	0.0000	0.0073	0.0000
	(50,50)	0.0048	0.0000	0.0030	0.0000
	(15,25)	-0.0041	0.0000	0.0018	0.0000
	(15,50)	-0.0158	0.0002	-0.0126	0.0001

Table 9: Estimate of $R_{s,k}$ using $X \sim GLFRD(a_1,b_1,\alpha),$ and $Y \sim GLFRD(a_2,b_2,\beta)$

(s,k)	$R_{s,k}$	(n,m)	$\hat{R}^{ML}_{s,k}$	$\hat{R}^{B}_{s,k}$	$95\%ACI$
(1,3)	0.9230	(15,15)	0.9329	0.9290	(0.80239,1.06340)
		(25,25)	0.9280	0.9255	(0.82000,1.03590)
		(50,50)	0.9253	0.9240	(0.84638,1.00420)
		(15,50)	0.9258	0.9241	(0.78740,1.06740)
		(25,50)	0.9205	0.9213	(0.80480,1.03770)
(2,4)	0.8687	(15,15)	0.8852	0.8790	(0.75460,1.01570)
		(25,25)	0.8769	0.8730	(0.76890,0.98480)
		(50,50)	0.8726	0.8705	(0.79360,0.95150)
		(15,25)	0.8733	0.8708	(0.73160,1.01490)
		(25,50)	0.8646	0.8661	(0.78160,0.98100)

Table 10: Bias and MSE using $X \sim GLFRD(a_1, b_1, \alpha)$, and $Y \sim GLFRD(a_2, b_2, \beta)$

		MLE		Bayes	
(s,k)	(n,m)	Bias	MSE	Bias	MSE
(1,3)	(15,15)	0.0099	0.0000	0.0060	0.0000
	(25,25)	0.0050	0.0000	0.0002	0.0000
	(50,50)	0.0023	0.0000	0.0010	0.0000
	(15,25)	-0.0028	0.0000	0.0011	0.0000
	(25,50)	-0.0025	0.0000	-0.0017	0.0000
(2,4)	(15,15)	0.0165	0.0002	0.0103	0.0000
	(25,25)	0.0082	0.0000	0.0043	0.0000
	(50,50)	0.0039	0.0000	0.0018	0.0000
	(15,25)	0.0046	0.0000	0.0021	0.0000
	(25,50)	-0.0041	0.0000	-0.0026	0.0000

Table (1)-(6),case (3) is discussed, the bias decreases when n increases, the best estimator of $R_{s,k}$ when $n = 5$ because MSE= 0, the bias of Bayes estimator is smaller than the bias of maximum likelihood estimator in all cases, the smallest bias when $X \sim GLFRD(1, 2, 1), andY \sim GLFRD(1, 2, 3)$, so we can decide the Bayes estimator is the best estimator in our study. Table (7)-(8)case (2) is discussed, we use large samples and observe the performance of Bayes estimator and maximum likelihood estimator is similar.Table (9)-(10)case (1) is discussed, we use large samples and observe the performance of Bayes estimator and maximum likelihood estimator is similar.

6. Data analysis

In this section, we introduce two real data sets of the breaking strengths of jute fiber at two different gauge lengths to sure our proposed method can apply in practice. Two sets of real data shown as follows:

Data set I: Breaking strength of jute fiber length 10 mm (variable X)

693.73 − 704.66 − 323.83 − 778.17 − 123.06 − 637.66 − 383.43 − 151.48 − 108.94 − 50.16
671.49 − 183.16 − 257.44 − 727.23 − 291.27 − 101.15 − 376.42 − 163.40 − 141.38 − 700.74−
262.90 − 353.24 − 422.11 − 43.93 − 590.48 − 212.13 − 303.90 − 506.60 − 530.55 − 177.25

Data set II: Breaking strength of jute fiber length 20 mm (variable Y)

71.46 − 419.02 − 284.64 − 585.57 − 456.60 − 113.85 − 187.85 − 688.16 − 662.66 − 45.58−
578.62 − 756.70 − 594.29 − 166.49 − 99.72 − 707.36 − 765.14 − 187.13 − 145.96 − 350.70−
547.44 − 116.99 − 375.81 − 581.60 − 119.86 − 48.01 − 200.16 − 36.75 − 244.53 − 83.55

This data were first used by Xia et al [26] and were then re-used by Saracoglu et al [23]. Also used by Shahsanaei and Daneshkhah [25] to study the estimation of stress strength parameter for GLFRD under progressive type-II censoring and studied the validity of GLFRD for both data sets. Now, we used both data sets to compute $\hat{R}_{1,3}^{ML}$, $\hat{R}_{2,4}^{ML}$ and the asymptotic confidence intervals in proposed three cases. The result shown in Table (11). Also Figures (1), (2) and (3) show the validity of GLFRD for both data sets.

