

Critical State Response of Granular Soils: Probabilistic Predictive Models Incorporating Grain Size, Distribution and Shape Effects

Kemal Önder ÇETİN^{1*}

Elife ÇAKIR²

Ahmet Seyit AKSOY³

ABSTRACT

The critical state locus (CSL), expressed in the void ratio (e) – natural logarithmic mean effective stress ($\ln p'$) space, is typically defined by slope (λ_e) and intercept parameters (Γ_1), the latter corresponding to the void ratio at a reference mean effective stress of unity. The angle of shearing resistance at the critical state (ϕ'_{cs}) relates the shear strength to effective normal stresses. These parameters are fundamentally governed by intrinsic properties of granular soils. This study examines their dependence on particle-scale descriptors, namely median grain size (D_{50}), coefficient of uniformity (C_u), particle roundness (R), and sphericity (S). A database containing critical state parameters (CSP), and grain size, gradation, and shape descriptors for over two hundred sandy and silty non-plastic soils was compiled from available literature, supplemented with laboratory tests performed on Cine, Sile, Tekirdag, and Iskenderun silica sands from Türkiye, and Sabratha silt from Libya. Their CSPs were evaluated following the procedures outlined by Santamarina and Cho [1], modified by benefiting from photogrammetric techniques for monitoring the specimen volume. Particle roundness and sphericity were quantified from microscopic imaging using the method defined by Cho et al. [2]. Semi-empirical probabilistic predictive models were developed using a maximum-likelihood estimation framework. They suggest that ϕ'_{cs} decreases with increasing R , decreasing C_u , but log-linearly increases with increasing D_{50} . The intercept Γ_1 decreases linearly with increasing R and decreasing S and C_u , and exhibits a log-linear decrease with increasing D_{50} . The compressibility λ_e increases with increasing D_{50} , whereas

Note:

- This paper was received on April 30, 2026 and accepted for publication by the Editorial Board on October 1, 2026.
- Discussions on this paper will be accepted by xxxxxxxx xx, xxxx.
- <https://doi.org/>

1 Middle East Technical University, Civil Engineering Department, Ankara, Türkiye
ocetin@metu.edu.tr - <https://orcid.org/0000-0003-0540-2247>

2 Middle East Technical University, Civil Engineering Department, Ankara, Türkiye
cakire@metu.edu.tr - <https://orcid.org/0000-0002-9773-3440>

3 GeoDestek Ltd. Corp., Middle East Technical University, Teknokent Halıcı Yazılımevi, Ankara, Türkiye
aaksoy@geodestek.com - <https://orcid.org/0000-0002-7707-410X>

* Corresponding author

it decreases with better grading. It exhibits a log-linear decrease with increasing R and S . The model performances, evaluated through analyses of residuals, indicate unbiased predictions with well-quantified levels of model precision. The resulting models provide quantitative insight into the relationships between micro-scale particle characteristics and macro-scale critical state behavior. They offer practical means for preliminary estimation of *CSPs* by explicitly incorporating variability in grain size, gradation, and shape metrics of granular soils.

Keywords: Critical state parameters, particle angularity, particle sphericity, angle of shearing resistance, median grain size, coefficient of uniformity.

1. INTRODUCTION

The critical state (*CS*) is defined as the condition at which soil behaves as a rate-independent frictional material, exhibiting constant effective stress, volume, and shear resistance, while deforming continuously to larger strains. Its locus (*CSL*) is commonly represented by a linear relationship in the void ratio-natural logarithmic mean effective stress ($e - \ln p'$) space. Given in Eqn. 1, it is characterized by two parameters: the slope (λ_e) and the intercept (Γ_1). The intercept Γ_1 corresponds to the void ratio at a reference mean effective stress of unity, while the subscript “ e ” in λ_e denotes that the slope is defined in terms of natural logarithmic ($\ln$) of p' .

$$e_{cs} = \Gamma_1 - \lambda_e \cdot \ln(p') \quad (1)$$

The stress-strain responses of soils under both drained and undrained (constant-volume) loading conditions are fundamentally governed by their initial distance from the *CSL*, expressed as the void-ratio difference ($e_0 - e_{cs}$), widely referred to as the state parameter, ψ . The ψ controls whether soil exhibits contractive or dilative behavior, and thus governs the evolution of shear strength, stiffness degradation, volumetric strain, or pore pressure generation during loading. Soils located above the *CSL* (positive ψ) tend to contract and soften, which may lead to instability and flow liquefaction under undrained loading, whereas those located below the *CSL* (negative ψ) dilate and exhibit strain-hardening behavior. Consequently, the soil state relative to the *CSL* provides a unifying framework for interpreting soil behavior by linking their initial state to deformation mechanisms, strength mobilization, and failure modes. Hence, it is critical to identify the *CSL* of soils.

Soil properties are commonly classified into: (i) intrinsic properties, such as mineral composition; particle shape, including roughness, roundness, and sphericity; grain size distribution, consistency limits, etc.; and (ii) state properties, including void ratio, moisture content, confining stress, overconsolidation ratio, etc. Intrinsic and state properties jointly govern the macro-scale mechanical response of soils [2]-[14], whereas critical state behavior is governed solely by intrinsic properties [15]-[17].

Mineralogy, weathering processes, and geological formation mechanisms play a key role in shaping intrinsic soil properties, namely particle size distributions and particle morphology [1]. Particle morphology is commonly represented by three independent descriptors acting at different scales: roughness, roundness, and sphericity [1], [7], [18]. Roughness describes

small-scale surface texture and is difficult to quantify reliably due to the fractal nature of particle surfaces; consequently, it is generally not widely used for quantitative evaluations [1]. Roundness characterizes the sharpness of particle corners, with higher roundness values indicating smoother grains, while sphericity describes the overall particle form relative to a spherical shape and decreases with increasing elongation or flatness. These descriptors influence the mechanics of particle contact, packing efficiency, and fabric evolution, and therefore govern the critical state response of granular soils. A schematic illustration on how to evaluate these particle shape descriptors, is presented in Figure 1.

