



## A CONVERGENCE THEOREM IN GENERALIZED CONVEX CONE METRIC SPACES

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**ABSTRACT.** The aim of this work is to establish convergence theorem of a new iteration process for a finite family of  $I$ -asymptotically quasi-nonexpansive mappings and a finite family of asymptotically quasi-nonexpansive mappings in generalized convex cone metric spaces. Our result is valid in the whole space, whereas the results given in [4, 5] are valid in a nonempty convex subset of a convex cone metric space. Our convergence results generalize and refine not only result of Gunduz [6] but also results of Lee [4, 5] and Temir [9].

### 1. INTRODUCTION

Fixed point theory plays an important role in applications of many branches of mathematics and applied sciences. The study of metric fixed point theory has been at the centre of vigorous research activity. There has been a number of generalizations of the usual notion of a metric space. One such generalization is a cone metric space introduced and studied by Huang and Zhang [2], in 2007. The idea of cone metric spaces is based on replacing the set of real numbers by an ordered Banach space in definition of metric spaces. Huang and Zhang [2] modified definitions of some concepts such as convergence of sequences, Cauchy sequences, and completeness in this space. They also proved some fixed point theorems of contractive mappings on complete cone metric spaces using assumption of the normality of a cone. After that a series of articles have been dedicated to existence and uniqueness of fixed point of different type mappings in cone metric spaces. In [4], Lee introduced the concept of convex cone metric spaces by combining idea of cone metric space and convex metric space defined by Takahashi [1], and started iterative approximation of fixed points of nonlinear mappings. Gunduz [7] studied convergence of a new multistep iteration for a finite family of asymptotically quasi-nonexpansive mappings in convex cone metric spaces. Result of Gunduz [7]

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is valid in the whole space, whereas the results of Lee [4, 5] are valid in a nonempty convex subset of a convex cone metric spaces.

The aim of this work is to study convergence of a new iteration process for a finite family of  $I$ -asymptotically quasi-nonexpansive mappings and a finite family of asymptotically quasi-nonexpansive mappings in generalized convex cone metric spaces. Our convergence results generalize and refine not only result of Gunduz [6] but also result of paper given in his references.

Throughout this article, we use the notation  $F(T)$  for the set of fixed points of a mapping  $T$  and  $F := (\bigcap_{i=1}^r F(T_i)) \cap (\bigcap_{i=1}^r F(I_i))$  for the set of common fixed points of two finite families of mappings  $\{T_i : i \in J\}$  and  $\{I_i : i \in J\}$ , where  $J$  is set of first  $r$  natural numbers.

## 2. PRELIMINARIES

In this section, we need to recall some basic notations, definitions, and necessary results and examples from existing literature.

In 1970, Takahashi [1] introduced the concept of convexity in a metric space  $(X, d)$  as follows.

**Definition 1.** [1] *A convex structure in a metric space  $(X, d)$  is a mapping  $W : X^2 \times [0, 1] \rightarrow X$  satisfying, for all  $x, y, u \in X$  and all  $\lambda \in [0, 1]$ ,*

$$d(u, W(x, y; \lambda)) \leq \lambda d(u, x) + (1 - \lambda) d(u, y).$$

Let  $E$  be a normed vector space, then the following definitions can be found in [2].

**Definition 2.** [2] *A nonempty subset  $P$  of  $E$  is called a cone if  $P$  is closed,  $P \neq \{\theta\}$ , for  $a, b \in \mathbb{R}^+ = [0, \infty)$  and  $x, y \in P$ ,  $ax + by \in P$  and  $P \cap \{-P\} = \{\theta\}$ . We define a partial ordering  $\preceq$  in  $E$  as  $x \preceq y$  if  $y - x \in P$ .  $x \ll y$  indicates that  $y - x \in \text{int}P$  and  $x \prec y$  means that  $x \preceq y$  but  $x \neq y$ . A cone  $P$  is said to be solid if its interior  $\text{int}P$  is nonempty. A cone  $P$  is said to be normal if there exists a positive number  $k$  such that for  $x, y \in P$ ,  $\theta \preceq x \preceq y$  implies  $\|x\| \leq k\|y\|$  or equivalently, if  $(\forall n) x_n \preceq y_n \preceq z_n$  and  $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} z_n = x$  imply  $\lim_{n \rightarrow \infty} y_n = x$ . The least positive number  $k$  is called the normal constant of  $P$ .*

It is clear that  $k \geq 1$ . There exist cones which are not normal.

