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Uniqueness of solutions of boundary value problems at resonance

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Abstract

In this paper, the method of upper and lower solutions is employed to obtain uniqueness of solutions for a boundary value problem at resonance. The shift method is applied to show the existence of solutions. A monotone iteration scheme is developed and sequences of approximate solutions are constructed that converge monotonically to the unique solution of the boundary value problem at resonance. Two examples are provided in which explicit upper and lower solutions are exhibited.

Keywords: Boundary value problem at resonance; Shift method; Upper and lower solutions; Monotone convergence.
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1. Introduction

The method of upper and lower solutions and monotone methods have been useful in the study of boundary value problems for nonlinear ordinary differential equations. For many problems, the associated Green's function has fixed sign that agrees with a maximum principle or an anti-maximum principle [7]. Then monotonicity of iterates can occur naturally by assuming the nonlinearity is monotone with respect to the unknown function or the monotonicity of iterates can be forced by various methods. We refer the reader to [7, 8, 9, 10] for discussions and applications of maximum or anti-maximum principles or to [4, 11, 12, 14, 25] for discussion and applications of the so-called isotone operators.

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The method of quasilinearization, introduced by Bellman [5, 6] in the 1960s, offers a numerical method to approximate solutions of nonlinear problems with sequences of solutions of linear problems. Under suitable hypotheses, the sequences of approximate solutions converge monotonically and quadratically. In many applications, the iterates converge to a unique solution of the boundary value problem. The quasilinearization method has been particularly useful in the study of boundary value problems for ordinary differential equations and we cite a number of those applications here [1, 3, 13, 15, 19, 20, 21, 22, 23, 26]. In these works, the monotonicity is obtained rather delicately and the uniqueness of solutions plays a key role in obtaining the monotonicity.

To obtain the monotonicity of the iterates in, for example, [1, 3, 13, 15, 19, 23], a standard hypothesis is that the nonlinear term is increasing as a function of the unknown function. This hypothesis is used only to show the uniqueness of solutions. In this work, we shall assume the standard hypothesis and we shall assume in addition that the nonlinear term is increasing as a function of the derivative of the unknown function. The new hypothesis is only employed in the analysis at a boundary point. It is of interest to note that the new hypothesis is used to show the uniqueness of solutions and to show the existence of solutions.

Recently [2], the method of quasilinearization was applied to a two-point boundary value problem for an ordinary differential equation at resonance. In this article we shall consider a new two-point boundary value problem at resonance and we shall construct the monotone iteration scheme associated with the method quasilinearization. One key contribution of this work is that the nonlinear term depends on the unknown function and the derivative of the unknown function. In [2], a shift argument [17] is employed to obtain existence of solution. The shift that is employed depends on the unknown function. In this work, we shall apply the shift argument with a shift that depends on the derivative of the unknown function. In doing so, we shall successfully construct the monotone method; however, we currently cannot verify quadratic convergence. We shall leave for future work the application of the shift argument employed in [2] with the intention to obtain quadratic convergence.

The paper is organized as follows. In Section 2 we shall first employ the method of upper and lower solutions and under suitable hypotheses obtain the uniqueness of solutions of a two-point boundary value problem at resonance for a second order ordinary differential equation. In Section 3, we shall apply the shift argument and obtain the existence of that unique solution. In Section 4, we shall construct the monotone method. Sections 2, 3 and 4 apply to a problem where the nonlinear term depends on the unknown function and the derivative of the unknown function. In Section 5, we shall show how the methods of Sections 2, 3 and 4 apply to a problem where the nonlinear term only depends on the unknown function. We close in Section 6 with two examples in which upper and lower solutions are explicitly exhibited.

The application of the method of quasilinearization to boundary value problems at resonance is not new; see [27, 28]. The motivation and development here is different than that in [27] or [28], since uniqueness of solutions is a key feature in this work and multiplicity of solutions is key in [27] or [28].

2. Uniqueness of solutions

Assume $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous. We shall consider the boundary value problem

$$y''(t) = f(t, y(t), y'(t)), \quad 0 \le t \le 1, \tag{2.1}$$

$$y(0) = 0, \quad y'(0) = y'(1). \tag{2.2}$$

The boundary value problem (2.1), (2.2) is at resonance since the linear functions, $y = ct, c \in \mathbb{R}$, are solutions of the homogeneous problem $y'' = 0$ and satisfy the homogeneous boundary conditions (2.2).

With the notation $f(t, y_1, y_2)$, we begin with the assumption that f is increasing in y_1 for each $(t, y_2) \in [0, 1] \times \mathbb{R}$ and f is increasing in y_2 for each $(t, y_1) \in [0, 1] \times \mathbb{R}$ to obtain results for the uniqueness of solutions of the boundary value problem (2.1), (2.2).

Theorem 2.1. *Assume* $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ *is continuous,* $\frac{\partial}{\partial y_1} f = f_{y_1} : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ *is continuous and* $\frac{\partial}{\partial y_2} f = f_{y_2} : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ *is continuous. Assume* $f_{y_1} > 0$ *on* $[0, 1] \times \mathbb{R}^2$ *and assume* $f_{y_2} > 0$ *on* $[0, 1] \times \mathbb{R}^2$*. Then solutions of the boundary value problem* (2.1), (2.2) *are unique, if they exist.*

Proof. Assume for the sake of contradiction that $y(t)$ and $z(t)$ denote two distinct solutions of the boundary value problem (2.1), (2.2). Assume without loss of generality that $y-z$ has a positive maximum at $t_0 \in [0,1]$. (If this is not the case, then $z-y$ has a positive maximum at some $t_0 \in [0,1]$.)

First, assume, $t_0 \in (0,1)$. Then $(y-z)''(t_0) \leq 0$. However, y and z each satisfy (2.1), and so,

$$(y-z)''(t_0) = f(t_0, y(t_0), y'(t_0)) - f(t_0, z(t_0), z'(t_0)) > 0 \tag{2.3}$$

since $y'(t_0) = z'(t_0)$ and f_{y_1} is positive on $[0,1] \times \mathbb{R}^2$. This is a contradiction, and we shall refer to this contradiction as the usual contradiction. Thus, $y-z$ does not have a positive maximum at $t_0 \in (0,1)$.

Second, assume $t_0 = 0$ and recall the boundary condition $y(0) = z(0) = 0$. Thus, $y-z$ does not have a positive maximum at $t_0 = 0$.

Third, assume $t_0 = 1$. Then $(y-z)'(1) \geq 0$.

If $(y-z)'(1) = 0$, then $y'(0) = z'(0)$. Then y and z both satisfy the initial value problem (2.1) with initial conditions

$$y(0) = 0, \quad y'(0) = z'(0),$$

and so, $y-z$ does not have a positive maximum at $t_0 = 1$ by the uniqueness of solutions of initial value problems.

Finally assume $(y-z)'(1) > 0$. It is in this case that the new hypothesis $f_{y_2} > 0$ on $[0,1] \times \mathbb{R}$ is employed. Note that $(y-z)'(0) = (y-z)'(1) > 0$ and we claim that $(y-z)'$ does not change sign in $(0,1)$. Assume for the sake of contradiction that $(y-z)'$ does change sign and let $c \in (0,1)$ such that $(y-z)'(c) < 0$. On the interval, $[0,c]$, $y-z$ has a local minimum at 0 and a local minimum at c. Thus, $y-z$ has an absolute maximum (on the interval $[0,c]$) at $\tau \in (0,c)$. Since $(y-z)(0) = 0$ it follows that $(y-z)(\tau) > 0$. Then, as in the case $t_0 \in (0,1)$, we obtain $(y-z)''(\tau) \leq 0$ and $(y-z)''(\tau) > 0$ producing the usual contradiction.

