Some results in metric fixed point theory

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Abstract

This is a survey of results mainly in metric fixed point theory, including the Darbo–Sadovskiĭ theorem using measures of noncompactness. Various different proofs are presented for some of the most important historical results. Furthermore many examples and remarks are added to illustrate the topics of the paper.

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1. Introduction

Fixed point theory is a major branch of nonlinear functional analysis because of its wide applicability. Numerous questions in physics, chemistry, biology, and economics lead to various nonlinear differential and integral equations.

There are two fundamental results, namely Banach's fixed point theorem and Darbos's fixed point theorem

The classical Banach contraction principle [2] of Banach's theorem is one of the most useful results in metric fixed point theory. Due to its applications in mathematics and other related disciplines, this principle has been generalized in many directions. Extensions of Banach's contraction principle have been obtained either by generalizing the distance properties of the underlying domain or by modifying the contractive condition on the mappings.

Darbo's fixed point theorem [17] of 1955 uses the condensing principle connected to Kuratowski's measure of noncompactness α [33] of 1930; it is a very important generalization of Schauder's fixed point theorem,

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and includes the existence part of Banach's fixed point theorem. Other measures of noncompactness were introduced by Goldenštein, Goh'berg and Markus [GGM1], the ball or Hausdorff measure of noncompactness, which was later studied by Goldenštein and Markus [GGM2] in 1968, Istrăţesku [27] in 1972, and others. Apparently Goldenštein, Goh'berg and Markus were not aware of Kuratowski's and Darbo's work. It is surprising that Darbo's theorem was almost never noticed and applied until in the 1970's mathematicians working in functional analysis, operator theory and differential equations started to apply Darbo's theorem and developed the theory connected with measures of noncompactness. These measures of noncompactness are studied in detail and their use is discussed, for instance, in the monographs [AKP, 53, 3, 28, 34, 35].

2. Banach contraction principle

In this section we are going to study the famous Banach fixed point theorem, usually called the Banach contraction principle. This principle from 1922 marks the beginning of the fixed point theory in metric spaces.

We also present several different proofs of Banach's contraction principle

Definition 2.1. Let (X, d) be a metric space. A mapping $f : X \to X$ is a *contraction* if there exists some $q \in [0, 1)$ such that

$$d(fx, fy) \leq q \cdot d(x, y), \text{ for all } x, y \in X. \tag{2.1}$$

We observe that every contraction is a continuous mapping. The following theorem shows the existence and uniqueness of a fixed point of an arbitrary contraction on a complete metric space. It is important to mention that there exists a continuous mapping without fixed point property.

Theorem 2.2 (Banach; Banach contraction principle)**.**
If (X, d) is a complete metric space and $f : X \to X$ is a contraction, then the mapping f has a unique fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary. We define a sequence (x_n) in X such that $x_n = f(x_{n-1})$ for $(n \in \mathbb{N}$, and prove that (x_n) is a Cauchy sequence, hence convergent in the complete metric space X.
We obtain for any $n \in \mathbb{N}$,

$$d(x_n, x_{n+1}) = d(f(x_{n-1}), f(x_n)) \leq q \cdot d(x_{n-1}, x_n)$$
$$\leq \cdots \leq q^n \cdot d(x_0, x_1),$$

and therefore, if $m > n$,

$$d(x_n, x_m) \leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \leq \sum_{k=n}^{m-1} q^k d(x_0, x_1) \leq \frac{q^n}{1-q} d(x_0, x_1). \tag{2.2}$$

Since $0 \leq q < 1$, it follows that $\lim_{n,m\to\infty} d(x_n, x_m) = 0$, hence (x_n) is a Cauchy sequence. Moreover, X is a complete metric space, and so there exists some $x \in X$ such that $\lim_{n\to\infty} x_n = x$.
We show $f(x) = x$ by estimating $d(x_n, f(x))$ for $n \in \mathbb{N}$;

$$0 \leq d(x_n, f(x)) = d(f(x_{n-1}), f(x)) \leq q \cdot d(x_{n-1}, x),$$

implies $\lim_{n\to\infty} d(x_n, f(x)) = 0$, and by the uniqueness of the limit of any convergent sequence in a metric space, we conclude $f(x) = x$.
It remains to prove that such x is uniquely determined. We assume $f(y) = y$ for some $y \in X$, $y \neq x$, then

$$d(x, y) = d(f(x), f(y)) \leq q \cdot d(x, y)$$

and $(1-q)d(x, y) \leq 0$, which contradicts our assumption because $0 < 1 - q \leq 1$.   □