**Example 1.** [3] *Let  $E = C_{\mathbb{R}}^1[0, 1]$  with  $\|x\| = \|x\|_{\infty} + \|x'\|_{\infty}$  on  $P = \{x \in E : x(t) \geq 0\}$ . This cone is not normal. Consider, for example,  $x_n(t) = \frac{t^n}{n}$  and  $y_n(t) = 1$ . Then  $\theta \preceq x_n \preceq y_n$ , and  $\lim_{n \rightarrow \infty} y_n = \theta$ , but  $\|x_n\| = \max_{t \in [0, 1]} \left| \frac{t^n}{n} \right| + \max_{t \in [0, 1]} |t^{n-1}| = \frac{1}{n} + 1 > 1$ ; hence  $x_n$  does not converge to zero. Thus  $P$  is a nonnormal cone.*

**Definition 3.** [2] *Let  $X$  be a nonempty set. A mapping  $d : X \times X \rightarrow (E, P)$  is called a cone metric if (i) for  $x, y \in X$ ,  $\theta \preceq d(x, y)$  and  $d(x, y) = \theta$  iff  $x = y$ , (ii) for  $x, y \in X$ ,  $d(x, y) = d(y, x)$  and (iii) for  $x, y, z \in X$ ,  $d(x, y) \preceq d(x, z) + d(z, y)$ .*

A nonempty set  $X$  with a cone metric  $d : X \times X \rightarrow (E, P)$  is called a cone metric space denoted by  $(X, d)$ , where  $P$  is a solid normal cone.

Since each metric space is a cone metric space with  $E = \mathbb{R}$  and  $P = [0, +\infty)$ , the concept of a cone metric space is more general than that of a metric space.

**Example 2.** [2] Let  $E = \mathbb{R}^2$ ,  $P = \{(x; y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$ ,  $X = \mathbb{R}$  and  $d : X \times X \rightarrow E$  defined by  $d(x, y) = (|x - y|, \alpha |x - y|)$ , where  $\alpha \geq 0$  is a constant. Then  $(X, d)$  is a cone metric space with normal cone  $P$  where  $k = 1$ .

**Definition 4.** A sequence  $\{x_n\}$  in a cone metric space  $(X, d)$  is said to converge to  $x \in X$  and is denoted as  $\lim_{n \rightarrow \infty} x_n = x$  or  $x_n \rightarrow x$  (as  $n \rightarrow \infty$ ) if for any  $c \in \text{int}P$ , there exists a natural number  $N$  such that for all  $n > N$ ,  $c - d(x_n, x) \in \text{int}P$ . A sequence  $\{x_n\}$  in  $(X, d)$  is called a Cauchy sequence if for any  $c \in \text{int}P$ , there exists a natural number  $N$  such that for all  $n, m > N$ ,  $c - d(x_n, x_m) \in \text{int}P$ . A cone metric space  $(X, d)$  is said to be complete if every Cauchy sequence converges.

In other words,  $\{x_n\}$  is said to converge to  $x$ , if there exists a natural number  $N$  such that  $d(x_n, x) \ll c$  for all  $n > N$  and for any  $c \in E$  with  $\theta \ll c$ .  $\{x_n\}$  is called a Cauchy sequence in  $X$ , if there exists a natural number  $N$  such that  $d(x_n, x_m) \ll c$  for all  $n, m > N$  and for any  $c \in E$  with  $\theta \ll c$ .

**Proposition 1.** [2] Let  $\{x_n\}$  be a sequence in a cone metric space  $(X, d)$  and  $P$  be a normal cone. Then

- (1)  $\{x_n\}$  converges to  $x$  in  $X$  if and only if  $d(x_n, x) \rightarrow \theta$  (as  $n \rightarrow \infty$ ) in  $E$ .
- (2)  $\{x_n\}$  is a Cauchy sequence if and only if  $d(x_n, x_m) \rightarrow \theta$  (as  $n, m \rightarrow \infty$ ) in  $E$ .