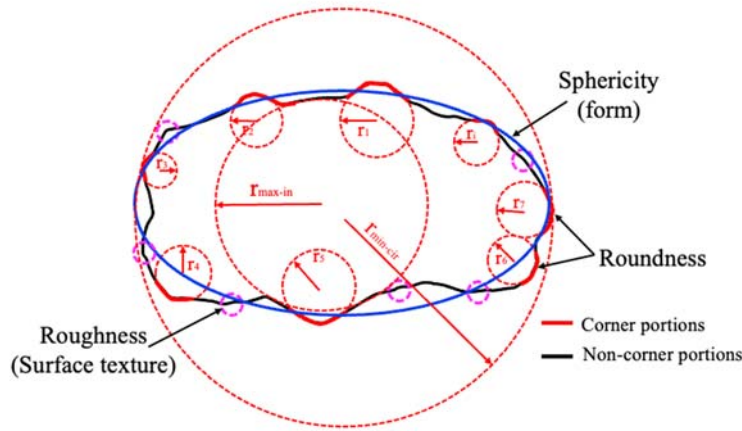


Figure 1 – Particle shape characterization at different scales, and identifying corner and non-corner irregularities of a particle outline (adapted from [19])

The influence of particle size, gradation, and shape on geomechanical behavior of granular soils has long been recognized [20]-[22], although early assessments were largely qualitative. Advances in experimental methods and image-based characterization techniques have enabled more rigorous quantitative evaluations. Recent studies have examined the relationships between particle shape parameters and some critical state properties [2], [23]-[26].

Among these studies, Santamarina and Cascante [8] suggested that ϕ'_{cs} increases with surface roughness, although direct experimental evidence remains limited. Cho et al. [2] reported that both ϕ'_{cs} and Γ_1 increase with decreasing roundness, a trend also confirmed by Guo and Su [15]. Rouse et al. [27] developed linear relationships between R and both the angle of repose and the friction angle at maximum contraction using a dataset of approximately 30 samples from 21 different sand types. Yang and Wei [28] showed that ϕ'_{cs} decreases with increasing R in clean sands and introduced the concept of combined roundness for sand-fines mixture, demonstrating its correlation with ϕ'_{cs} . Jia et al. [29] developed predictive models for the CSP, critical state stress ratio (M), Γ_1 , and λ_e as functions of R using a dataset of 78 sands, while reporting negligible effects of S , C_u , and D_{50} . Arda and Cinicioglu [30] proposed a multilinear regression model relating ϕ'_{cs} to S , $S \cdot R$, D_{50} , and C_u using data from 71 sands. Cheri Taiba et al. [31] developed expressions relating overall roundness (OR) to the packing

density (e_{max} and e_{min}) and granulometric characteristics (D_{50} and C_u). More recently, Kuncar et al. [32] demonstrated that roundness, rather than sphericity, governs the overall stress–strain response at large strains, and it plays a more significant role in controlling both ϕ'_{cs} and Γ_1 .

In addition to laboratory experiments, the effects of particle shape and gradation on critical state parameters have been studied using the discrete element method (DEM). Xie et al. [33] reported that more elongated particles (lower aspect ratio, AR) exhibit higher shear resistance, resulting in a higher critical state stress ratio. Nguyen et al. [34] showed that the critical state stress ratio increases with decreasing S and R , while the CSL intercept (Γ_1) and curvature parameters decrease with increasing sphericity. Nie et al. [35] adopted the three-dimensional overall regularity (OR) index to characterize particle shape and developed linear relationships between OR and the critical state parameters (Γ_1 , λ_e , and ϕ'_{cs}), demonstrating that both M and the Γ_1 decrease as particles become more regular and spherical. Adesina et al. [36] used two-dimensional DEM simulations to independently vary particle convexity, sphericity, and aspect ratio, showing that the critical state shear strength decreases with increasing particle convexity and sphericity, whereas it increases with increasing aspect ratio.

Nevertheless, a unified framework with a comprehensive database that correlates particle-scale descriptors (grain size, gradation, and particle shape) with the corresponding macro-scale critical state parameters does not currently exist. Motivated by this, the present study compiled an extensive database of two hundred thirty-three, non-plastic, sandy-silty soils from the literature, for which critical state parameters and associated intrinsic descriptors were determined by laboratory tests. In addition, silica sand samples from Cine, Sile, Tekirdag, and Iskenderun in Turkiye, as well as a non-plastic silt sample from Sabratha, Libya, were tested to determine the critical state loci and corresponding critical state effective angle of shearing resistance, ϕ'_{cs} . It should be noted that, although the tested samples are named after the cities or regions from which they were collected, the reported properties are representative only of the tested materials and should not be considered representative of all sand deposits within those regions. The compiled database is subsequently used to develop predictive relationships for the complete set of critical state parameters (Γ_1 , λ_e , and ϕ'_{cs}) as functions of sphericity, roundness, gradation, and median particle size.

2. GRAIN SIZE DISTRIBUTION AND PARTICLE SHAPE DESCRIPTORS

The grain size distribution characteristics are quantified using the median particle size (D_{50}), and the coefficient of uniformity (C_u), which are typically determined in accordance with ASTM D6913 / D6913M [37]. Sphericity (S) and roundness (R) are selected as particle shape descriptors. The quantitative evaluations of S and R descriptors are non-trivial and therefore needs further discussion.

S is a dimensionless particle-shape descriptor that quantifies the degree to which the shape of a particle approaches that of a sphere. It is a widely used parameter in soil mechanics, sedimentology, and granular mechanics to characterize particle morphology and to interpret its influence on particle packing, shear strength, dilatancy, and critical state behavior. Strictly speaking, the accurate determination of S requires measurement of the three-dimensional surface area and volume of a particle. However, such measurements are rarely practical and

are therefore commonly approximated using simplified two-dimensional image-based proxies [38], [39]-[42]. The specific definition of S adopted in this study is given in Eqn. 2, where $r_{max,in}$ and $r_{min,cir}$ denote the radii of the maximum inscribed circle and the minimum circumscribed circle of a particle, respectively, illustrated in Figure 1.

$$S = \frac{r_{max,in}}{r_{min,cir}} \quad (2)$$

R is a particle-shape descriptor that quantifies the smoothness of particle corners and edges, largely independent of the particle's overall form or elongation. It is a key morphological parameter in granular soil mechanics, sedimentology, and more specifically liquefaction studies, as it governs contact behavior, abrasion history, and frictional resistance. Wadell [43] defined R as the ratio of the average radius of curvature of a particle's corners to the radius of the largest inscribable circle within the particle outline. In practice, R is evaluated by examining each corner of a particle's two-dimensional outline and matching it against reference templates to identify the largest circle that can be fitted within the corner curvature. The definition of R adopted in this study is formulated in Eqn. 3, where $\bar{r}$, as illustrated in Figure 1, denotes the arithmetic mean of the radii of curvatures corresponding to the grain's protrusions.

$$R = \frac{\bar{r}}{r_{max,in}} \quad (3)$$

In contrast to relatively more straightforward determination of S , the quantification of R is considerably more complex when using basic image analysis techniques. Estimation of Wadell's roundness requires the identification of individual particle corners and an assessment of their local curvature (i.e.: sharpness). In non-automated, manual evaluations, this procedure inherently involves a degree of subjectivity, as the identification of corners relies on analyst judgment and experience.