Corollary 2.3. *Let $f : X \to X$ be a q–contraction on a complete metric space X and $z \in X$ be the fixed point of the function f. Then we have*

(1) the sequence $(f^n(x))$ converges for each $x \in X$ and converges to z;

(2) $d(x, z) \leq 1/(1 - q) \cdot d(x, f(x))$;

(3) $d(f^n(x), z) \leq q^n/(1 - q) \cdot d(x, f(x))$;

(4) $d(f^{n+1}(x), z) \leq q \cdot d(f(x), x)$;

(5) $d(f^{n+1}(x), z) \leq q/(1 - q) \cdot d(f^n(x), f^{n+1}(x))$.

    Proof. We only prove the second and third condition, the proofs of the other conditions are analogous.

$$(2)\ d(x, z) = \lim_{n \to \infty} d(x, f^n(x))$$

$$\leq \lim_{n \to \infty} \sum_{k=0}^{n-1} d(f^k(x), f^{k+1}(x))$$

$$= \sum_{k=0}^{\infty} d(f^k(x), f^{k+1}(x))$$

$$\leq \sum_{k=0}^{\infty} q^k d(x, f(x))$$

$$= \frac{1}{1 - q} \cdot d(x, f(x))$$

(3) It follows from $f(z) = z$, that
$f^n(z) = z$, and from the first part of the prove, we have

$$d(f^n(x), z) = d(f^n(x), f^n(z)) \leq q^n d(x, z) \leq \frac{q^n}{1 - q} \cdot d(x, f(x)). \qquad \square$$

Remark 2.4. There exist various approaches to the Banach fixed point theorem, but the proof above gives a method of how to find the fixed point for a contraction f. It is also known as *Picard's iteration method* or *fixed point iteration*. It is based on the idea of defining a sequence of successive iterations. We start with any $x_0 \in X$ and define $x_n = f(x_{n-1})$ for $n \in \mathbb{N}$. The proof presented above guarantees the existence of a limit $\lim_{n \to \infty} x_n = x \in X$ such that $f(x) = x$. If we let $m \to \infty$ in (2.2), then

$$d(x_n, x) \leq \frac{q^n}{1 - q} d(x_0, x_1),$$

and this is an estimate for the error made by approximating the solution x by the n–th iteration x_n.

    We now present a few proofs of Theorem 2.2.

    Proof of Theorem 2.2 (Joseph and Kwack [29]). Let $c = \inf\{d(x, f(x)) : x \in X\}$. If $c > 0$, then $c/q > c$ and there exists $x \in X$ such that

$$d(f(x), f(f(x))) \leq q \cdot d(x, f(x)) < c,$$

which is a contradiction. Hence we must have $c = 0$. Let (x_n) be a sequence in X such that $d(x_n, f(x_n)) \to 0$ as $n \to \infty$. We show that (x_n) is a Cauchy sequence, since

$$d(x_n, x_m) \leq d(x_n, f(x_n)) + d(f(x_n), f(x_m)) + d(f(x_m, x_m))$$

implies

$$(1 - q)d(x_n, x_m) \leq d(x_n, f(x_n)) + d(x_m, f(x_m)).$$

Hence there exists $p \in X$ such that $\lim_{n \to \infty} x_n = p$, and $\lim_{n \to \infty} d(x_n, f(x_n)) = 0$ implies $\lim f(x_n) = p$. It follows from $d(f(x_n), f(p)) \leq q d(x_n, p)$ that $\lim_{n \to \infty} f(x_n) = f(p)$, hence $f(p) = p$. The uniqueness of the fixed point 0f the function f follows from the contractive condition of f. $\qquad \square$

Proof of Theorem 2.2 (Palais [42]). Let $x_1, x_2 \in X$. Then we have

$$d(x_1, x_2) \leq d(x_1, f(x_1)) + d(f(x_1), f(x_2)) + d(f(x_2), x_2),$$

that is,

$$(1 - q)d(x_1, x_2) \leq d(x_1, f(x_1)) + d(f(x_2), x_2).$$

Hence we obtain the *fundamental contraction inequality*

$$d(x_1, x_2) \leq \frac{1}{1 - q} \cdot [d(x_1, f(x_1)) + d(x_2, f(x_2))], \text{ for all } x_1, x_2 \in X. \tag{2.3}$$

If x_1 and x_2 are fixed points of the function f, then it follows from (2.3) that $x_1 = x_2$, that is, the contraction can have at most one fixed point.