**Definition 5.** [4] Let  $(X, d)$  be a cone metric space. A mapping  $W : X^2 \times [0, 1] \rightarrow X$  is called a convex structure on  $X$  if  $d(W(x, y, \lambda), u) \preceq \lambda d(x, u) + (1 - \lambda) d(y, u)$  for all  $x, y, u \in X$  and  $\lambda$  in  $[0, 1]$ . A cone metric space  $(X, d)$  with a convex structure  $W$  is called a convex cone metric space and denoted as  $(X, d, W)$ . A nonempty subset  $C$  of a convex cone metric space  $(X, d, W)$  is said to be convex if  $W(x, y, \lambda) \in C$  for all  $x, y \in C$  and  $\lambda \in [0, 1]$ .

**Example 3.** Let  $(X, d)$  be a cone metric space as in Example 2. If  $W(x, y, \lambda) = \lambda x + (1 - \lambda)y$ , then  $(X, d)$  is a convex cone metric space. Hence, this concept is more general than that of a convex metric space.

Definition 1 can be extended as follows: A mapping  $W : X^3 \times [0, 1]^3 \rightarrow X$  is said to be a convex structure on  $X$ , if it satisfies the following condition: For any  $(x, y, z; a, b, c) \in X^3 \times [0, 1]^3$  with  $a + b + c = 1$ , and  $u \in X$ :

$$d(u, W(x, y, z; a, b, c)) \leq ad(u, x) + bd(u, y) + cd(u, z).$$

If  $(X, d)$  is a metric space with a convex structure  $W$ , then  $(X, d)$  is called a generalized convex metric space. A nonempty subset  $C$  of a generalized convex

metric space  $X$  is said to be convex if  $W(x, y, z; a, b, c) \in C$ ,  $\forall(x, y, z) \in C^3$ ,  $\forall(a, b, c) \in [0, 1]^3$  with  $a + b + c = 1$ .

Every linear normed space is a generalized convex metric space with a convex structure  $W(x, y, z; a, b, c) = ax + by + cz$ , for all  $x, y, z \in X$  and  $a, b, c \in [0, 1]$  with  $a + b + c = 1$ . But there exist some convex metric spaces which can not be embedded into any linear normed spaces (see, Gunduz and Akbulut [8]).

Considering generalized convex metric space together with cone metric space, any one can be defined generalized convex cone metric spaces as follow:

**Definition 6.** [4] Let  $(X, d)$  be a cone metric space. A mapping  $W : X^3 \times [0, 1]^3 \rightarrow X$  is called a convex structure on  $X$  if  $d(u, W(x, y, z; a, b, c)) \preceq ad(u, x) + bd(u, y) + cd(u, z)$  for all  $x, y, z, u \in X$  and  $a, b, c \in [0, 1]$  with  $a + b + c = 1$ . A cone metric space  $(X, d)$  with a convex structure  $W$  is called a generalized convex cone metric space and denoted as  $(X, d, W)$ . A nonempty subset  $C$  of a generalized convex cone metric space  $(X, d, W)$  is said to be convex if  $W(x, y, z; a, b, c) \in C$  for all  $x, y, z, \in C$  and  $a, b, c \in [0, 1]$  with  $a + b + c = 1$ .

**Remark 1.** If we take  $E = \mathbb{R}$ ,  $P = [0, +\infty)$  and  $\|\cdot\| = |\cdot|$ , then generalized convex cone metric spaces coincide with generalized convex metric spaces.

Now we give definition of some mappings which will be used later.

**Definition 7.** Let  $(X, d)$  be a cone metric space with a solid cone  $P$  and  $T, I : (X, d) \rightarrow (X, d)$  be two mapping. The mapping  $T$  is said to be

- (1) asymptotically nonexpansive if there exists  $u_n \in [1, \infty)$  for all  $n \in \mathbb{N}$  with  $\lim_{n \rightarrow \infty} u_n = 1$  such that

$$d(T^n x, T^n y) \preceq u_n d(x, y) \text{ for all } x, y \in X \text{ and } n \in \mathbb{N}.$$

- (2) asymptotically quasi-nonexpansive if  $F(T) \neq \emptyset$  and there exists  $u_n \in [1, \infty)$  for all  $n \in \mathbb{N}$  with  $\lim_{n \rightarrow \infty} u_n = 1$  such that

$$d(T^n x, p) \preceq u_n d(x, p) \text{ for all } x \in X, p \in F(T) \text{ and } n \in \mathbb{N}.$$

- (3)  $I$ -asymptotically nonexpansive if there exists a sequence  $\{v_n\} \subset [0, \infty)$  with  $\lim_{n \rightarrow \infty} v_n = 0$  such that

$$d(T^n x, T^n y) \preceq (1 + v_n) d(I^n x, I^n y)$$

for all  $x, y \in X$  and  $n \geq 1$ .