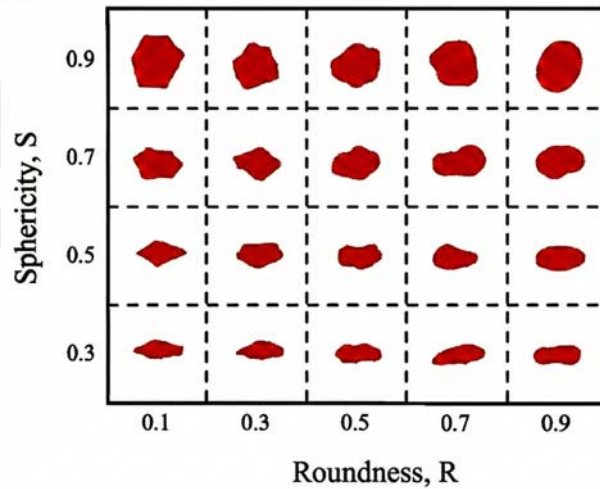


Figure 2 - Particle shape determination – sphericity, S , and roundness, R (after [2])

Table 1 – An illustrative summary of roundness and sphericity evaluations

Sand/Silt Type	$r_{min,cir}$	$r_{max,in}$	r_i	Location on the S-R chart
Iskenderun				
Sile				
Cine				
Tekirdag				
Sabratha				

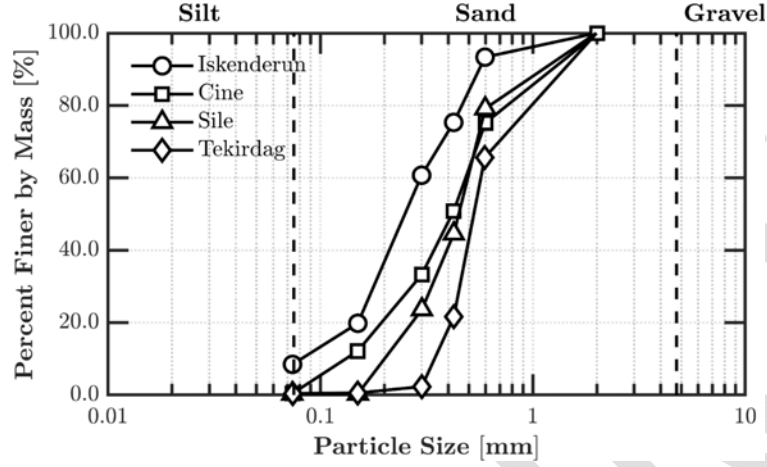


Figure 3 – Grain size distribution curves for coarse grained (sandy) soil samples

Table 2 – A summary of particle shape, grain size distribution characteristics, and critical state parameters

	Sabratha Silt	Cine Sand	Sile Sand	Iskenderun Sand	Tekirdag Sand
USCS Classification	ML	SP	SP	SP-SM	SP
D_{10} (mm)	-	0.12	0.20	0.09	0.33
D_{30} (mm)	-	0.26	0.33	0.18	0.44
D_{50} (mm)	-	0.38	0.44	0.24	0.51
D_{60} (mm)	-	0.46	0.50	0.30	0.54
Coefficient of uniformity, C_u	-	3.83	2.50	3.33	1.64
Coefficient of curvature, C_c	-	1.20	1.10	1.20	1.10
% Fines, FC	63.31	0.07	0.01	7.04	0.03
Sphericity, S	0.64 ± 0.11	0.60 ± 0.10	0.58 ± 0.07	0.60 ± 0.09	0.60 ± 0.10
Roundness, R	0.48 ± 0.11	0.37 ± 0.08	0.28 ± 0.09	0.24 ± 0.08	0.25 ± 0.08
Particle Shape Classification	Sub-angular	Sub-angular	Sub-angular	Angular	Angular
ϕ'_{cs} (°)	31.0	33.0	33.5	31.2	33.5
Γ_1	1.011	0.941	0.654	0.877	0.774
λ_e	0.095	0.124	0.031	0.098	0.045

Despite its labor-intensive and inherently subjective nature, R remains as a widely used parameter for quantifying grain shape characteristics in granular materials [2], [3], [7], [9], [40], [44]–[45]. To facilitate practical estimation of Wadell's roundness, reference charts illustrating idealized particle shapes were introduced in the mid-twentieth century, allowing visual comparison with particles observed under optical microscopy. Notable chart systems were developed by Krumbein [46], Powers [47], and Krumbein and Sloss [38]. Figure 2 illustratively presents the chart proposed by Krumbein and Sloss [38]. Although chart-based classification involves greater subjectivity than Wadell's original template-based approach, it has been extensively used across geotechnical engineering, soil science, agriculture, powder engineering, pavement engineering, and geology due to its practicality.

In this study, particle sphericity and roundness were determined using images taken by a digital microscope with 1600-pixel resolution. For each soil sample, a representative number of grains, typically 10 to 15, was randomly selected to evaluate estimates of particle shape characteristics. The selected particles were considered sufficient to obtain representative average roundness and sphericity values for each sample; however, it is acknowledged that a larger particle population would reduce sampling uncertainty. Particle images were analyzed using the accompanying image analysis software, which served only as a visualization and measurement tool. Roundness and sphericity were determined manually by the operator by fitting the reference circles to each particle following the conventional Krumbein and Sloss [38] method.

Representative examples of the visual and optical measurements of the radii, along with the corresponding calculated S and R values for all examined grains, are presented in Table 1. In the $S - R$ plots, diamonds indicate the representative (mean) value for the soil sample, whereas dots represent the measurements for the individual grains.