Let $x \in X$, $n, m \in \mathbb{N}$, and $x_1 = f^n(x)$ and $x_2 = f^m(x)$. We obtain from (2.3)

$$d(f^n(x), f^m(x)) \leq \frac{1}{1 - q} \cdot [d(f^n(x), f(f^n(x))) + d(f^m(x), f(f^m(x)))] \tag{2.4}$$

$$\leq \frac{q^n + q^m}{1 - q} \cdot d(x, f(x)). \tag{2.5}$$

Since $0 \leq q < 1$, it follows that $\lim_{n \to \infty} q^n = 0$, hence $d(f^n(x), f^m(x)) \to 0$ as $n \to \infty$ and $m \to \infty$. Therefore the Cauchy sequence $(f^n(x))$ converges, that is, there exists $p \in X$ such that $\lim_n f^n(x) = p$. Because of the continuity of the function f, we have $f(p) = f(\lim_n f^n(x)) = \lim_n f(f^n(x)) = p$. We note that letting $m \to \infty$ in (2.4), we obtain

$$d(f^n(x), p) \leq \frac{q^n}{1 - q} \cdot d(x, f(x)). \qquad \square \tag{2.6}$$

Proof of Theorem 2.2 (Boyd and Wong [6]). We define $\varphi(x) = d(x, f(x))$ for $x \in X$. Since f is a contraction, the function $\varphi : X \to \mathbb{R}$ is continuous and $\varphi(f^n(x)) \to 0$ as $n \to \infty$, for each $x \in X$. We put

$$C_m = \left\{ x \in X : \varphi(x) \leq \frac{1}{m} \right\}.$$

It follows from the conditions above that C_m is a closed and nonempty subset of X for each $m = 1, 2, \ldots$. Now we estimate the diameter of the set C_m. Let $x, y \in C_m$. Then we have

$$d(x, y) \leq d(x, f(x)) + d(f(x), f(y)) + d(f(y), y) \leq \frac{2}{m} + q d(x, y),$$

hence

$$\text{diam} C_m \leq \frac{2}{m(1 - q)}.$$

Since each C_m is a closed, nonempty subset of X, $C_1 \supset C_2 \supset C_3 \supset \ldots$ and $\text{diam} C_m \to 0$ as $m \to \infty$, it follows by Cantor's intersection theorem $\bigcap_m C_m = \{\xi\}$.

Since $f(C_m) \subset C_m$ for each m, it follows that ξ is a fixed point of the function f, and clearly the fixed point is unique. (We note $f(\{\xi\}) = f(\bigcap_m C_m) \subset \bigcap_m f(C_m) \subset \bigcap_m C_m = \{\xi\}$.)

We have for each $x \in X$

$$d(f^n(x), \xi) = d(f^n(x), f^n(\xi)) \leq q^n d(x, \xi) \to 0 \ (n \to \infty).$$

Since

$$d(x, \xi) \leq d(x, f(x)) + d(f(x), f(\xi)) \leq d(x, f(x)) + q d(x, \xi),$$

it follows that

$$d(x,\xi) \leq \frac{d(x,f(x))}{1-q}.$$

Hence we again have the estimate

$$d(f^n(x),\xi) \leq \frac{q^n}{1-q} \cdot d(x,f(x)). \qquad \qquad \square \qquad (2.7)$$

Corollary 2.5. *Let S be a closed subset of a complete metric space (X,d) and $f : S \to S$ be a contraction. For an arbitrary point $x_0 \in S$, the iterative sequence $x_n = f(x_{n-1})$ $(n \in \mathbb{N})$ converges to the fixed point of the mapping f.*

The following example will show that the statement in Corollary 2.5 does not hold without the assumption that the set S is closed, in general.

Example 2.6. Let d be the natural metric on $\mathbb{R}$ defined by $d(x,y) = |x - y|$ for all $x, y \in \mathbb{R}$, and $S = B_0(1) = \{x \in \mathbb{R} : |x| < 1\}$. Then the mapping

$$f : S \to S \text{ with } f(x) = \frac{x+1}{2},$$

is a contraction without a fixed point in S.