- (4)  $I$ -asymptotically quasi nonexpansive if  $F(T) \cap F(I) \neq \emptyset$  and there exists a sequence  $\{v_n\} \subset [0, \infty)$  with  $\lim_{n \rightarrow \infty} v_n = 0$  such that

$$d(T^n x, p) \preceq (1 + v_n) d(I^n x, p)$$

for all  $x \in X$  and  $p \in F(T) \cap F(I)$  and  $n \geq 1$ .

- (5)  $I$ -uniformly Lipschitz if there exists  $\Gamma > 0$  such that

$$d(T^n x, T^n y) \preceq \Gamma d(I^n x - I^n y), \quad x, y \in X \text{ and } n \geq 1.$$

**Remark 2.** From the above definition, it follows that if  $F(T)$  is nonempty, then an asymptotically nonexpansive mapping is asymptotically quasi-nonexpansive. Also, an  $I$ -asymptotically nonexpansive mapping is  $I$ -uniformly Lipschitz with the Lipschitz constant  $\Gamma = \sup \{1 + v_n : n \geq 1\}$  and an  $I$ -asymptotically nonexpansive mapping with  $F(T) \cap F(I) \neq \emptyset$  is  $I$ -asymptotically quasi nonexpansive. However, the converse of these claims are not true in general. It is easy to see that if  $I$  is identity mapping, then  $I$ -asymptotically nonexpansive mappings and  $I$ -asymptotically quasi nonexpansive mappings coincide with asymptotically nonexpansive mappings and asymptotically quasi nonexpansive mappings, respectively.

In [6], Gunduz used the Ishikawa iteration process with error terms to prove some convergence results in a convex metric space. We can modify his process in accordance with our purpose as follow:

Let  $(X, d)$  be a generalized convex cone metric space with convex structure  $W$ ,  $\{T_i : i \in J\} : X \rightarrow X$  be a finite family of  $I_i$ -asymptotically quasi-nonexpansive mappings and  $\{I_i : i \in J\} : X \rightarrow X$  be a finite family of asymptotically quasi-nonexpansive mappings. Suppose that  $\{u_n\}$  and  $\{v_n\}$  are two bounded sequences (with respect to cone metric  $d$ ) in  $X$  and  $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\hat{\alpha}_n\}, \{\hat{\beta}_n\}, \{\hat{\gamma}_n\}$  are six sequences in  $[0, 1]$  such that  $\alpha_i + \beta_n + \gamma_n = 1 = \hat{\alpha}_n + \hat{\beta}_n + \hat{\gamma}_n$  for  $n \in \mathbb{N}$ . For any given  $x_1 \in X$ , iteration process  $\{x_n\}$  defined by,

$$\begin{aligned} x_{n+1} &= W(x_n, I_i^n y_n, u_n; \alpha_n, \beta_n, \gamma_n), \\ y_n &= W(x_n, T_i^n x_n, v_n; \hat{\alpha}_n, \hat{\beta}_n, \hat{\gamma}_n), \quad n \geq 1, \end{aligned} \tag{2.1}$$

where  $n = (k-1)r + i$ ,  $i = i(n) \in J$  is a positive integer and  $k(n) \rightarrow \infty$  as  $n \rightarrow \infty$ . Thus, (2.1) can be expressed in the following form:

$$\begin{aligned} x_{n+1} &= W(x_n, I_{i(n)}^{k(n)} y_n, u_n; \alpha_n, \beta_n, \gamma_n), \\ y_n &= W(x_n, T_{i(n)}^{k(n)} x_n, v_n; \hat{\alpha}_n, \hat{\beta}_n, \hat{\gamma}_n), \quad n \geq 1. \end{aligned}$$

Let's give with a proposition.