The gradation of four sand samples was determined through laboratory testing conducted in accordance with ASTM D6913/D6913M-17 [37]. Key gradation parameters: the coefficient of uniformity (C_u), coefficient of curvature (C_c), and effective particle sizes (D_{10} , D_{30} , and D_{60}), were evaluated. The resulting grain size distribution curves and associated gradation characteristics for coarse-grained soil samples are presented in Figure 3 and Table 2, along with a summary of the mean $\pm$ standard deviation particle shape descriptors.

3. CRITICAL STATE DESCRIPTORS

Several experimental methods have been employed over the years to evaluate the critical state descriptors of granular soils. Ring shear [48] and direct shear tests [49] have been performed on various sands; however, an accurate definition of the stress state within ring shear and direct shear specimens remains challenging [50]. Triaxial testing has also been widely used in earlier years of geotechnical engineering profession to develop and verify the critical state soil mechanics concepts, as critical state can often be reached at relatively lower levels of global strains in undrained testing of loose soil samples [51]–[52]. Nevertheless, it is not without limitations. In drained triaxial tests, the critical state may not be reached within axial strain levels of approximately 10–20%, which are typically the practical limits for reliable specimen area corrections. Undrained tests are further complicated by poro-elastic effects and the potential for strain localization. Direct simple shear and torsional shear tests are more suitable due to their ability to impose large, uniform shear strains without the

development of stress non-uniformities or premature localization. Unlike triaxial compression tests, simple and torsional shear modes maintain a well-defined stress path and constant principal stress orientation, enabling soils to reach and sustain critical state conditions. As a result, critical state parameters can be evaluated more reliably. However, these tests demand specialized equipment, careful specimen preparation, and longer testing durations, resulting in increased experimental complexity and cost.

To address these challenges, Santamarina and Cho [1] proposed a simplified testing procedure, which eliminated the need for a confining pressure chamber. Despite its practical advantages, this method requires full saturation of the specimen, often achieved under vacuum-induced suction, in order to monitor specimen volume during drained loading. Achieving complete saturation can be challenging, particularly for coarse-grained soils. Accordingly, in the present study, the simplified testing procedure was modified by eliminating the need for full saturation. Instead, specimen volume changes during drained axial loading were calculated using photogrammetric techniques coupled with three-dimensional surface reconstruction. The photogrammetric measurements were performed before loading and after completion of the large-deformation stage. Continuous volume measurements during shearing were not performed because they would require image acquisition and three-dimensional reconstruction at each deformation increment, substantially increasing the experimental effort and processing time. Accordingly, the proposed methodology was intended to characterize the initial and final specimen geometries rather than the continuous evolution of specimen volume during loading. Once volumetric changes diminish under continued loading (approximately strain levels over 40%), the specimen is judged to have reached the critical state. Figure 4 illustrates a specimen ready to be loaded axially and subjected to axial strain levels over 40%, respectively.



Figure 4 – Soil specimen ready to be loaded axially and subjected to axial strain levels over 40%

The main stages of the experimental procedure are summarized below. In addition, a detailed description of the modified testing methodology is also provided in Aksoy [53]:

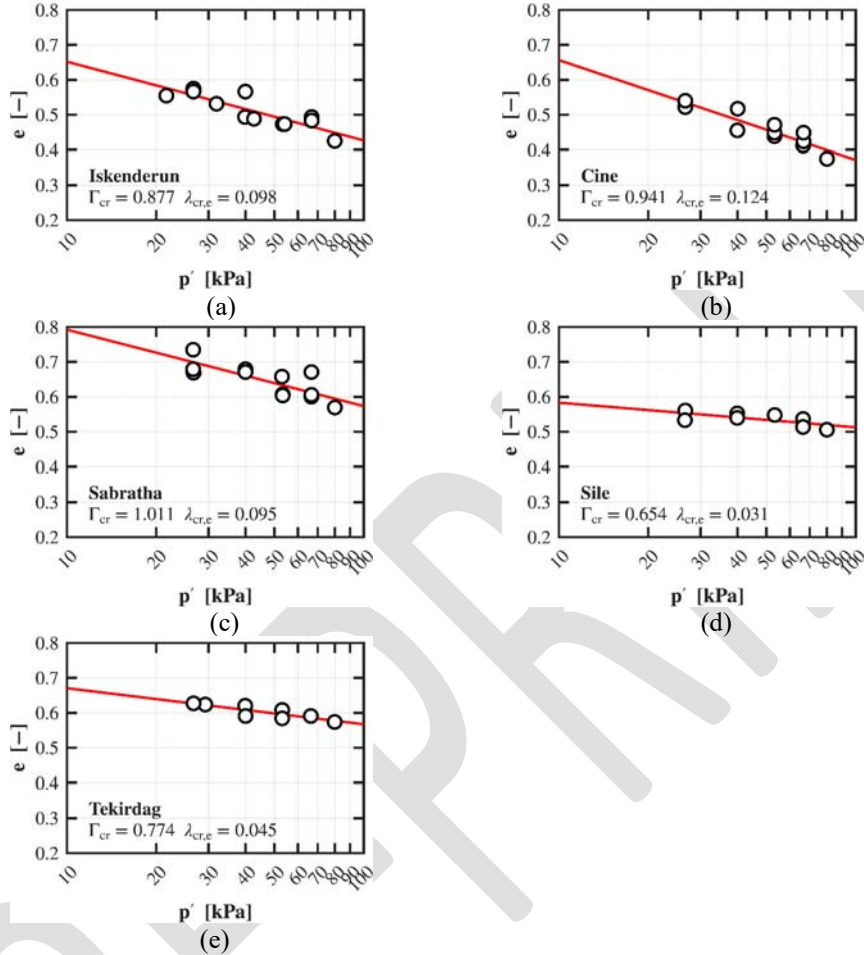


Figure 5 – Critical state loci of tested specimens

1. Oven-dried samples were compacted in a split mold employed for a conventional triaxial specimen. Uniform specimens are prepared in a medium-dense state to ensure a dilative response, producing a bulged specimen shape in drained loading. As the tests are conducted on oven-dried samples subjected to drained loading, saturation is not required.
2. A vacuum pump equipped with a regulator valve is used to apply the desired confining pressure.
3. Axial strain increments are applied using a hydraulic loading frame, capable of axial strain levels more than 40 %.
4. The deformed specimen geometry (and corresponding volume) is determined using photogrammetric techniques coupled with three-dimensional surface

reconstruction. When necessary, for enhanced accuracy, surface irregularities are smoothed through manual massaging of the specimen.