So the claim that $(y-z)'(t)$ does not change sign in $(0,1)$ is true and $(y-z)'(t) \geq 0$ on $[0,1]$. This implies that $(y-z)(t)$ is increasing on $[0,1]$ and since $y(0) = z(0) = 0$ and $(y-z)'(0) > 0$, it follows that $(y-z)(t) > 0$ on $(0,1]$. For $t \in (0,1]$ consider

$$(y-z)''(t) = f(t, y(t), y'(t)) - f(t, z(t), z'(t)).$$

Since $y(t) > z(t)$ and $y'(t) \geq z'(t)$, the hypotheses $f_{y_1} > 0$ on $[0,1] \times \mathbb{R}$ and $f_{y_2} > 0$ on $[0,1] \times \mathbb{R}$ imply $(y-z)''(t) > 0$ for $t \in (0,1]$. In particular, $(y-z)'$ is a strictly increasing function in t and $(y-z)'(0) < (y-z)'(1)$ which contradicts the boundary condition $(y-z)'(0) = (y-z)'(1)$. Thus, $y-z$ does not have a positive maximum at $t_0 = 1$.

We conclude that $y(t) \leq z(t)$ for $0 \leq t \leq 1$. A completely analogous argument gives that $z(t) \leq y(t)$ for $0 \leq t \leq 1$. Thus, solutions of (2.1), (2.2) are unique, if they exist. □

Definition 2.2. *We say $\alpha \in C^2[0,1]$ is a lower solution of the boundary value problem* (2.1), (2.2) *if* $\alpha(0) = 0$, $\alpha'(0) = \alpha'(1)$ *and*

$$\alpha''(t) \geq f(t, \alpha(t), \alpha'(t)), \quad 0 \leq t \leq 1.$$

We say $\beta \in C^2[0,1]$ is an upper solution of the boundary value problem (2.1), (2.2) *if* $\beta(0) = 0$, $\beta'(0) = \beta'(1)$ *and*

$$\beta''(t) \leq f(t, \beta(t), \beta'(1)), \quad 0 \leq t \leq 1.$$

Theorem 2.3. *Assume $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, $\frac{\partial}{\partial y_1} f = f_{y_1} : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and $\frac{\partial}{\partial y_2} f = f_{y_2} : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous. Assume $f_{y_1} > 0$ on $[0,1] \times \mathbb{R}^2$ and assume $f_{y_2} > 0$ on $[0,1] \times \mathbb{R}^2$. Assume α is a lower solution of the boundary value problem* (2.1), (2.2) *and assume β is an upper solution of the boundary value problem* (2.1), (2.2)*. Then*

$$\alpha(t) \leq \beta(t), \quad 0 \leq t \leq 1.$$

Proof. With the exception of the case $t_0 = 1$, $(\alpha - \beta)'(1) = 0$, the proof of this theorem is simply obtained by replacing y with α, z with β and

$$(y-z)''(t) = f(t, y(t), y'(t)) - f(t, z(t), z'(t)),$$

with

$$(\alpha-\beta)''(t) \geq f(t_0, \alpha(t), \alpha'(t)) - f(t_0, \beta(t), \beta'(t))$$

in the proof of Theorem 2.1. We produce the details for the case $t_0 = 1$, $(\alpha - \beta)'(1) = 0$ here.

Assume $\alpha - \beta$ has a positive maximum at $t_0 = 1$ and assume $(\alpha - \beta)'(1) = 0$. Then $(\alpha - \beta)''(1) \leq 0$. However, since α and β are lower and upper solutions, respectively, of (2.1), (2.2), and $f_{y_1} > 0$,

$$(\alpha-\beta)''(1) \geq f(1, \alpha(1), \alpha'(1)) - f(1, \beta(1), \beta'(1)) > 0,$$

producing the usual contradiction. □

Remark 2.4. *We point out here that Theorem 2.1 follows as an immediate corollary of Theorem 2.3 since a solution of boundary value problem* (2.1), (2.2) *is both an upper solution of the boundary value problem* (2.1), (2.2) *and a lower solution of the boundary value problem* (2.1), (2.2).

Remark 2.5. *The argument for the case $t_0 = 1$, $(\alpha - \beta)'(1) = 0$, in the proof of Theorem 2.3 can be used for the case $t_0 = 1$, $(y - z)'(1) = 0$, in the proof of Theorem 2.1.*

3. Existence of solutions

To obtain existence of solutions, we shall apply the shift argument [17]. Assume $\lambda > 0$ and consider the shifted equation

$$y''(t) - \lambda y'(t) = g(t, y(t), y'(t)) = f(t, y(t), y'(t)) - \lambda y'(t), \quad 0 \leq t \leq 1. \tag{3.1}$$

The boundary value problem, (3.1), (2.2) is not at resonance for any $\lambda > 0$ since Theorem 2.1 (modified with the hypothesis $f_{y_1} \geq 0$ on $[0,1] \times \mathbb{R}$) applies to the homogeneous problem

$$y''(t) - \lambda y'(t) = 0, \quad 0 \leq t \leq 1,$$

with boundary conditions (2.2), if we rewrite $y''(t) - \lambda y'(t) = 0$ in the form $y''(t) = \lambda y'(t)$. Thus, the Green's function for the boundary value problem (3.1), (2.2) can be constructed and has the form

$$G(\lambda; t, s) = \begin{cases} \frac{e^{\lambda t} e^{\lambda(1-s)} - e^{\lambda(1-s)}}{\lambda(1-e^{\lambda})}, & 0 \leq t \leq s \leq 1, \\ \\ \frac{e^{\lambda t} e^{\lambda(1-s)} - e^{\lambda(1-s)}}{\lambda(1-e^{\lambda})} + \frac{e^{\lambda(t-s)} - 1}{\lambda}, & 0 \leq s \leq t \leq 1. \end{cases} \tag{3.2}$$

Note that

$$\frac{\partial}{\partial t} G(\lambda; t, s) = G_t(\lambda; t, s) = \begin{cases} \frac{e^{\lambda t} e^{\lambda(1-s)}}{(1-e^{\lambda})}, & 0 \leq t \leq s \leq 1, \\ \\ \frac{e^{\lambda t} e^{\lambda(1-s)}}{(1-e^{\lambda})} + e^{\lambda(t-s)}, & 0 \leq s \leq t \leq 1. \end{cases} \tag{3.3}$$

We observe the following properties of $G(\lambda; t, s)$.

Theorem 3.1. *Let $G(\lambda; t, s)$ denote the Green's function of the boundary value problem* (3.1), (2.2). *Then*

- $G(\lambda; t, s) < 0$, $(t, s) \in (0, 1] \times [0, 1]$,
- $G_t(\lambda; t, s) < 0$, $(t, s) \in [0, 1] \times [0, 1]$,

- $\max\{\max_{0\leq t\leq 1}\int_0^1 |G(\lambda;t,s)|ds, \max_{0\leq t\leq 1}\int_0^1 |G_t(\lambda;t,s)|ds\} = \frac{1}{\lambda}$.

Proof. The term $\frac{e^{\lambda t}e^{\lambda(1-s)}-e^{\lambda(1-s)}}{\lambda(1-e^{\lambda})} < 0$ on $[0,1]\times[0,1]$. The term

$$\frac{e^{\lambda t}e^{\lambda(1-s)}-e^{\lambda(1-s)}}{\lambda(1-e^{\lambda})} + \frac{e^{\lambda(t-s)}-1}{\lambda}$$

is decreasing in t for $0\leq s\leq t\leq 1$ and negative at $t=s$. And so, $G(\lambda;t,s)<0$, on $[0,1]\times[0,1]$. The sign of $G_t(\lambda;t,s)$ is determined similarly.