Table 11: Data analysis results (case 3)

Scale parameters	(s,k)	$(\hat{a_1}^{ML},\hat{b_1}^{ML},\hat{a_2}^{ML},\hat{b_2}^{ML},\hat{\alpha}^{ML},\ \hat{\beta}^{ML})$	$\hat{R}_{s,k}^{ML}$	$95\%ACI$
known	(1,3)	(1,2,1,2,6,9.53)	0.8265	(0.7539,0.8990)
	(2,4)		0.7131	(0.3777,0.9999)
unknown	(1,3)	(5.0575,5.1680,5.2315,5.3458,4.2884,4.4359)	0.7562	(0.5277,0.9840)
	(2,4)		0.6090	(0.4111,0.8068)

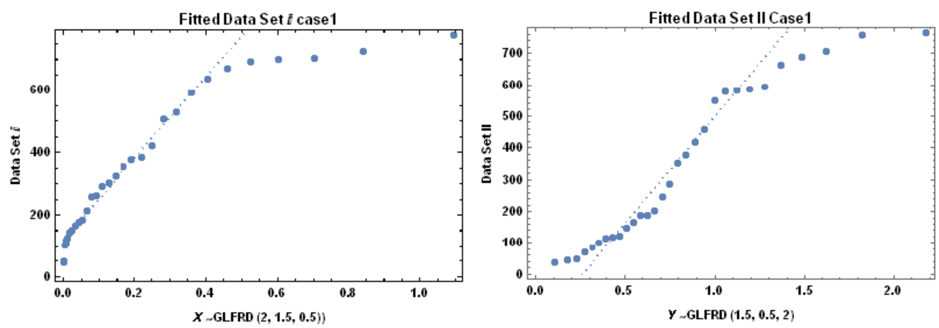

Figure 1: Fitted Generalized Linear Failure Rate Distribution for Data Set I and Data Set II (Case 1).

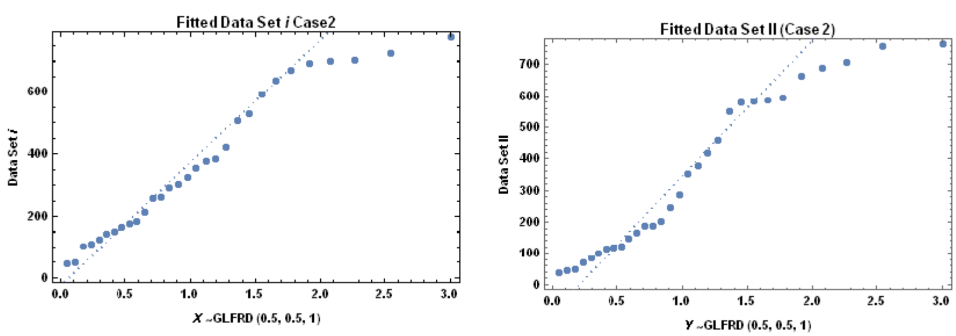

Figure 2: Fitted Generalized Linear Failure Rate Distribution for Data Set I and Data Set II (Case 2)

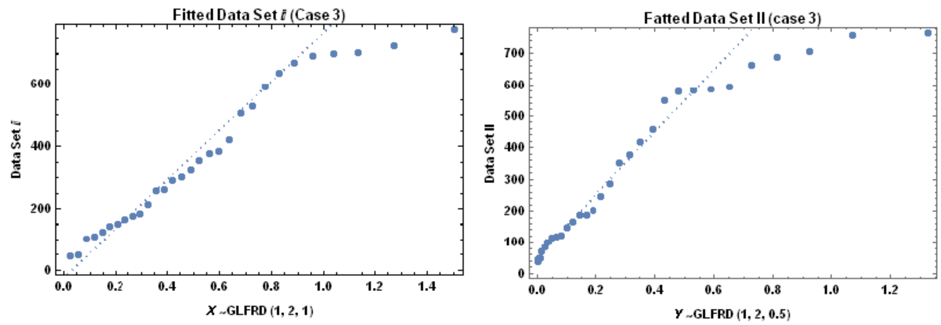

Figure 3: Fitted Generalized Linear Failure Rate Distribution for Data Set I and Data Set II (Case 3)

7. Conclusions

In this paper, we study the multicomponent stress-strength reliability for GLFRD. We obtain different estimators as MLE and Bayes estimator. We find the MLE is simplest in computation and Bayes estimator computes using Lindley approximation. Finally, we find $\hat{R}^{B}_{s,k}$ is better than $\hat{R}^{ML}_{s,k}$.

Acknowledgements The author thanks very much the referee for his (her) comments and corrections.

References

[1] Al-Mutairi, D. K., Ghitany, M. E. and Kundu, D. *Inferences on stress-strength reliability from Lindley distributions*, Commun. Statist. Theory Methods **42(8)** , 1443-1463, 2013.