5. Axial strain increments continue until a constant volume response is reached, assuring critical state.
6. The next confining pressure increment is applied on the reformed specimen, and Steps 3 through 5 are repeated.

The critical state loci determined using the outlined procedure are presented in Figure 5. For each soil sample, multiple tests were conducted to ensure the robustness of the methodology and to verify the accuracy of *CSPs*.

The effective angle of shearing resistance at critical state was determined following the methodology proposed by Santamarina and Cho [1], schematically illustrated in Figure 6. Specimens were prepared using a transparent plexiglass container with dimensions of $10 \times 20 \times 50$ cm. The soil was poured into the container filled with water, after which the container was sealed and gently shaken to form a uniform water–soil slurry. The slurry was then allowed to settle, producing the loosest possible water-sedimented state. Subsequently, the container was carefully rotated to an inclination exceeding 60° and gradually returned to the vertical position. During this stage, excessive movement or vibration was avoided to prevent water oscillation and disturbance of the loose soil fabric, which could lead to unintended densification.

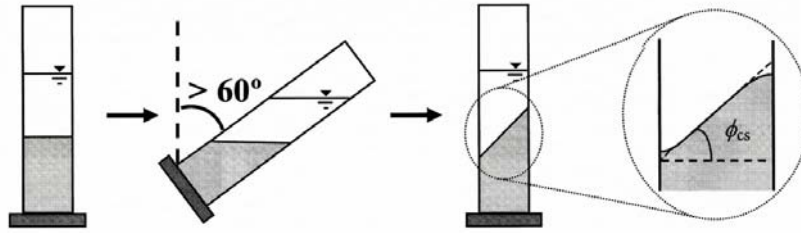


Figure 6 – Simplified method to determine the critical state angle of shearing resistance (adapted from [1])

After particle deposition and flow stops, the inclination of the steepest stable slope is measured to determine the critical state angle of shearing resistance, as illustrated in Figures 6 and 7. Measurements were repeated multiple times to verify the repeatability of the procedure and the consistency of the results. Repeatability of this method was also investigated in detail by Santamarina and Cho [1]. In addition to repeatability, Santamarina and Cho [1] compared the angle of shearing resistance at critical state obtained using the simplified method with those determined from conventional triaxial tests and reported good agreement between the two approaches. Accordingly, the angle of shearing resistance at critical state reported in the literature and determined using conventional laboratory tests were considered suitable for inclusion in the database and subsequent development of the predictive model.

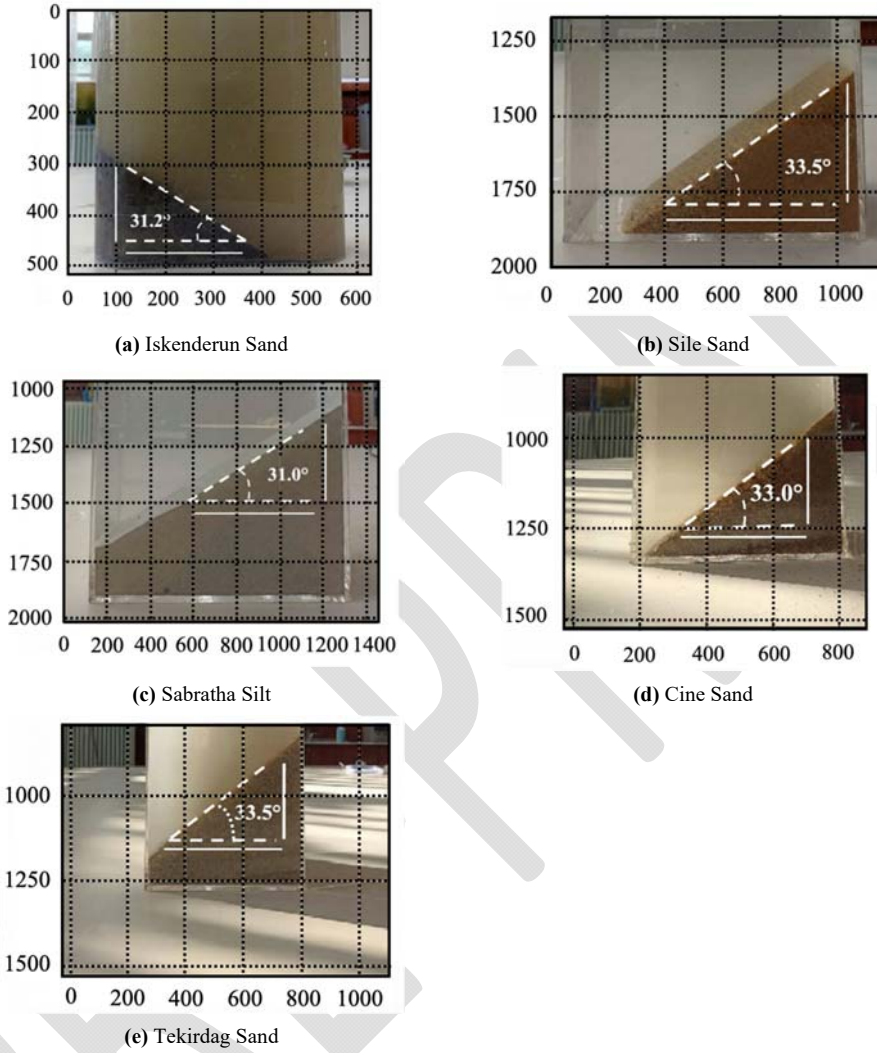


Figure 7 – Determination of effective angles of shearing resistance at critical state

4. RESULTING DATABASE AND DATA TRENDS

In addition to these five soil samples tested, a database was compiled comprising 232 sands reported in the literature. The resulting database is provided in the electronic supplement. Of the 232 sands, 123 include the evaluations for ϕ'_{cs} , Γ_1 , and λ_e ; 71 for both Γ_1 and λ_e , and 44 for only ϕ'_{cs} . To determine the descriptors governing the critical state characteristics, the dependent variables ϕ'_{cs} , Γ_1 , and λ_e were visually examined in relation to the independent variables of median particle size (D_{50}), coefficient of uniformity (C_u), roundness (R), and

sphericity (S), as illustrated in Figure 8. On the same figure, data trends expressed by simple linear regression are shown by dashed lines. It should be noted that the reported $CSPs$, D_{50} , C_u , R , and S values were compiled directly from the original literature sources. Among these parameters, roundness and sphericity are inherently influenced by the adopted measurement methodology, including image resolution, particle-size fraction, number of particles evaluated, and operator judgement. Consequently, methodological differences among studies may contribute to the scatter observed in the proposed empirical relationships and should be recognized as a limitation of the compiled database.