Note that $\int_0^1 G(\lambda;t,s)ds$ is the solution of the boundary value problem (2.1), (2.2) for $f\equiv 1$. Thus, $\int_0^1 G(\lambda;t,s)ds = \frac{-t}{\lambda}$. Since $G(\lambda;t,s)<0$, it follows that $\int_0^1 |G(\lambda;t,s)|ds = \frac{t}{\lambda}$. So, $\max_{0\leq t\leq 1}\int_0^1 |G(\lambda;t,s)|ds = \frac{1}{\lambda}$. Since $\int_0^1 G(\lambda;t,s)ds = \frac{-t}{\lambda}$, then $\int_0^1 G_t(\lambda;t,s)ds = \frac{-1}{\lambda}$. Thus, $G_t(\lambda;t,s)<0$ implies $\max_{0\leq t\leq 1}\int_0^1 |G_t(\lambda;t,s)|ds = \frac{1}{\lambda}$.

□

Remark 3.2. *If we write*

$$G(\lambda;t,s) = \int_0^t G_t(\lambda;r,s)dr \tag{3.4}$$

then $G(\lambda;t,s)$ is represented as the convolution of Green's functions for lower order problems. $G_t(\lambda;t,s)$ denotes the Green's function for the periodic boundary value problem

$$u'(t)-\lambda u(t)=h, \quad u(0)=u(1)$$

and the function

$$K(t,s) = \begin{cases} 0, & 0\leq t\leq s\leq 1,\\ 1, & 0\leq s\leq t\leq 1,\end{cases}$$

denotes the Cauchy function for the initial value problem

$$u'(t)=k, \quad u(0)=0.$$

Then

$$G(\lambda;t,s) = \int_0^1 K(t,r)G_t(\lambda;r,s)dr.$$

Periodic boundary value problems have been studied extensively and we refer the reader to the recent monograph [7] for an authoritative account. The proof of Theorem 3.1 can be obtained as a corollary of the observation of (3.4) *as a convolution of Green's functions for lower order problems.*

Prior to obtaining existence of solutions of the boundary value problem (2.1), (2.2) we state without proof versions of the Kamke convergence criterion for solutions of initial value problems for ordinary differential equations [16, 18]. We also state a version of the Schauder fixed point theorem [24].

Definition 3.3. *Assume $g(t,y_1,y_2):[0,1]\times\mathbb{R}^2\to\mathbb{R}^2$ is continuous. We say g satisfies a Nagumo condition with respect to y_2 if, for each $M>0$, there exists h_M defined on $(0,\infty)$ such that*

$$|g(t,y_1,y_2)|\leq h_M(|y_2|), \qquad \textit{for all } (t,y_1,y_2)\in[0,1]\times[-M,M]\times\mathbb{R},$$

and such that

$$\int_0^\infty \frac{s}{h_M}(s)ds = +\infty.$$

An important consequence of the Nagumo condition is that if g satisfies a Nagumo condition, then solutions of $y''(t) = g(t, y(t), y'(t))$ either extend to the interval $[0, 1]$ or the functional value $y(t)$ becomes unbounded on its maximal interval of existence. We refer the reader to [16] or [18] for further details. The next theorem is a version of the Kamke convergence criterion and again we refer the reader to [16] or [18] for further details.

Theorem 3.4. *For $k \in \{0, 1, \dots\}$ assume the functions $f_k(t, y_1, y_2)$ are continuous on $[0, 1] \times \mathbb{R}^2$ and assume that there exists $f(t, y_1, y_2)$ such that*

$$\lim_{k\to\infty} f_k(t, y_1, y_2) = f(t, y_1, y_2)$$

uniformly on compact subsets of $[0, 1] \times \mathbb{R}^2$. Assume f satisfies a Nagumo condition in y_2. Assume that $\{t_k\}_{k=0}^{\infty} \subset [0, 1]$ such that $\lim_{k\to\infty} t_k = t^*$. Assume for each integer $k \geq 0$, y_k is a solution of $y''(t) = f_k(t, y(t), y'(t))$ which is defined on $[0, 1]$. Assume further that there exist $y_1, y_2 \in \mathbb{R}$ such that $\lim_{k\to\infty} y_k^{i-1}(t_k) = y_i$, $i = 1, 2$. Then there exists a subsequence $\{y_{k_j}\}$ of $\{y_k\}$ and a solution y of

$$y''(t) = f(t, y(t), y'(t)), \quad 0 \leq t \leq 1,$$

such that $y^{(i-1)}(t^*) = y_i$, $i = 1, 2$ and $y_{k_j}^{(i-1)}(t)$ converges uniformly to $y^{(i-1)}(t)$ on $[0, 1]$, $i = 1, 2$.

We shall employ the Kamke criterion in the form of the following corollary which is proved in [15].

Corollary 3.5. *For $k \in \{0, 1, \dots\}$ assume the functions $f_k(t, y_1, y_2)$ are continuous on $[0, 1] \times \mathbb{R}^2$ and assume that there exists $f(t, y_1, y_2)$ such that*

$$\lim_{k\to\infty} f_k(t, y_1, y_2) = f(t, y_1, y_2)$$

uniformly on compact subsets of $[0, 1] \times \mathbb{R}^2$. Assume f satisfies a Nagumo condition in y_2. Assume that, for $k = 0, 1, \dots$, $y_k(t)$ is a solution of $y''(t) = f_k(t, y(t), y'(t))$ and assume $y_k(t)$ satisfies the boundary conditions (2.2). Assume that $\{y_k\}$ is monotone decreasing and bounded below by a continuously differentiable function $\alpha(t)$. Then there exists a solution y of

$$y''(t) = f(t, y(t), y'(t)), \quad 0 \leq t \leq 1,$$

such that $y_k^{(i-1)}(t)$ converges uniformly to $y^{(i-1)}(t)$, $i = 1, 2$.

Our final preliminary result is the following version of the Schauder fixed point theorem.

Theorem 3.6. *If $\mathcal{U}$ is a closed convex subset of a Banach space $\mathcal{B}$, if $T : \mathcal{U} \to \mathcal{U}$ is continuous on $\mathcal{U}$, and if $\overline{T(\mathcal{U})}$ is a compact subset of $\mathcal{B}$, then T has a fixed point in $\mathcal{U}$.*

We are now in a position to provide sufficient conditions for the existence of a solution of the boundary value problem (2.1), (2.2). It is of interest to note that the hypothesis $f_{y_1} > 0$ on $[0, 1] \times \mathbb{R}^2$ is not employed and the hypothesis $f_{y_2} > 0$ on $[0, 1] \times \mathbb{R}^2$ will be employed at the boundary point $t_0 = 1$. In particular, the following theorem makes no claims to uniqueness of solutions.

Theorem 3.7. *Assume $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, $\frac{\partial}{\partial y_1} f = f_{y_1} : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and $\frac{\partial}{\partial y_2} f = f_{y_2} : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous. Assume $f_{y_2} > 0$ on $[0, 1] \times \mathbb{R}^2$. Assume f satisfies a Nagumo condition in y_2. Assume α is a lower solution of the boundary value problem (2.1), (2.2) and assume β is an upper solution of the boundary value problem (2.1), (2.2) and assume*

$$\alpha(t) \leq \beta(t), \quad 0 \leq t \leq 1.$$

Then there exists a solution y of (2.1), (2.2) satisfying

$$\alpha(t) \leq y(t) \leq \beta(t), \quad 0 \leq t \leq 1.$$

Proof. Let $\lambda > 0$ and first define a truncation, F, of $g(t, y(t), y'(t)) = f(t, y(t), y'(t)) - \lambda y'(t)$ by

$$F(t, y(t), y'(t)) = \begin{cases} f(t, \beta(t), y'(t)) - \lambda y'(t) + \frac{y(t)-\beta(t)}{1+y(t)-\beta(t)}, & y(t) > \beta(t), \\ f(t, y(t), y'(t)) - \lambda y'(t), & \alpha(t) \le y(t) \le \beta(t), \\ f(t, \alpha(t), y'(t)) - \lambda y'(t) + \frac{y(t)-\alpha(t)}{1+\alpha(t)-y(t)}, & y(t) < \alpha(t). \end{cases}$$