**Lemma 1.** [10] Let  $\{a_n\}, \{b_n\}$  and  $\{c_n\}$  be three nonnegative sequences satisfying

$$\sum_{n=0}^{\infty} b_n < \infty, \quad \sum_{n=0}^{\infty} c_n < \infty, \quad a_{n+1} = (1 + b_n) a_n + c_n, \quad n \geq 0.$$

Then

- i)  $\lim_{n \rightarrow \infty} a_n$  exists,
- ii) if either  $\liminf_{n \rightarrow \infty} a_n = 0$  or  $\limsup_{n \rightarrow \infty} a_n = 0$ , then  $\lim_{n \rightarrow \infty} a_n = 0$ .

### 3. MAIN RESULTS

Using the steps in the proof of [6, Proposition 1.9.], we can prove easily the next proposition which plays a key role in the proof of our main result.

**Proposition 2.** *Let  $(X, d)$  be a generalized convex cone metric space with a solid cone  $P$  and convex structure  $W$ ,  $\{T_i : i \in J\} : X \rightarrow X$  be a finite family of  $I_i$ -asymptotically quasi-nonexpansive mappings, and  $\{I_i : i \in J\} : X \rightarrow X$  be a finite family of asymptotically quasi-nonexpansive mappings with  $F := (\bigcap_{i=1}^r F(T_i)) \cap (\bigcap_{i=1}^r F(I_i)) \neq \emptyset$ . Then, there exist a point  $p \in F$  and sequences  $\{k_n\}, \{l_n\} \subset [0, \infty)$  with  $\lim_{n \rightarrow \infty} k_n = \lim_{n \rightarrow \infty} l_n = 0$  such that*

$$d(T_i^n x, p) \preceq (1 + k_n)d(I_i^n x, p) \text{ and } d(I_i^n x, p) \preceq (1 + l_n)d(x, p)$$

for all  $x \in K$ , for each  $i \in I$ .

We now prove convergence theorem of the iterative scheme (2.1) in generalized convex cone metric spaces.

**Theorem 1.** *Let  $(X, d, W)$  be a generalized convex cone metric space with a cone metric  $d : X \times X \rightarrow (E, P)$ , where  $P$  is a solid normal cone with the normal constant  $k$ . Let  $\{T_i : i \in J\} : X \rightarrow X$  be a finite family of  $I_i$ -asymptotically quasi-nonexpansive mappings and  $\{I_i : i \in J\} : X \rightarrow X$  be a finite family of asymptotically quasi-nonexpansive mappings with  $F \neq \emptyset$ . Suppose that  $\sum_{n=1}^{\infty} k_n < \infty$ ,  $\sum_{n=1}^{\infty} l_n < \infty$  and  $\{x_n\}$  is as in (2.1) with  $\{\gamma_n\}, \{\hat{\gamma}_n\}$  satisfying  $\sum_{n=1}^{\infty} \gamma_n < \infty$  and  $\sum_{n=1}^{\infty} \hat{\gamma}_n < \infty$ . (i) If  $\{x_n\}$  converges to a point in  $F$ , then  $\liminf_{n \rightarrow \infty} d(x_n, F) = \theta$ . (ii)  $\{x_n\}$  converges to a point in  $F$ , if  $X$  is complete and  $\liminf_{n \rightarrow \infty} d(x_n, F) = \theta$ .*

*Proof.* We prove only (ii), since (i) is obvious. Let  $p \in F$ . Since  $\{u_n\}$  and  $\{v_n\}$  are bounded sequences with respect to cone metric  $d$  in  $X$ , there exists  $M \succeq \theta$  such that  $\max\{\sup_{n \geq 1} d(u_n, p), \sup_{n \geq 1} d(v_n, p)\} \preceq M$ . Considering Proposition 2 and (2.1), we have

$$\begin{aligned} d(y_n, p) &= d\left(W\left(x_n, T_i^n x_n, v_n; \hat{\alpha}_n, \hat{\beta}_n, \hat{\gamma}_n\right), p\right) \\ &\preceq \hat{\alpha}_n d(x_n, p) + \hat{\beta}_n d(T_i^n x_n, p) + \hat{\gamma}_n d(v_n, p) \\ &\preceq \hat{\alpha}_n d(x_n, p) + \hat{\beta}_n (1 + k_n) d(I_i^n x_n, p) + \hat{\gamma}_n M \\ &\preceq \hat{\alpha}_n d(x_n, p) + \hat{\beta}_n (1 + k_n) (1 + l_n) d(x_n, p) + \hat{\gamma}_n M \\ &\preceq \left(1 + \hat{\beta}_n (k_n + l_n + k_n l_n)\right) d(x_n, p) + \hat{\gamma}_n M \end{aligned} \quad (3.1)$$