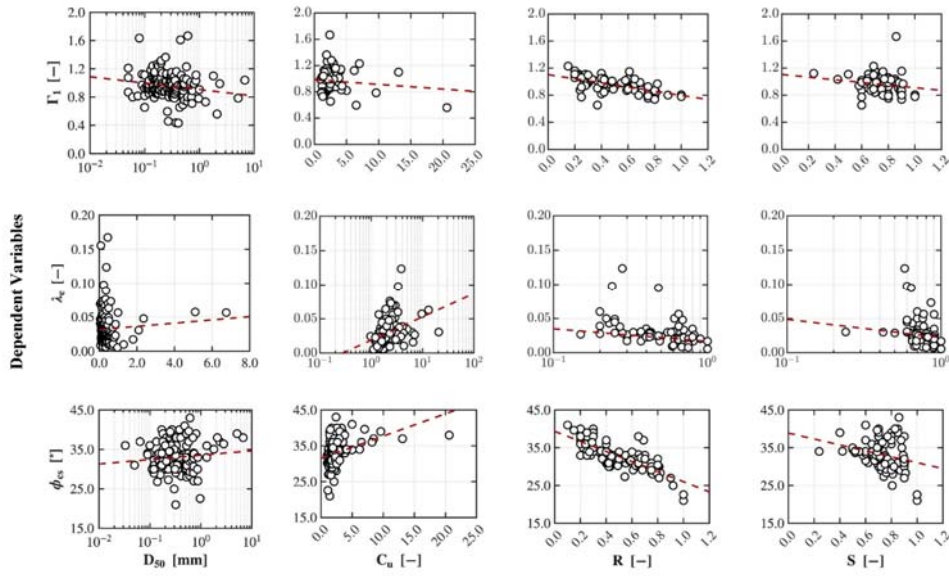


Figure 8 - Dependency of critical state parameters on D_{50} , C_u , R , and S

Visual inspection of these trends suggests the following:

- Γ_1 shows a negative log-linear correlation with D_{50} and a negative linear correlation with C_u , R , and S .
- A weak but positive linear correlation is observed between λ_e and D_{50} . Conversely, λ_e exhibits negative log-linear correlations with R and S , respectively. A positive but weak log-linear correlation with C_u is identified.
- A positive log-linear correlation exists between ϕ'_{cs} and D_{50} , whereas negative linear correlations are observed with R and S . In addition, ϕ'_{cs} exhibits a positive linear relationship with the C_u .

5. PROBABILISTIC PREDICTIVE MODELS AND VERIFICATION

Inspired by these observational trends, and after evaluating several alternative functional forms, the limit state functions, $g(\cdot)$, presented in Eqns. 4, 5, and 6, were adopted for ϕ'_{cs} , Γ_1 , and λ_e .

$$g_\phi(\phi'_{cs}, D_{50}, C_u, R) = \phi'_{cs} - \theta_{1,\phi} \cdot \ln(D_{50}) - \theta_{2,\phi} \cdot C_u + \theta_{3,\phi} \cdot R - \theta_{4,\phi} + \varepsilon_\phi \quad (4)$$

$$g_\Gamma(\Gamma_1, D_{50}, C_u, R, S) = \Gamma_1 - \theta_{1,\Gamma} \cdot \ln(D_{50}) - \theta_{2,\Gamma} \cdot C_u - \theta_{3,\Gamma} \cdot R - \theta_{4,\Gamma} \cdot S - \theta_{5,\Gamma} + \varepsilon_\Gamma \quad (5)$$

$$g_\lambda(\lambda_e, D_{50}, C_u, R, S) = \lambda_e - \theta_{1,\lambda} \cdot D_{50} - \theta_{2,\lambda} \cdot C_u - \theta_{3,\lambda} \cdot \ln(R) - \theta_{4,\lambda} \cdot \ln(S) - \theta_{5,\lambda} + \varepsilon_\lambda \quad (6)$$

These functional forms not only provide sufficient flexibility to capture the outlined observational trends, but also yield higher overall likelihood values, indicating a statistically superior fit to the dataset. Nevertheless, it should be noted that Eqns. 4-6 include a random model-correction term ε , which differs for each model. This term accounts for (i) the possible influence of unobserved explanatory variables and (ii) the likelihood that the adopted mathematical form may deviate from the ideal functional representation of the underlying physics. It is reasonable and appropriate to assume that ε is normally distributed with zero mean, as this ensures that the resulting models remain unbiased [54].

Using a maximum likelihood estimation framework, the model coefficients were obtained, and the resulting probabilistic predictive models are presented in Eqns. 7 through 9. In Eqns. 7 and 8, soil grading effects (i.e.: well-graded vs. poorly-graded) on ϕ'_{cs} and Γ_1 are represented directly through C_u parameter. However, using C_u in the λ_e model, as formulated in Eqn. 6, produced a statistically superior model (higher likelihood) that conflicted with the basics of soil mechanics, suggesting a more compressible (higher λ_e) response for well-graded soils. This inconsistency was attributed to limited number of well-graded soil tests in the resulting database. To mitigate this, Eqn. 6 is slightly modified by adopting a dummy variable I_{C_u} , defined as 0 for poorly graded soils ($C_u < 6$) and 1 for well-graded soils ($C_u \geq 6$). This form, given in Eqn. 9, reduced the influence of overrepresented poorly-graded soils in the database and yielded more consistent predictions with soil mechanics basics. Therefore, it was adopted despite its slight statistical inferiority.

$$\phi'_{cs} = 0.8259 \cdot \ln(D_{50}) + 0.5866 \cdot C_u - 13.3298 \cdot R + 39.0616 \pm 2.00 \quad (7)$$

$$\Gamma_1 = -0.0026 \cdot \ln(D_{50}) + 0.0233 \cdot C_u - 0.2845 \cdot R + 0.0490 \cdot S + 1.0141 \pm 0.0966 \quad (8)$$

$$\lambda_e = 0.0041 \cdot D_{50} - 0.0259 \cdot I_{Cu} - 0.0180 \cdot \ln(R) - 0.0226 \cdot \ln(S) + 0.0075 \pm 0.0172 \quad (9)$$

The residuals of the models, defined as the difference between the predicted and measured values, are presented in Figure 9. Residuals distributed around a zero-mean indicate that the resulting predictive models are unbiased. In addition, plus and minus three standard deviations of model error ($\pm 3 \sigma_\varepsilon$) centered at zero-mean are plotted on the same figure, and the bounds successfully capture nearly all data points. The relatively large scatter reflects the inherent high variability in critical state responses as functions of D_{50} , C_u , R , and S .