Now, let $L > 0$ be such that

$$|\alpha'(t)| \le L, \quad |\beta'(t)| \le L, \quad 0 \le t \le 1.$$

Define a further truncation F_k, for $k = 0, 1, \dots$ by

$$F_k(t, y(t), y'(t)) = \begin{cases} F(t, y(t), L + k), & y'(t) > L + k, \\ F(t, y(t), y'(t)), & |y'(t)| \le L + k, \\ F(t, y(t), L + k), & y'(t) < -L - k. \end{cases}$$

For each integer $k \ge 0$, $F_k(t, y_1, y_2)$ is continuous and bounded on $[0, 1] \times \mathbb{R}^2$ and

$$\lim_{k \to \infty} F_k(t, y_1, y_2) = F(t, y_1, y_2)$$

uniformly on compacts subsets of $[0, 1] \times \mathbb{R}^2$. More importantly, in order to apply Corollary 3.5 note that

$$\lim_{k \to \infty} F_k(t, y_1, y_2) + \lambda y_2 = F(t, y_1, y_2) + \lambda y_2$$

uniformly on compacts subsets of $[0, 1] \times \mathbb{R}^2$. For each integer $k \ge 0$, define an operator

$$T_k : C^1[0, 1] \to C^1[0, 1]$$

by

$$T_k y(t) = \int_0^1 G(\lambda; t, s) F_k(s, y(s), y'(s)) ds$$

where $G(\lambda; t, s)$ is given by (3.2). The purpose of the Green's function is to provide the following equivalent statements. A function $y \in C^2[0, 1]$ is a solution of the boundary value problem

$$y''(t) - \lambda y'(t) = F_k(t, y(t), y'(t)), \quad 0 \le t \le 1, \tag{3.5}$$

with boundary conditions (2.2) if, and only if, $y \in C^1[0, 1]$ and

$$y(t) = \int_0^1 G(\lambda; t, s) F_k(s, y(s), y'(s)) ds, \quad 0 \le t \le 1.$$

That is, $y \in C^2[0, 1]$ is a solution of the boundary value problem (3.5), (2.2) if, and only if, $y \in C'[0, 1]$ and

$$y(t) = T_k y(t), \quad 0 \le t \le 1.$$

Since $G(\lambda; t, s)$ and $G_t(\lambda; t, s)$ are continuous on $[0, 1] \times [0, 1]$ if follows that $T_k : C^1[0, 1] \to C^1[0, 1]$. Since the further truncation, F_k, is bounded and continuous on $[0, 1] \times \mathbb{R}^2$ it is a straightforward application of the Schauder fixed point theorem to show that the boundary value problem (3.5), (2.2) has a solution. To see this, let

$$M = \sup\{|F_k(t, y_1, y_2)| : 0 \le t \le 1, y \in \mathbb{R}\}$$

and recall

$$\frac{1}{\lambda} = \max\{\max_{0 \le t \le 1} \int_0^1 |G(\lambda; t, s)| ds, \max_{0 \le t \le 1} \int_0^1 |G_t(\lambda; t, s)| ds\}.$$

If $y \in C^1[0, 1]$, with

$$||y||_1 = \max\{\max_{0 \le t \le 1} |y'(t)|, \max_{0 \le t \le 1} |y'(t)|\},$$

then $T_k y \in C^1[0, 1]$ and

$$||T_k y||_1 \le \frac{M}{\lambda}.$$

Define
$$\mathcal{U} = \{y \in C^1[0,1] : ||y||_1 \leq \frac{M}{\lambda}\}.$$
Then $\mathcal{U}$ is a closed convex subset of $C^1[0,1]$ and $T_k : \mathcal{U} \to \mathcal{U}$. To show that $\overline{T(\mathcal{U})}$ is compact, we apply the Arzela-Ascoli Theorem to each set $\{(T_k y)' : y \in \mathcal{U}\}$ and $\{(T_k y) : y \in \mathcal{U}\}$. Each of the sets $\{(T_k y)' : y \in \mathcal{U}\}$ and $\{(T_k y) : y \in \mathcal{U}\}$ is uniformly bounded by $\frac{M}{\lambda}$. Since $\{(T_k y)' : y \in \mathcal{U}\}$ is uniformly bounded, an application of the mean value theorem implies that $\{(T_k y) : y \in \mathcal{U}\}$ is equicontinuous. Moreover, note that if $y \in \mathcal{U}$ then
$$(T_k y)''(t) = F_k(t, y(t), y'(t)) + \lambda y'(t), \quad 0 \leq t \leq 1,$$
and so, if $y \in \mathcal{U}$,
$$|(T_k y)''(t)| \leq 2M.$$
Thus, $\{(T_k y)'' : y \in \mathcal{U}\}$ is uniformly bounded and a further application of the mean value theorem implies that $\{(T_k y)' : y \in \mathcal{U}\}$ is equicontinuous. Thus, $\overline{T(\mathcal{U})}$ is compact and the Schauder fixed point theorem implies there exists a fixed point, $y_k \in \mathcal{U}$ of the operator T_k. Let y_k denote a fixed point of T_k and so y_k satisfies the boundary value problem
$$y''(t) - \lambda y'(t) = F_k(t, y(t), y'(t)), \quad 0 \leq t \leq 1,$$
$$y(0) = 0, \quad y'(0) = y'(1).$$

We now argue that
$$\alpha(t) \leq y_k(t) \leq \beta(t), \quad 0 \leq t \leq 1.$$
Details are similar to the proof of Theorem 2.1 and we highlight the differences in the details due to the truncation, F_k.

To see that $y_k(t) \leq \beta(t)$, $0 \leq t \leq 1$, first assume without loss of generality that $y_k - \beta$ has positive maximum at $t_0 \in (0,1)$. Then $(y_k - \beta)''(t_0) \leq 0$. Moreover, $y_k'(t_0) = \beta'(t_0)$ implies $|y_k'(t_0)| \leq L$; in particular,
$$F_k(t_0, y_k(t_0), y_k'(t_0)) = F(t_0, y_k(t_0), y_k'(t_0)).$$
Then
$$\begin{aligned}(y_k - \beta)''(t_0) &\geq f(t_0, \beta(t_0), y_k'(t_0)) - \lambda y_k'(t_0) \\ &+ \frac{y_k(t_0) - \beta(t_0)}{1 + y_k(t_0) - \beta(t_0)} + \lambda y_k'(t_0) - f(t_0, \beta(t_0), \beta(t_0)) \\ &= \frac{y_k(t_0) - \beta(t_0)}{1 + y_k(t_0) - \beta(t_0)} > 0,\end{aligned} \tag{3.6}$$
producing the usual contradiction. So, $t_0 \notin (0,1)$. It is important to note that the hypothesis $f_{y_1} > 0$ on $[0,1] \times \mathbb{R}^2$ is not required to obtain the usual contradiction.

Second, $t_0 \neq 0$ by the boundary condition (2.2).

Finally assume $y_k - \beta$ has a positive maximum at $t_0 = 1$ and so $(y_k - \beta)'(1) \geq 0$. If $(y_k - \beta)'(1) = 0$, then $(y_k - \beta)''(1) \leq 0$. Then the calculation in (3.6) is valid at $t_0 = 1$ and $(y_k - \beta)''(1) > 0$, which produces the usual contradiction.