and

$$\begin{aligned} d(x_{n+1}, p) &= d\left(W\left(x_n, I_i^n y_n, u_n; \alpha_n, \beta_n, \gamma_n\right), p\right) \\ &\preceq \alpha_n d(x_n, p) + \beta_n d(I_i^n y_n, p) + \gamma_n d(u_n, p) \\ &\preceq \alpha_n d(x_n, p) + \beta_n (1 + l_n) d(y_n, p) + \gamma_n M. \end{aligned} \quad (3.2)$$

Substituting (3.1) into (3.2),

$$\begin{aligned}
d(x_{n+1}, p) &\preceq \alpha_n d(x_n, p) + \beta_n (1 + l_n) d(y_n, p) + \gamma_n M \\
&\preceq \alpha_n d(x_n, p) + \beta_n (1 + l_n) \left(1 + \hat{\beta}_n (k_n + l_n + k_n l_n)\right) d(x_n, p) \\
&\quad + \beta_n (1 + l_n) \hat{\gamma}_n M + \gamma_n M \\
&\asymp \alpha_n d(x_n, p) + \beta_n (1 + l_n) d(x_n, p) \\
&\quad + \beta_n (1 + l_n) \hat{\beta}_n (k_n + l_n + k_n l_n) d(x_n, p) \\
&\quad + (\beta_n (1 + l_n) \hat{\gamma}_n + \gamma_n) M \\
&\asymp \left[1 + \beta_n l_n + \beta_n \hat{\beta}_n (1 + l_n) (k_n + l_n + k_n l_n)\right] d(x_n, p) \\
&\quad + (\beta_n (1 + l_n) \hat{\gamma}_n + \gamma_n) M.
\end{aligned}$$

Thus we obtain

$$d(x_{n+1}, p) \preceq [1 + \kappa_n] d(x_n, p) + t_n M \quad (3.3)$$

where  $\kappa_n = \beta_n l_n + \beta_n \hat{\beta}_n (1 + l_n) (k_n + l_n + k_n l_n)$  and  $t_n = (\beta_n (1 + l_n) \hat{\gamma}_n + \gamma_n)$  with  $\sum_{n=1}^{\infty} \kappa_n < \infty$  and  $\sum_{n=1}^{\infty} t_n < \infty$ . Hence, by the normality of  $P$ , we have for the normal constant  $k > 0$

$$\|d(x_{n+1}, F)\| \leq k [1 + \kappa_n] \|d(x_n, F)\| + k t_n \|M\| \quad (3.4)$$

Lemma 1 and (3.4) imply that the  $\lim_{n \rightarrow \infty} \|d(x_n, F)\|$  exists.

Now  $\liminf_{n \rightarrow \infty} \|d(x_n, F)\| = 0$  implies  $\lim_{n \rightarrow \infty} \|d(x_n, F)\| = 0$ .

Next, we show that the sequence  $\{x_n\}$  is a Cauchy sequence. Taking into account that the inequality  $1 + x \leq e^x$  for all  $x \geq 0$ , and (3.4), therefore we have

$$\|d(x_{n+1}, p)\| \leq k \exp\{\kappa_n\} \|d(x_n, p)\| + k \|M\| t_n. \quad (3.5)$$

Hence, for any positive integers  $n, m$ , from (3.5) it follows that

$$\begin{aligned}
\|d(x_{n+m}, p)\| &\leq k_1 \exp\{\kappa_{n+m-1}\} \|d(x_{n+m-1}, p)\| + k_1 t_{n+m-1} \|M\| \\
&\leq k_1 \exp\{\kappa_{n+m-1}\} [k_2 \exp\{\kappa_{n+m-2}\} \|d(x_{n+m-2}, p)\| \\
&\quad + k_2 t_{n+m-2} \|M\|] + k_1 t_{n+m-1} \|M\| \\
&= k_1 k_2 \exp\{\kappa_{n+m-1}\} \exp\{\kappa_{n+m-2}\} \|d(x_{n+m-2}, p)\| \\
&\quad + k_1 k_2 \exp\{\kappa_{n+m-1}\} t_{n+m-2} \|M\| + k_1 t_{n+m-1} \|M\| \\
&\leq \dots \\
&\leq \prod_{j=1}^m k_j \exp\left\{\sum_{i=n}^{n+m-1} \kappa_i\right\} \|d(x_n, p)\| \\
&\quad + \prod_{j=1}^m k_j \exp\left\{\sum_{i=n}^{n+m-1} \kappa_i\right\} \sum_{i=n}^{n+m-1} t_i \|M\| \\
&\leq BG \|d(x_n, p)\| + BG \sum_{i=n}^{n+m-1} t_i \|M\|,
\end{aligned}$$