[2] Bhattacharyya, G. K. and Johnson, R. A. *Estimation of reliability in a multicomponent stress-strength model*, J. Amer. Statist. Assoc., **69**, 966-970, 1974.

[3] Birnbaum, Z.W. *On a use of the Mann-Whitney statistic. In: Proceedings of Third Berkeley Symposium on Mathematical Statistics and Probability*, **1**, 13-17, Universityof California Press, Berkeley, CA., 1956

[4] Chen, M.H., Shao, Q.M., Ibrahim and J.G. *Monte Carlo methods in Bayesian computation*, Springer-Verlag, New York, 2000 .

[5] Eryilmaz, S. *Multivariate stress-strength reliability model and its evaluation for coherent structures*, J. Multivariate Anal., **99**, 1878-1887, 2008.

[6] Gentle, JE. *Random number generation and Monte Carlo methods*, Springer, New York, 1998.

[7] Ghitany, M. E., Al-Mutairi, D. K. and Aboukhamseen, S. M. *Estimation of the reliability of a stress-strength system from power Lindley distributions*, Commun. Statist. Simul. Comput., **44** ,118-136, 2015.

[8] Jeffrey, H. *Theory of probability*, 3rd ed., Oxford University Press, 1961.

[9] Kotz, S., Lumelskii, Y., and Pensky, M. *The stress-strength model and its generalization: theory and applications*, World Scientific, Singapore, 2003.

[10] Krishnamoorthy, K., Mukherjee, S. and Guo, H. *Inference on reliability in two-parameter exponential stress-strength model*, Metrika, **6** ,261-273, 2007.

[11] Kundu, D. and Gupta, R. D. *Estimation of $P(X < Y)$ for the generalized exponential distribution*, Metrika, **61**, 291-308, 2005.

[12] Kundu, D. and Gupta, R. D. *Estimation of p(Y<X) for Weibull distribution*, IEEE Trans. Rel., **55(2)**, 270-280, 2006.

[13] Kundu, D. and Raqab, M. Z. *Estimation of* $R = P(Y < X)$ *for three-parameter Weibull distribution*, Statist. Probability Lett.,**79**, 1839-1846, 2009.

[14] Lindley, D. V. *Approximation Bayesian methods, Trabajos de Estadistica*,**21**, 223-237, 1980.

[15] Mudholkar, G.S., Srivastava, D.K. and Freimer, M. *The exponentiated Weibull family; a reanalysis of the bus motor failure data*, Technometrics, **37**, 436 - 445, 1995.

[16] Pakdaman, Z. and Ahmadi, J. *Stress- Strength reliability for*$P[X_{r:n_1,k:n_2}]$ *in exponential case, Journal of the Turkish statistical association*, **6(3)**, 92-102, 2013.

[17] Pandey, M., Uddin, M. B. and Ferdous, J. *Reliability estimation of an s-out-of-k system with non-identical component strengths: the Weibull case*, Reliab. Eng. Sys. Saf., **36**,109-116.

[18] Rao, G. S. a and Kantam,R. R. L. *Estimation of reliability in multicomponent stress-strength model: Log-logistic distribution*, Electron. J. Appl. Statist. Anal, **3(2)**, 75-84, 2010.

[19] Rao, G. S. *Estimation of reliability in multicomponent stress-strength model based on generalized exponential distribution*, Colombian J.Statist., **35(1)**, 67-76, 2012.

[20] Rao, G. S. and Kantam,R. R. L., Rosaiah, K. and Reddy, J. P. *Estimation of reliability in multicomponent stress-strength model based on inverse Rayleigh distribution*, J. Statist. Appl. Probability, **3**, 261-267, 2013.

[21] Raqab, M. Z. and Kundu, D. *Comparison of different estimators of* $p(Y < X)$ *for a scaled Burr type X distribution*, Commun. Statist. Simul. Comput., **34(2)**, 465 - 483, 2006.

[22] Robert, C.P. and Casella, G. *Monte Carlo statistical methods*, Second Edition, Springer, NewYork , 2004.

[23] Saracoglu, B., Kinaci, I. and Kundu, D. *On estimation of* $R = p(Y < X)$ *for exponential distribution under progressive type-II censoring, Journal of Statistical Computation and Simulation*, **82(5)**, 729-744, 2012.

[24] Sarhan, A. and Kundu, D. *Generalized linear failure rate distribution*, Commun. Statist. Theory Methods, **38**, 642 - 660, 2009.

[25] Shahsanaei, F and Daneshkhah, A. *Estimation of stress strength model in generalized linear failure rate distribution*, ArXiv Preprint 1312.0401 v1, 2013.

[26] Xia, Z. P., Yu, J. Y. , Cheng, L. D. , Liu, L. F. and Wang, W. M. *Study on the breaking strength of jute fibers using modified Weibull distribution*, Journal of Compos- ites Part A: Applied Science and Manufacturing, **40**, 54-59, 2009.