For the final case, assume $t_0 = 1$ and $(y_k - \beta)'(1) > 0$. As in the proof of Theorem 2.1, we show there $(y_k - \beta)'(t)$ does not change sign in $(0,1)$. If $(y_k - \beta)'(t)$ does change sign, there exist $0 < \tau < c < 1$ such that $(y_k - \beta)(t)$ has a positive absolute maximum on $[0,c]$ at τ and so, $(y_k - \beta)''(\tau) \leq 0$. The calculation in (3.6) at $t_0 = \tau$ produces the usual contradiction and so $(y_k - \beta)'(t)$ does not change sign in $(0,1)$. As in the proof of Theorem 2.1, $(y_k - \beta)'(t) > 0$ for $t \in [0,1]$ implies $(y_k - \beta)(t)$ is increasing; in particular, $(y_k - \beta)(t) > 0$, for $t \in (0,1]$.

To see that $(y_k - \beta)'(t)$ is monotone increasing (which contradicts $(y_k - \beta)'(0) = (y_k - \beta)'(1)$ and hence completes the proof) we argue that $(y_k - \beta)''(t) > 0$ for $0 \leq t \leq 1$. Note that $(y_k - \beta)'(t) > 0$ implies $y_k'(t) > L + k$ or $|y_k'(t)| \leq L + k$. If $y_k'(t) > L + k$ then
$$\begin{aligned}(y_k - \beta)''(t) &\geq f(t, \beta(t), L + K) - \lambda(L + k) + \frac{y_k(t) - \beta(t)}{1 + y_k(t) - \beta(t)} \\ &+ \lambda y_k'(t) - f(t, \beta(t), \beta'(t)) \\ &\geq \frac{y_k(t) - \beta(t)}{1 + y_k(t) - \beta(t)} > 0\end{aligned}$$

and if $|y_k'(t)| \le L + k$ then

$$\begin{aligned}(y_k-\beta)''(t) &\ge f(t,\beta(t),y_k'(t)) - \lambda y_k'(t) + \frac{y_k(t)-\beta(t)}{1+y_k(t)-\beta(t)}\\ &+ \lambda y_k'(t) - f(t,\beta(t),\beta'(t))\\ &\ge \frac{y_k(t)-\beta(t)}{1+y_k(t)-\beta(t)} > 0.\end{aligned}$$

So, $(y_k-\beta)'$ is monotone increasing which implies the contradiction $(y_k-\beta)'(0) < (y_k-\beta)'(1)$.

Thus, $y_k-\beta$ does not have a positive maximum on $[0,1]$ and $y_k(t) \le \beta(t)$, $0 \le t \le 1$. Similarly, $\alpha(t) \le y_k(t)$, $0 \le t \le 1$ and we conclude

$$\alpha(t) \le y_k(t) \le \beta(t), \quad 0 \le t \le 1.$$

For each $k = 0,1,\dots$, there exists $t_k \in (0,1)$ such that

$$|y_k'(t_k)| = |y_k(1)-y_k(0)| \le \max\{|\beta(0)-\alpha(1)|, |\beta(1)-\alpha(0)|\}.$$

Thus, each sequence $\{y_k(t_k)\}$ and $\{y_k'(t_k)\}$ is bounded. Hence, there exists a subsequence of $\{t_k\}$, relabeled as the original sequence, and $y_1, y_2 \in \mathbb{R}$, such that

$$\lim t_k = t^*, \quad \lim y_k^{(i-1)}(t_k) = y_i, \quad i = 1,2.$$

Theorem 3.4 applies and there exists a solution, y, of $y''(t) - \lambda y'(t) = F(t,y(t),y'(t))$ on $[0,1]$ and a subsequence of $\{y_k\}$ converging in $C^1[0,1]$ to y. Thus

$$\alpha(t) \le y_k(t) \le \beta(t), \quad 0 \le t \le 1$$

implies

$$\alpha(t) \le y(t) \le \beta(t), \quad 0 \le t \le 1$$

and $F(t,y(t),y'(t)) = g(t,y(t),y'(t))$. Moreover, y satisfies (2.2) (since each y_k satisfies (2.2)). Thus, y is a solution of the boundary value problem (3.1), (2.2) which implies y is a solution of the boundary value problem (2.1), (2.2). □

4. The monotone method

In this section we develop the monotone method. The construction is modeled after the construction in [15].

Theorem 4.1. *Assume $f : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ is continuous, $\frac{\partial}{\partial y_1} f = f_{y_1} : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ is continuous and $\frac{\partial}{\partial y_2} f = f_{y_2} : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ is continuous. Assume $f_{y_1} > 0$ on $[0,1]\times\mathbb{R}^2$ and assume $f_{y_2} > 0$ on $[0,1]\times\mathbb{R}^2$. Assume f satisfies a Nagumo condition in y_2. Assume further that $\frac{\partial^2}{\partial y_1^2} f = f_{y_1y_1} : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ is continuous. Assume α_0 is a lower solution of the boundary value problem* (2.1), (2.2) *and assume β_0 is an upper solution of the boundary value problem* (2.1), (2.2)*. Then there exists a unique solution y of* (2.1), (2.2) *satisfying*

$$\alpha_0(t) \le y(t) \le \beta_0(t), \quad 0 \le t \le 1.$$

Moreover, there exist sequences $\{\alpha_n\}, \{\beta_n\}$ of lower and upper solutions, respectively, of the boundary value problem (2.1), (2.2)*, each of which converges in $C[0,1]$ to the unique solution y, of the boundary value problem* (2.1), (2.2) *and satisfy*

$$\alpha_n(t) \le \alpha_{n+1}(t) \le y(t) \le \beta_{n+1}(t) \le \beta_n(t), \quad 0 \le t \le 1.$$

Proof. The existence of a unique solution y satisfying

$$\alpha_0(t) \le y(t) \le \beta_0(t), \quad 0 \le t \le 1,$$

follows from Theorems 2.1, 2.3 and 3.7.

Let $F(t,y_1):[0,1]\times\mathbb{R}\to\mathbb{R}$ be such that $F, F_{y_1}, F_{y_1y_1}$ are continuous on $[0,1]\times\mathbb{R}$ and

$$F_{y_1y_1}(t,y_1)\geq 0,\quad (t,y_1)\in[0,1]\times\mathbb{R}. \tag{4.1}$$

Define $\phi(t,y_1,y_2)=F(t,y_1)-f(t,y_1,y_2)$ on $[0,1]\times\mathbb{R}^2$. It follows from (4.1) that if $y_1,z_1\in\mathbb{R}$, then $F(t,z_1)\geq F(t,y_1)+F_{y_1}(t,y_1)(z_1-y_1)$. Thus, for $y_1,z_1,y_2,z_2\in\mathbb{R}$,

$$f(t,z_1,z_2)\geq f(t,y_1,y_2)+F_{y_1}(t,y_1)(z_1-y_1)-(\phi(t,z_1,z_2)-\phi(t,y_1,y_2)). \tag{4.2}$$

The proof proceeds by induction on n and so for the base case argument, define the function

$$\begin{aligned}h(t,y_1,y_2;\alpha_0,\beta_0,\alpha_0')&=f(t,\alpha_0(t),\alpha_0'(t))+F_{y_1}(t,\beta_0(t))(y_1-\alpha_0)(t)\\&\quad-(\phi(t,y_1,y_2)-\phi(t,\alpha_0(t),\alpha_0'(t))\end{aligned}$$

and

$$\begin{aligned}k(t,y_1,y_2;\beta_0,\beta_0')&=f(t,\beta_0(t),\beta_0'(t))+F_{y_1}(t,\beta_0(t))(y_1-\beta_0)(t)\\&\quad-(\phi(t,y_1,y_2)-\phi(t,\beta_0(t),\beta_0'(t)).\end{aligned}$$

First, consider the boundary value problem

$$y''(t)=h(t,y(t),y'(t);\alpha_0,\beta_0,\alpha_0'),\quad 0\leq t\leq 1, \tag{4.3}$$

with boundary conditions (2.2). Since f satisfies a Nagumo condition with respect to y_2 it is readily seen that h satisfies a Nagumo condition with respect to y_2. Moreover, $h_{y_2}=f_{y_2}>0$ on $[0,1]\times\mathbb{R}^2$.