Banach's fixed point theorem has wide and diverse applications, for instance, in solving various kinds of equations, inclusions, etc.

Example 2.7. If X is a Banach space, $A, B \in \mathcal{B}(X)$, A is an invertible operator and $\|B - A\| \cdot \|A^{-1}\| < 1$, then the invertibility of B follows from Banach's fixed point theorem.

Proof. It is sufficient to show that, for any $y \in X$, the equation $Bx = y$ has a unique solution $x \in X$. We choose an arbitrary point y in X. If $Bx_0 = y$ for some $x_0 \in X$, then

$$y = Bx_0 = (B - A)x_0 + Ax_0 \text{ and } A^{-1}y = A^{-1}(B - A)x_0 + x_0.$$

We put $z = A^{-1}y$ and $C = A^{-1}(B - A)$. Then we have $x_0 = z - Cx_0$.
The idea is to show that the function $f : X \to X$ defined by $f(x) = z - Cx$ for $x \in X$ is a contraction and x_0 is its fixed point.
The following inequalities hold for all $x, y \in X$

$$\|f(x) - f(y)\| = \|C(x - y)\| \leq \|A^{-1}\| \cdot \|B - A\| \cdot \|x - y\|.$$

Since $\|A^{-1}\| \cdot \|B - A\| < 1$, f is a contraction and x_0 is the unique fixed point of f.
Based on a few elements of an iterative sequence $(f^n(x))$,

$$z - Cx, \ z - C(z - Cx) = z - Cx + C^2x, \ z - Cx + C^2x - C^3x, \ldots$$

we may assume, and then easily prove that, because of $\|C\| < 1$, this sequence converges to $z - Cx + C^2x - C^3x + \cdots$.
We observe that if $A = I$ and $\|C\| < 1$, then $I - C + C^2 - C^3 + \cdots$ is an inverse of $I + C$. $\qquad \square$

The following corollary shows a relation between f^n and f in the case when f^n is a contraction.

Corollary 2.8 (Bryant [10])**.** *If (X,d) is a complete metric space and $f : X \to X$ is a mapping such that f^n is a contraction for some $n \geq 1$, then f has a unique fixed point in X.*

Proof. By Banach's fixed point theorem, there exists a unique $z \in X$ such that $f^n(z) = z$. Since $f^n(f(z)) = f(f^n(z)) = f(z)$, it follows that $f(z) = z$. Every fixed point of f is, at the same time, a fixed point of f^n, thus z is the unique fixed point of f. □

As observed in [10], the mapping f mentioned in Corollary 2.8 need not be continuous as in Theorem 2.2.

Example 2.9 (Bryant [10]). We define $f : [0,2] \to [0,2]$ by $f(x) = 1$ for $x \in [0,1)$, and $f(x) = 2$ for $x \in [1,2]$. Then $f^2(x) = 2$ for $x \in [0,2]$ and $f^2 : [0,2] \to [0,2]$ is a contraction although f is not continuous.

Since the proof of Banach's theorem is based on an iterative sequence for a point $x \in X$, the next reasonable step in the research was to check local properties and modify this result.

Theorem 2.10. *Let (X,d) be a complete metric space and $B_r(x_0) = \{x \in X : d(x, x_0) < r\}$ be the open ball in X for some $x_0 \in X$ and $r > 0$. Also let $f : B_r(x_0) \to X$ be a contraction, that is,*

$$d(f(x), f(y)) \le q \cdot d(x, y), \ (x, y \in B_r(x_0)) \ \text{for some } q \in [0, 1) \tag{2.8}$$

and

$$d(f(x_0), x_0) < (1 - q)r. \tag{2.9}$$

Then the mapping f has a unique fixed point in $B_r(x_0)$.

Proof. We choose $r_0 \in [0, r)$ such that (2.9) holds. Then $f : \overline{B}_{r_0}(x_0) \to \overline{B}_{r_0}(x_0)$, where $\overline{B}_{r_0}(x_0)$ is the closure of $B_{r_0}(x_0)$, since, for any $x \in \overline{B}_{r_0}(x_0)$,

$$d(f(x), x_0) \le d(f(x), f(x_0)) + d(f(x_0), x_0)$$
$$\le q \cdot d(x, x_0) + (1 - q)r_0 \le r_0.$$

Hence f has a unique fixed point $z \in \overline{B}_{r_0}(x_0)$. It easily follows from (2.8) that z is the unique fixed point of f in $B_r(x_0)$. □

3. Darbo's fixed point theorem

If the contractive condition of f in Theorem 2.2 is relaxed, that is, if we consider so–called *nonexpansive mappings* f, that is, functions $f : X \to X$ satisfying

$$d(f(x), f(y)) \le d(x, y) \ \text{for all } x, y \in X,$$

then Banach's fixed point theorem need no longer hold.