where  $B = \prod_{j=1}^m k_j$ ,  $G = \exp \left\{ \sum_{i=n}^{n+m-1} \kappa_i \right\} < \infty$  and  $k_i$  is corresponding normal constant for  $i = 1, 2, \dots, m$ .

Since  $\lim_{n \rightarrow \infty} \|d(x_n, F)\| = 0$  and  $\sum_{n=1}^{\infty} t_n < \infty$ , for any given positive real number  $\varepsilon$ , there exists a natural number  $N_0 \in \mathbb{N}$  such that  $\|d(x_n, F)\| \leq \frac{\varepsilon}{2(1+BG)}$  and  $\sum_{n=1}^{\infty} t_n < \frac{\varepsilon}{2BG\|M\|}$  for  $n \geq N_0$ . In particular, there exist a point  $p_1 \in F$  such that  $\|d(x_n, p_1)\| \leq \frac{\varepsilon}{2(1+BG)}$  for  $n \geq N_0$ . Consequently, for any  $n \geq n_0$  and for all  $m \geq 1$  we have

$$\begin{aligned} \|d(x_{n+m}, x_n)\| &\leq \|d(x_{n+m}, p_1)\| + \|d(x_n, p_1)\| \\ &\leq (1+BG)\|d(x_n, p_1)\| + BG \sum_{i=n}^{n+m-1} t_i \|M\| \\ &\leq (1+BG)\frac{\varepsilon}{2(1+BG)} + BG\frac{\varepsilon}{2BG\|M\|} \|M\| = \varepsilon. \end{aligned}$$

This implies that  $\{x_n\}$  is a Cauchy sequence in  $X$ , therefore, it converges to some point  $q$  in the complete space  $X$ .

Finally, we show that  $q \in F$ . Let  $\{q_n\}$  be a sequence in  $F$  such that  $q_n \rightarrow q$ . Since

$$\begin{aligned} d(q, T_i q) &\leq d(q, q_n) + d(q_n, I_i q) \\ &= d(q, q_n) + d(I q_n, I_i q) \\ &\leq d(q, q_n) + (1+l_n)d(q_n, q), \end{aligned}$$

taking limit in above inequality, we have  $q \in \bigcap_{i=1}^r F(I_i)$  for all  $i \in I$ . Similarly,  $q \in \bigcap_{i=1}^r F(T_i)$ . So  $q \in F$ , which means that  $F$  is closed. Since  $d(q, F) = d(\lim_{n \rightarrow \infty} x_n, F) = \lim_{n \rightarrow \infty} d(x_n, F) = \theta$  by Propostion 1 (i), we have  $q \in F$ . In other words,  $\{x_n\}$  converges to a common fixed point in  $F$ .  $\square$

**Remark 3.** We get Theorem 2.2. of Gunduz [6] restricting the normed linear space  $(E, P)$  to a real number system  $(\mathbb{R}, [0, \infty))$  from Theorem 1. Additionally to this restriction taking the metric space  $(X, d)$  to a Banach space with  $W(x, y, z; \alpha, \beta, \gamma) = \alpha x + \beta y + \gamma z$ , and  $\gamma_n = \hat{\gamma}_n = 0$  for all  $n \in \mathbb{N}$ , we get a generalization of corresponding result of Temir [9].

**Remark 4.** We want to point out that our theorem generalizes the result of Temir [9] in two ways: (i) from a closed convex subset of Banach spaces to general setup of generalized convex cone metric space, (ii) a finite family of  $I_i$ -asymptotically nonexpansive mappings to a finite family of  $I_i$ -asymptotically quasi-nonexpansive mappings.

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