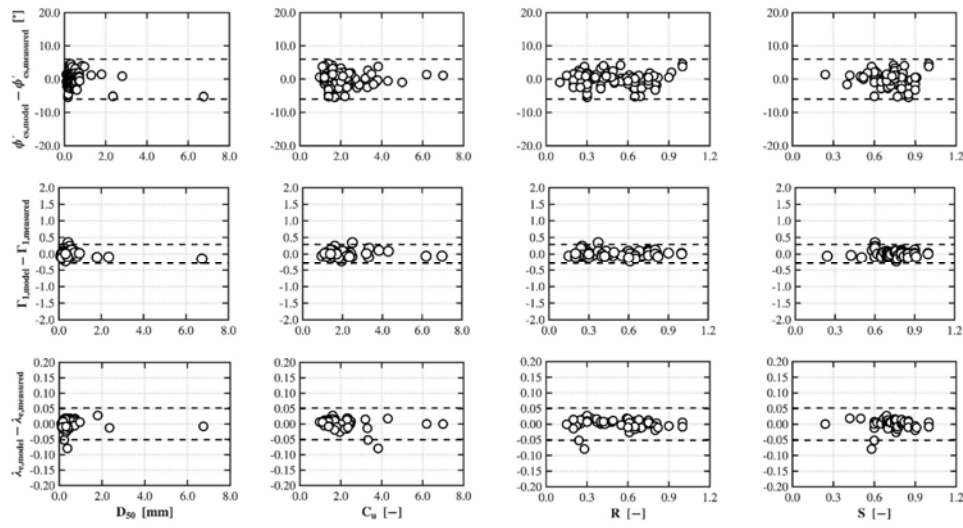


Figure 9 – Residuals of the models

To evaluate the sensitivity of the predictor variables, tornado plots were developed, which are presented in Figure 10. The results indicate that the ϕ'_{cs} response is most sensitive to variations in R , consistent with previous studies [2,29-30]. This is consistent with the role of roundness in controlling interparticle contact characteristics: particles with sharper corners and more angular shapes mobilize greater interlocking and frictional resistance, thereby producing higher critical state shear strength. The Γ_1 model is also sensitive to changes in R , whereas its influence is less pronounced in the λ_e model, suggesting that particle shape plays a stronger role in governing critical state shear resistance than compressibility. S has a minor role in the Γ_1 model, but a more noticeable effect on λ_e , while no significant relationship is observed between S and ϕ'_{cs} .

Γ_1 model is not very sensitive to changes in median grain size D_{50} , indicating that grain size alone does not strongly control CSL , once particle shape and gradation effects are accounted for. In contrast, variations in D_{50} affect λ_e more significantly, consistent with the interpretation that size-dependent compressibility behavior is fundamentally governed by particle crushing mechanisms [55]. As D_{50} increases, the crushing strength of individual grains decreases [56]. Moreover, soils composed of larger-sized grains tend to have a smaller

number of contact points, as compared to those composed of finer-grains, resulting in higher stresses concentrated at these contact points under a given macroscopic effective stress level [57]. Although particle crushing is often associated with elevated volumetric stress levels in the literature, the presence of shear stresses jointly acting with elevated confining stresses may ease grain breakage [58]. Consequently, soils with larger D_{50} may experience crushing-induced volumetric contractions at lower mean effective stresses, leading to a more compressible response.

As illustrated in Figure 10, λ_e is also observed to be sensitive to the changes in the coefficient of uniformity C_u . This is consistent with fundamental soil mechanics principles: broader gradation promotes denser packing and reduced compressibility under loading, whereas the influence of grain size becomes secondary and manifests mainly through fabric development and void structure. Finally, it should be noted once again that the current database includes a limited number of descriptors for well-graded soils, and further validation of conclusions regarding gradation effects may benefit from future additional data.

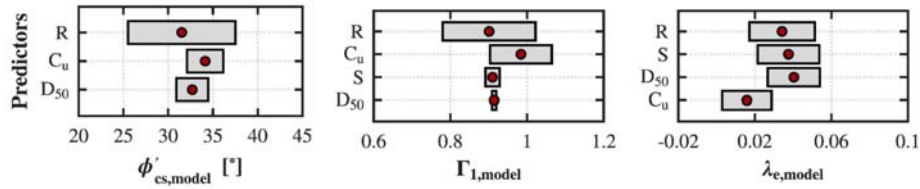


Figure 10 – Tornado plots

Table 3 – Comparisons of critical-state angle of shearing resistance predictions

Input parameters	The critical-state angle of shearing resistance, ϕ'_{cs} (°) estimated by		
	This study	Jia et al. [29]	Arda and Cinicioglu [30]
$D_{50}=0.09$ $C_u=0.97$ $R=0.10$ $S=0.24$ (Database minimums)	36.3	43.4	32.2
$D_{50}=0.52$ $C_u=2.08$ $R=0.49$ $S=0.72$ (Database means)	33.3	31.8	33.3
$D_{50}=6.75$ $C_u=7.00$ $R=1.00$ $S=1.00$ (Database maximums)	31.4	27.7	38.3

Table 3 compares the critical-state angle of shearing resistance predicted by the model proposed in this study with the predictions of Jia et al. [29] and Arda and Cinicioglu [30], using the minimum, median, and maximum values of the database parameters as inputs.