We shall show that α_0 and β_0 are lower and upper solutions of the boundary value problem (4.3), (2.2) and apply Theorem 3.7 to obtain a solution, $\alpha_1(t)$, of the boundary value problem (4.3), (2.2) satisfying

$$\alpha_0(t)\leq\alpha_1(t)\leq\beta_0(t),\quad 0\leq t\leq 1.$$

Note that

$$h(t,\alpha_0(t),\alpha_0'(t);\alpha_0,\beta_0,\alpha_0')=f(t,\alpha_0(t),\alpha_0'(t)),\quad 0\leq t\leq 1,$$

and so,

$$\alpha_0''(t)\geq f(t,\alpha_0(t),\alpha_0'(t))=h(t,\alpha_0(t),\alpha_0'(t);\alpha_0,\beta_0,\alpha_0'),\quad 0\leq t\leq 1.$$

Moreover, (4.2) implies since

$$\begin{aligned}f(t,\beta_0(t),\beta_0'(t))&\leq f(t,\alpha_0(t),\alpha_0'(t))-F_{y_1}(t,\beta_0(t))(\alpha_0-\beta_0)(t)\\&\quad+(\phi(t,\alpha_0(t),\alpha_0'(t))-\phi(t,\beta_0(t),\beta_0'(t))\\&=h(t,\beta_0(t),\beta_0'(t);\alpha_0,\beta_0,\alpha_0'),\quad 0\leq t\leq 1.\end{aligned}$$

By assumption, each of α_0 and β_0 satisfy the boundary conditions, (2.2) and so, α_0 and β_0 are lower and upper solutions of the boundary value problem (4.3), (2.2). Thus, by Theorem 3.7 there exists a solution, $\alpha_1(t)$, of the boundary value problem (4.3), (2.2) satisfying

$$\alpha_0(t)\leq\alpha_1(t)\leq\beta_0(t),\quad 0\leq t\leq 1.$$

Second, consider the boundary value problem

$$y''(t)=k(t,y(t),y'(t);\beta_0,\beta_0'),\quad 0\leq t\leq 1, \tag{4.4}$$

with boundary conditions (2.2). Again k satisfies a Nagumo condition with respect to y_2. Moreover, $k_{y_2}=f_{y_2}>0$ on $[0,1]\times\mathbb{R}^2$. Note that

$$f(t,\beta_0(t),\beta_0'(t))=k(t,\beta_0(t),\beta_0'(t);\beta_0,\beta_0'),\quad 0\leq t\leq 1,$$

and so, β_0 is an upper solution of the boundary value problem (4.4), (2.2). Moreover, (4.2) readily implies that

$$f(t,\alpha_0(t),\alpha_0'(t)) \geq k(t,\alpha_0(t),\alpha_0'(t);\beta_0,\beta_0'), \quad 0 \leq t \leq 1,$$

and so, α_0 is a lower solution of the boundary value problem (4.4), (2.2). Thus, by Theorem 3.7 there exists a solution, $\beta_1(t)$, of the boundary value problem (4.3), (2.2) satisfying

$$\alpha_0(t) \leq \beta_1(t) \leq \beta_0(t), \quad 0 \leq t \leq 1.$$

For the final step of the base case argument, we show that α_1 and β_1 are lower and upper solutions, respectively, of the boundary value problem (2.1), (2.2) and then it will follow by Theorem 2.3 that $\alpha_1(t) \leq \beta_1(t)$, $0 \leq t \leq 1$, and so

$$\alpha_0(t) \leq \alpha_1(t) \leq \beta_1(t) \leq \beta_0(t), \quad 0 \leq t \leq 1.$$

Employ (4.1) and (4.2) and obtain

$$\begin{aligned}
\alpha_1''(t) &= h(t,\alpha_1(t),\alpha_1'(t);\alpha_0,\beta_0,\alpha_0') \\
&= f(t,\alpha_0(t),\alpha_0'(t)) + F_{y_1}(t,\beta_0(t))(\alpha_1(t)-\alpha_0(t)) \\
&- (\phi(t,\alpha_1(t),\alpha_1'(t)) - \phi(t,\alpha_0(t),\alpha_0'(t)) \\
&\geq f(t,\alpha_1(t),\alpha_1'(t)) + F_{y_1}(t,\alpha_1(t))(\alpha_0(t)-\alpha_1(t)) \\
&- (\phi(t,\alpha_0(t),\alpha_0'(t)) - \phi(t,\alpha_1(t),\alpha_1'(t)) \\
&+ F_{y_1}(t,\beta_0(t))(\alpha_1(t)-\alpha_0(t)) - (\phi(t,\alpha_1(t),\alpha_1'(t)) - \phi(t,\alpha_0(t),\alpha_0'(t))) \\
&= f(t,\alpha_1(t),\alpha_1'(t)) + (F_{y_1}(t,\beta_0(t)) - F_{y_1}(t,\alpha_1(t)))(\alpha_1(t)-\alpha_0(t)) \\
&\geq f(t,\alpha_1(t),\alpha_1'(t).
\end{aligned}$$

In a similar way, (4.2) implies that

$$\beta_1''(t) \leq f(t,\beta_1(t),\beta_1'(t)), \quad 0 \leq 1 \leq t.$$

Apply Theorem 2.3 to obtain $\alpha_1(t) \leq \beta_1(t)$, $0 \leq t \leq 1$, and so

$$\alpha_0(t) \leq \alpha_1(t) \leq \beta_1(t) \leq \beta_0(t), \quad 0 \leq t \leq 1.$$

For the induction hypotheses, assume that each of the sequences,

$$\{\alpha_k\}_{k=1}^n \quad \text{and} \quad \{\beta_k\}_{k=1}^n,$$

have been constructed inductively such that for each k,

$$\begin{aligned}
h(t,y_1,y_2;\alpha_k,\beta_k,\alpha_k') = f(t,\alpha_k(t),\alpha_k'(t)) + F_{y_1}(t,\beta_k(t))(y_1-\alpha_k)(t)) \\
- (\phi(t,y_1,y_2) - \phi(t,\alpha_k(t),\alpha_k'(t))
\end{aligned}$$

and

$$\begin{aligned}
k(t,y_1,y_2;\beta_k,\beta_k') = f(t,\beta_k(t),\beta_k'(t)) + F_{y_1}(t,\beta_k(t))(y_1-\beta_k)(t)) \\
- (\phi(t,y_1,y_2) - \phi(t,\beta_k(t),\beta_k'(t)),
\end{aligned}$$

α_k is a solution of the boundary value problem

$$y''(t) = h(t,y(t),y'(t);\alpha_{k-1},\beta_{k-1},\alpha_{k-1}'), \quad 0 \leq t \leq 1,$$

with boundary conditions (2.2) and β_k is a solution of the boundary value problem

$$y''(t) = k(t,y(t),y'(t);\beta_k,\beta_k'), \quad 0 \leq t \leq 1,$$

with boundary conditions (2.2). Assume α_k, β_k, $k = 1, \ldots, n$ denote a lower solution and an upper solution, respectively of (2.1), (2.2) and for each $k = 0, \ldots, n$,

$$\alpha_{k-1}(t) \leq \alpha_k(t) \leq y(t) \leq \beta_k(t) \leq \beta_{k-1}(t), \quad 0 \leq t \leq 1,$$

where y is the unique solution of the boundary value problem (2.1), (2.2).