In 1965, Browder proved a fixed point theorem for nonexpansive maps.

Theorem 3.1 (Browder's fixed point theorem). *Let X be a Banach space, C be a convex and bounded subset of X and $T : C \to C$ be a nonexpansive map. If X is either a Hilbert space, or a uniformly convex or a reflexive Banach space, then T has a fixed point.*

This result uses the convexity hypothesis which is more usual in topological fixed point theory, and the geometric properties of Banach spaces commonly used in linear functional analysis.

The following Brouwer fixed point theorem should be considered in a different setting.

Theorem 3.2 (Brouwer's fixed point theorem). *Every continuous map from the closed unit ball of $\mathbb{R}^n$ into itself has a fixed point.*

Remark 3.3. In the case of one variable, the Brouwer fixed point theorem is the following:
Every continuous function of the interval $[-1, 1]$ onto itself has a fixed point.
or equivalently
Every continuous function of the interval $[-1, 1]$ onto itself intersects the main diagonal at some point.

One cannot expect uniqueness of the fixed point in Brouwer's theorem (Theorem 3.2), in general. So we must consider the non–empty set $\mathcal{F}(f)$ of fixed points of a function f. If f is continuous, then the set

$$\mathcal{F}(f) = \ker(f - \text{id}) = (f - \text{id})^{-1}(\{0\}), \text{ where id is the identity},$$

is closed. It is natural to study what other properties the set $\mathcal{F}(f)$ has. The following theorem shows that no other special features can be inferred, since we will see that for any given non–empty closed subset of the closed unit ball $\overline{B}_1^n(0)$ of Euclidean $\mathbb{R}^n$ there exists a continuous function $f : \overline{B}_1^n(0) \to \overline{B}_1^n(0)$ which has $\mathcal{F}(f)$ as the set of its fixed points.

Theorem 3.4. *Let $F \neq \emptyset$ be a closed subset of $\overline{B}_1^n(0)$. Then there exists a continuous function $f : \overline{B}_1^n(0) \to \overline{B}_1^n(0)$ with $F = \mathcal{F}(f)$.*

Proof. For every $x \in \overline{B}_1^n(0)$, let $d(x, F) = \inf\{\|x - y\| : y \in F\}$. Obviously this function is continuous. We define the function $f : \overline{B}_1^n(0) \to \overline{B}_1^n(0)$ by

$$f(x) = \begin{cases} x - d(x, F)\dfrac{x - x_0}{\|x - x_0\|} & (x \neq x_0) \\ x_0 & (x = x_0), \end{cases}$$

where x_0 is an arbitrary point in F.

It is easy to show that f is well defined and continuous. Moreover $\mathcal{F}(f) = F$ and the theorem is proved.

An important generalization of Brouwer's fixed point theorem was obtained by Schauder.

Theorem 3.5 (Schauder's fixed point theorem). *Every continuous map from a nonempty, compact and convex subset C of a Banach space X into C has a fixed point.*

Clearly the conditions in the hypothesis are preserved if the norm of X is replaced by an equivalent norm, so Theorem 3.5 cannot be viewed as a metric fixed point theorem. Schauder's fixed point theorem can be used to prove Peano's existence theorem for the solution of systems of first order ordinary differential equations with initial conditions.

The situation is completely different when certain generalizations are considered, in particular those concerning *condensing maps*, where a condensing map is one under which the image of any set is – in a certain sense – more compact than the set itself. The degree of noncompactness of a set is measured by certain functions called *measures of noncompactness*.

Darbo's fixed point theorem, which uses *Kuratowski's measure of noncompactness α* mentioned in the introduction, is a generalization of Schauder's fixed point theorem.

Theorem 3.6 (Darbo's fixed point theorem). *Let C be a non–empty bounded, closed and convex subset of a Banach space X and α be the Kuratowski measure of noncompactness on X. If $T : C \to C$ is a continuous map such that there exists a constant $c \in [0, 1)$ with*

$$\alpha(T(Q)) \leq k \cdot \alpha(Q) \text{ for every } Q \subset C, \tag{3.1}$$

then T has a fixed point in C.