Moreover, Figure 11 compares the sensitivity of the angle of shearing resistance at critical state to the model parameters considered in the present study and in the models proposed by Jia et al. [29] and Arda and Cinicioglu [30]. In the present model, R exhibits the greatest influence on ϕ'_{cs} , followed by C_u and D_{50} . Similarly, the model of Arda and Cinicioglu [30] indicates that the interaction term $S \cdot R$ is the most influential variable, followed by D_{50} , C_u , and S . Thus, despite differences in model formulation, both models highlight the dominant role of roundness-related descriptors in controlling ϕ'_{cs} . The model of Jia et al. [29], which incorporates only R as a predictor, likewise emphasizes the importance of roundness. Compared with the model of Arda and Cinicioglu [30], the present model exhibits lower sensitivity to variations in D_{50} and C_u . These results further support the importance of roundness in predicting the angle of shearing resistance at critical state while indicating that the relative contribution of particle size and gradation depends on the adopted model formulation.

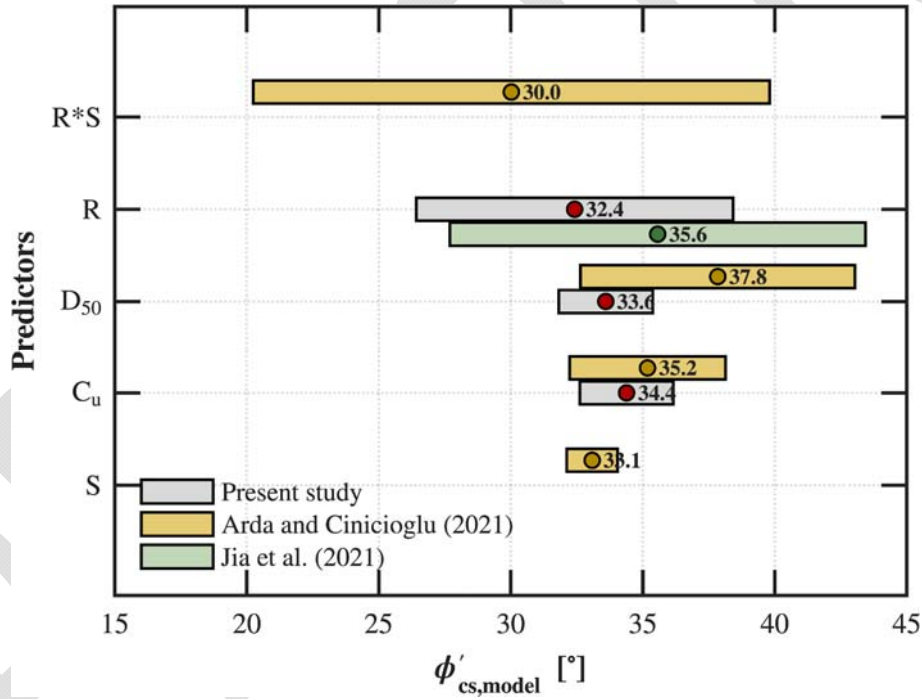


Figure 11 – Comparative tornado plots of the present study, Jia et al. [29], and Arda and Cinicioglu [30]

6. SUMMARY AND CONCLUSIONS

This study experimentally evaluated the effective angle of shearing resistance at the critical state and the associated critical state parameters for four alluvial sands from Türkiye, namely

Cine, Sile, Tekirdag, and Iskenderun sands; and the non-plastic Sabratha silt from Libya. The simplified testing framework of Santamarina and Cho [1] was adopted with key modifications: instead of full sample saturation and undrained loading, dry drained tests were performed, and specimen volume changes were monitored using photogrammetry coupled with three-dimensional surface reconstruction. Particle morphology was quantified through sphericity and roundness indices following Cho et al. [2].

In addition, an extensive database comprising 232 sands was compiled from the literature, providing a comprehensive basis for evaluating the influence of particle morphology and gradation on critical state behavior. Based on this database, probabilistic limit-state models were formulated for predicting ϕ'_{cs} , Γ_1 , and λ_e as functions of R , S , D_{50} , and C_u . Model coefficients were evaluated using a maximum-likelihood estimation framework, allowing the combined influence of particle scale descriptors on critical state behavior to be quantified within a unified predictive framework.

The resulting predictive relationships indicate that better-graded, coarser-grained soils composed of more angular particles mobilize higher critical state shear resistances, largely independent of variations in sphericity. For the intercept parameter Γ_1 , the predictive model indicates an increase with increasing C_u and S , and with decreasing D_{50} and R . The predictive λ_e model suggests that better-graded soils composed of more rounded and spherical grains exhibit a less compressible critical state response.

Residual analyses indicate that the proposed predictive models are essentially unbiased, with ± 3 standard deviation bounds encompassing nearly all data points. Sensitivity analyses demonstrated that R is the governing parameter affecting both ϕ'_{cs} and Γ_1 , highlighting the dominant role of particle angularity in controlling interparticle locking and critical void ratio characteristics. In contrast, compressibility behavior was found to be comparably sensitive to variations in all descriptors, namely roundness, sphericity, median grain size, and gradation. This conclusion explains the wider variability (scatter) in compressibility responses of soils as compared to their shearing resistances. In simpler terms, the variability in modulus of soils is significantly larger than the variability in their effective angles of shearing resistance. It also supports the role of particle rearrangement through interparticle slip and rotation and particle damage [55].

In contrast to previous studies, the proposed framework provides a unified probabilistic formulation for simultaneously predicting the complete set of critical state parameters from readily measurable particle-scale descriptors. However, these models are not intended to replace laboratory-based evaluations, and further refinement will require expansion of the database, particularly with additional well-graded soils and broader critical state coverage.

Symbols

CS: Critical state

CSL: Critical state locus

CSP: Critical state parameters

C_u : Coefficient of uniformity

C_c : Coefficient of curvature

D_{10} : Particle diameter corresponding to 10% passing (effective particle sizes)

D_{30} : Particle diameter corresponding to 30% passing

D_{50} : Particle diameter corresponding to 50% passing (median grain size)

D_{60} : Particle diameter corresponding to 60% passing

e : Void ratio

e_{cs} : Void ratio at critical state

$\ln p'$: Natural logarithmic mean effective stress

$r_{max,in}$: Radii of the maximum inscribed circle of a particle

$r_{min,cir}$: Radii of the minimum circumscribed circle of a particle

R : Particle roundness

S : Sphericity

I_1 : Intercept parameter

λ_e : Slope of the critical state line defined in terms of natural logarithmic of p'

ϕ'_{cs} : The angle of shearing resistance at the critical state

ψ : State parameter

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