To finish the induction argument, consider the ordinary differential equation

$$y''(t) = h(t, y(t), y'(t); \alpha_n, \beta_n, \beta_n'), \quad 0 \leq t \leq 1. \tag{4.5}$$

Note that

$$h(t, \alpha_n(t), \alpha_n'(t); \alpha_n, \beta_n, \alpha_n') = f(t, \alpha_n(t), \alpha_n'(t)), \quad 0 \leq t \leq 1$$

and (4.1) and (4.2) imply

$$h(t, \beta_n(t), \beta_n'(t); \alpha_n, \beta_n, \alpha_n') \geq f(t, \beta_n(t), \beta_n'(t)), \quad 0 \leq t \leq 1.$$

So, α_n, and β_n denote a lower and an upper solution of the boundary value problem (4.5), (2.2) respectively as well. Since h satisfies the hypotheses of Theorem 3.7, there exists a solution, $\alpha_{n+1}(t)$, of the boundary value problem (4.5), (2.2) satisfying

$$\alpha_n(t) \leq \alpha_{n+1}(t) \leq \beta_n(t), \quad 0 \leq t \leq 1.$$

Moreover, for $0 \leq t \leq 1$,

$$\alpha_{n+1}''(t) = h(t, \alpha_{n+1}(t), \alpha_{n+1}'(t); \alpha_n, \beta_n, \alpha_n') \geq f(t, \alpha_{n+1}(t), \alpha_{n+1}'(t)),$$

and α_{n+1} is a lower solution (2.1), (2.2).

Similarly, consider the ordinary differential equation

$$y''(t) = k(t, y(t), y'(t); \beta_n, \beta_n'), \quad 0 \leq t \leq 1. \tag{4.6}$$

Note that

$$k(t, \beta_n(t), \beta_n'(t); \beta_n, \beta_n') = f(t, \beta_n(t), \beta_n'), \quad 0 \leq t \leq 1,$$

and (4.2) implies

$$k(t, \alpha_n(t), \alpha_n'(t); \beta_n, \beta_n') \leq f(t, \alpha_n(t), \alpha_n'(t)), \quad 0 \leq t \leq 1.$$

So, α_n and β_n denote a lower and an upper solution of the boundary value problem (4.6), (2.2) respectively as well. Since k satisfies the hypotheses of Theorem 3.7, there exists a solution, $\beta_{n+1}(t)$, of the boundary value problem (4.6), (2.2) satisfying

$$\alpha_n(t) \leq \beta_{n+1}(t) \leq \beta_n(t), \quad 0 \leq t \leq 1.$$

Moreover, for $0 \leq t \leq 1$,

$$\beta_{n+1}''(t) = k(t, \beta_{n+1}(t), \beta_{n+1}'(t); \beta_n, \beta_n') \leq f(t, \beta_{n+1}(t), \beta_{n+1}'),$$

and β_{n+1} is an upper solution of (2.1), (2.2).

Finally, apply Theorem 2.3 and Theorem 3.7 to obtain

$$\alpha_n(t) \leq \alpha_{n+1}(t) \leq y(t) \leq \beta_{n+1}(t) \leq \beta_n(t), \quad 0 \leq t \leq 1,$$

where y is the unique solution of the boundary value problem (2.1), (2.2).

To complete the proof, $\{\alpha_n\}$ and $\{\beta_n\}$ are monotone sequences of continuous functions bounded above or below, respectively, on a compact domain. So by Dini's theorem, each converges uniformly to $\alpha(t), \beta(t)$

respectively on $[0,1]$. We can not apply Corollary 3.5 directly since neither $h(t,y_1,y_2;\alpha_n,\beta_n,\beta_n')$ nor $k(t,y_1,y_2;\beta_n,\beta_n')$ converge uniformly to $f(t,y_1,y_2)$ uniformly on compact domains of $[0,1]\times\mathbb{R}^2$. To see this, write

$$\begin{aligned}h(t,y_1,y_2;\alpha_n,\beta_n,\beta_n') = f(t,y_1,y_2) + F_{y_1}(t,\beta_n(t))(y_1-\alpha_n(t))\\ + F(t,\alpha_n(t)) - F(t,y_1)\end{aligned}$$

and

$$\begin{aligned}k(t,y_1,y_2;\beta_n,\beta_n') = f(t,y_1,y_2) + F_{y_1}(t,\beta_n(t))(y_1-\beta_n(t))\\ + F(t,\beta_n(t)) - F(t,y_1).\end{aligned}$$

Define

$$\begin{aligned}\hat{h}(t,y_1,y_2;\alpha_n,\beta_n,\beta_n') = f(t,y_1,y_2) + F_{y_1}(t,\beta_n(t))(\alpha_{n+1}(t)-\alpha_n(t))\\ + F(t,\alpha_n(t)) - F(t,\alpha_{n+1}(t))\end{aligned}$$

and

$$\begin{aligned}\hat{k}(t,y_1,y_2;\beta_n,\beta_n') = f(t,y_1,y_2) + F_{y_1}(t,\beta_n(t))(\beta_{n+1}(t)-\beta_n(t))\\ + F(t,\beta_n(t)) - F(t,\beta_{n+1}(t)).\end{aligned}$$

Corollary 3.5 applies to the boundary value problem

$$y''(t) = \hat{h}(t,y(t),y'(t);\alpha_n,\beta_n,\beta_n') \quad 0\le t\le 1, \tag{4.7}$$

with boundary conditions (2.2) and α_{n+1} is a solution of the boundary value problem (4.7), (2.2). Now by Corollary 3.5, $\{\alpha_n\}$ converges in $C^1[0,1]$ to y, the unique solution of the boundary value problem (2.1), (2.2). Similarly $\{\beta_n\}$ converges in $C^1[0,1]$ to y, the unique solution of the boundary value problem (2.1), (2.2) and the proof is complete. □

5. A simplified problem

We have proved the main results in this paper for the case where f depends on the derivative of the unknown function. The results are also valid if f is independent of the derivative of the unknown function.

Assume $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is continuous and consider the boundary value problem

$$y''(t) = f(t,y(t)), \quad 0\le t\le 1, \tag{5.1}$$

with the boundary conditions (2.2).

We state the following three theorems without proof as the proofs follow in a straightforward way from the proofs of Theorems 2.1, 2.3 and 3.7 very closely and are more simple.

Theorem 5.1. *Assume $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is continuous and $\frac{\partial}{\partial y_1}f = f_{y_1}:[0,1]\times\mathbb{R}\to\mathbb{R}$ is continuous. Assume $f_{y_1}>0$ on $[0,1]\times\mathbb{R}$. Then solutions of the boundary value problem* (5.1), (2.2) *are unique, if they exist.*

Definition 5.2. *We say $\alpha\in C^2[0,1]$ is a lower solution of the boundary value problem* (5.1), (2.2) *if $\alpha(0)=0$, $\alpha'(0)=\alpha'(1)$ and*

$$\alpha''(t)\ge f(t,\alpha(t)), \quad 0\le t\le 1.$$

We say $\beta\in C^2[0,1]$ is an upper solution of the boundary value problem (5.1), (2.2) *if $\beta(0)=0$, $\beta'(0)=\beta'(1)$ and*

$$\beta''(t)\le f(t,\beta(t)), \quad 0\le t\le 1.$$

Theorem 5.3. *Assume* $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ *is continuous and* $\frac{\partial}{\partial y_1} f = f_{y_1} : [0,1] \times \mathbb{R} \to \mathbb{R}$ *is continuous. Assume* $f_{y_1} > 0$ *on* $[0,1] \times \mathbb{R}$. *Assume* α *is a lower solution of the boundary value problem* (5.1), (2.2) *and assume* β *is an upper solution of the boundary value problem* (5.1), (2.2). *Then*

$$\alpha(t) \leq \beta(t), \quad 0 \leq t \leq 1.$$

Theorem 5.4. *Assume* $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ *is continuous and* $\frac{\partial}{\partial y_1} f = f_{y_1} : [0,1] \times \mathbb{R} \to \mathbb{R}$ *is continuous. Assume* $f_{y_1} > 0$ *on* $[0,1] \times \mathbb{R}$. *Assume* α *is a lower solution of the boundary value problem* (5.1), (2.2) *and assume* β *is an upper solution of the boundary value problem* (5.1), (2.2). *Then there exists a unique solution* y *of* (5.1), (2.2) *satisfying*

$$\alpha(t) \leq y(t) \leq \beta(t), \quad 0 \leq t \leq 1.$$

Recall that in the proof of Theorem 4.1, the function F satisfying (4.1) was arbitrary. To develop the monotone method for the boundary value problem, (5.1), (2.2), set $F = f$, and so, F is not arbitrary. Then the function $\phi \equiv 0$ and the proof of Theorem 4.1 applies to obtain the monotone method for the boundary value problem (5.1), (2.2). We specify the h and k functions employed in the proof of the following theorem.

$$h(t, y_1; \alpha_n, \beta_n) = f(t, \alpha_n(t)) + f_{y_1}(t, \beta_n(t))(y_1 - \alpha_n)(t))$$

and

$$k(t, y_1; \beta_n) = f(t, \beta_n(t)) + f_{y_1}(t, \beta_n(t))(y_1 - \beta_n)(t)).$$

Theorem 5.5. *Assume* $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ *is continuous and* $\frac{\partial}{\partial y_1} f = f_{y_1} : [0,1] \times \mathbb{R} \to \mathbb{R}$ *is continuous. Assume* $f_{y_1} > 0$ *on* $[0,1] \times \mathbb{R}$. *Assume further that* $\frac{\partial^2}{\partial y_1^2} f = f_{y_1 y_1} : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ *is continuous and*

$$f_{y_1 y_1}(t, y_1) \geq 0 \quad (t, y_1) \in [0,1] \times \mathbb{R}.$$

Assume α_0 *is a lower solution of the boundary value problem* (5.1), (2.2) *and assume* β_0 *is an upper solution of the boundary value problem* (5.1), (2.2). *Then there exists a unique solution* y *of* (5.1), (2.2) *satisfying*

$$\alpha_0(t) \leq y(t) \leq \beta_0(t), \quad 0 \leq t \leq 1.$$

Moreover, there exist sequences $\{\alpha_n\}, \{\beta_n\}$ *of lower and upper solutions, respectively, of the boundary value problem* (5.1), (2.2), *each of which converges in* $C[0,1]$ *to the unique solution* y, *of the boundary value problem* (5.1), (2.2) *and satisfy*

$$\alpha_n(t) \leq \alpha_{n+1}(t) \leq y(t) \leq \beta_{n+1}(t) \leq \beta_n(t), \quad 0 \leq t \leq 1.$$

6. Two examples

The method of upper and lower solutions is only as good as one's ability to exhibit the existence of upper and lower solutions. In general, algorithms do not exist to construct upper or lower solutions. It was shown in [4] that the shift method implies that the nontrivial solutions of the homogeneous boundary value problem at resonance provide excellent candidates as upper or lower solutions.

Theorem 6.1. *Assume* $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ *is continuous,* $\frac{\partial}{\partial y_1} f = f_{y_1} : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ *is continuous and* $\frac{\partial}{\partial y_2} f = f_{y_2} : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ *is continuous. Assume* $f_{y_1} > 0$ *on* $[0,1] \times \mathbb{R}^2$ *and assume* $f_{y_2} > 0$ *on* $[0,1] \times \mathbb{R}^2$. *Assume* f *satisfies a Nagumo condition in* y_2. *Assume further that* $\frac{\partial^2}{\partial y_1^2} f = f_{y_1 y_1} : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ *is continuous. Assume there exists a continuous and bounded function* $\sigma : [0,1] \times \mathbb{R} \to \mathbb{R}$ *such that* $|\sigma(t, y_2)| \leq M_1$ *on* $[0,1] \times \mathbb{R}$. *Assume there exists a nondecreasing function* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *such that*

$$|f(t, y_1, y_2) - \lambda y_2| \leqslant \sigma(t, y_2)\psi(|y_1|), \quad (t, y_1, y_2) \in [0,1] \times \mathbb{R}^2$$

and there exists $M > 0$ such that

$$\frac{\lambda M}{M_1\psi(M)} > 1.$$

Then there exist sequences $\{\alpha_n\}, \{\beta_n\}$ of lower and upper solutions, respectively, of the boundary value problem (2.1), (2.2), each of which converges in $C^1[0,1]$ to the unique solution y, of the boundary value problem (2.1), (2.2) and satisfy

$$\alpha_n(t) \leq \alpha_{n+1}(t) \leq y(t) \leq \beta_{n+1}(t) \leq \beta_n(t), \quad 0 \leq t \leq 1.$$

Now Theorem 4.1 applies.

Proof. To exhibit β_0, an upper solution, set

$$\beta_0 = Mt.$$

Then

$$\beta_0''(t) - \lambda\beta_0'(t) = -\lambda M \leq -M_1\psi(M) \leq f(t, \beta_0(t), \beta_0'(t)) - \lambda\beta_0'(t).$$

To exhibit α_0 a lower solution, set

$$\alpha_0(t) = -Mt.$$

□

To provide a more specific example, let $a(t) \in C[0,1]$, $a(t) > 0$, and set

$$f(t, y_1, y_2) = \lambda y_2 + a(t)\tan^{-1}(y_2 + 1)(y_1^3 + y_1 + 1).$$

Let $K = \frac{\pi}{2}\max_{0\leq t\leq 1}|a(t)|$. Then $M_1 = K$, and $\psi(y_1) = y_1^3 + y_1 + 1$. Consider the linear function, $u(M) = \lambda M$ and the cubic function $v(M) = K(M^3 + M + 1)$. If

$$\lambda > K((\frac{1}{2})^{\frac{2}{3}} + 1 + (\frac{1}{2})^{\frac{-1}{3}}),$$

then there exist $0 < M_1 < M_2$ such that $(u - v)(M_1) = (u - v)(M_2) = 0$ and $(u - v)(M) > 0$ for $M_1 < M < M_2$. Thus, Theorem 6.1 applies if

$$\lambda > K((\frac{1}{2})^{\frac{2}{3}} + 1 + (\frac{1}{2})^{\frac{-1}{3}}).$$

In a similar way, if the growth condition $|f(t, y_1, y_2) - \lambda y_2| \leqslant \sigma(t, y_2)\psi(|y_1|)$ is replaced by a boundedness condition, there exists $M > 0$ such that

$$|f(t, y_1, y_2) - \lambda y_2| \leqslant M, \quad (t, y_1, y_2) \in [0,1] \times \mathbb{R}^2$$

then upper and lower solutions are readily exhibited. Set $\beta_0 = \frac{M}{\lambda}t$ and set $\alpha_0 = -\beta_0$.

Theorem 6.2. *Assume $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, $\frac{\partial}{\partial y_1}f = f_{y_1} : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and $\frac{\partial}{\partial y_2}f = f_{y_2} : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous. Assume $f_{y_1} > 0$ on $[0,1] \times \mathbb{R}^2$ and assume $f_{y_2} > 0$ on $[0,1] \times \mathbb{R}^2$. Assume f satisfies a Nagumo condition in y_2. Assume further that $\frac{\partial^2}{\partial y_1^2}f = f_{y_1y_1} : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous. Assume there exists $M > 0$ such that*

$$|f(t, y_1, y_2) - \lambda y_2| \leqslant M, \quad (t, y_1, y_2) \in [0,1] \times \mathbb{R}^2.$$

Then there exist sequences $\{\alpha_n\}, \{\beta_n\}$ of lower and upper solutions, respectively, of the boundary value problem (2.1), (2.2), each of which converges in $C^1[0,1]$ to the unique solution y, of the boundary value problem (2.1), (2.2) and satisfy

$$\alpha_n(t) \leq \alpha_{n+1}(t) \leq y(t) \leq \beta_{n+1}(t) \leq \beta_n(t), \quad 0 \leq t \leq 1.$$

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