We will prove a generalization of Theorem 3.6, namely the Darbo–Sadovskiĭ theorem, in the next section.

4. Measures if noncompactness and the Darbo–Sadovskiĭ theorem

Darbo's fixed point theorem generalizes from compact sets to bounded and closed sets in infinite dimensional Banach spaces, and needs the additional hypothesis of the *condensing property* in (3.1). As is well known, when we pass from finite to infinite dimensional Banach spaces, bounded and closed subsets need not necessarily be compact. So it is natural to ask if Schauder's fixed point theorem (Theorem 3.5) holds in infinite dimensional Banach spaces for convex, closed and bounded subsets. The following example provides a strong negative answer to this question.

Example 4.1 (Kakutani)**.** There is a fixed point free continuous map on the unit ball of

$$\ell_2(\mathbb{Z}) = \left\{ x = (x_n) : \sum_{n \in \mathbb{Z}} |x_n|^2 < \infty \right\}.$$

Proof.   We consider $\ell_2(\mathbb{Z})$ with the standard Schauder basis $(e^{(n)})_{n \in \mathbb{Z}}$, where for each $n \in \mathbb{N}$, $e^{(n)}$ is the sequence with $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ for $k \neq n$, and with the natural norm given by

$$\|x\| = \|x\|_2 = \left(\sum_{n \in \mathbb{Z}} |x_n|^2 \right)^{1/2} \quad \text{for all } x \in \ell_2(\mathbb{Z}).$$

We write $\overline{B}_{\ell_2(\mathbb{Z})}$ for the closed unit ball in $\ell_2(\mathbb{Z})$. Every sequence $x = (x_n)_{n \in \mathbb{Z}} \in \ell_2(\mathbb{Z})$ has a unique representation $x = \sum_{n \in \mathbb{Z}} x_n e^{(n)}$. We define the left shift operator $U : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z})$ by

$$U(x) = \sum_{n \in \mathbb{Z}} x_n e^{(n+1)}.$$

The relation

$$x - U(x) = \sum_{n \in \mathbb{Z}} (x_n - x_{n-1}) e^{(n)} = c \cdot e^{(0)}$$

implies $x_n = x_0$ for all $n > 0$ and $x_n = x_1$ for all $n < 0$. For a sequence in $\ell_2(\mathbb{Z})$, this is only possible if $x_0 = x_1 = 0$. So $x - U(x)$ is a multiple of $e^{(0)}$ if and only if $x = 0$.
We define the map $T : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z})$ by

$$T(x) = (1 - \|x\|) e^{(0)} + U(x).$$

Then T maps $\overline{B}_{\ell_2(\mathbb{Z})}$ into $\overline{B}_{\ell_2(\mathbb{Z})}$, since we have for $\|x\| \leq 1$

$$\|T(x)\| \leq |1 - \|x\|| \cdot \|e^{(0)}\| + \|U(x)\| = (1 - \|x\|) + \|x\| = 1.$$

Finally, T is a fixed point free map. Indeed, if

$$x - T(x) = (1 - \|x\|) e^{(0)} + U(x),$$

then $x - U(x) = (1 - \|x\|)e^{(0)}$, which is clearly impossible if $x = 0$, and impossible if $x \neq 0$, as we have seen above.   □

To be able to prove the Darbo–Sadovskiĭ theorem we need to recall the concepts of measures of noncompactness, in particular, the Kuratowski measure of noncompactness, and their most important properties. The results presented here and their proofs can be found, for instance, in [53, 35, 36].

Since notion of a measure of noncompactness was originally introduced in metric spaces, we are going to give our axiomatic definition in this class of spaces as given in the monograph [53]. In the books [1] and [3], two different patterns are provided for the axiomatic introduction of measures of noncompactness in Banach spaces.

Definition 4.2. Let (X, d) be a complete metric space. A set function $\phi : \mathcal{M}_X \to [0, \infty)$ is called a *measure of noncompactness on fX*, if it satisfies the following conditions

> (MNC.1) $\phi(Q) = 0$ if and only if Q is relatively compact
>   (regularity)
> (MNC.2) $\phi(Q) = \phi(\overline{Q})$ for all $Q \in \mathcal{M}_X$
>   (invariance under closure)
> (MNC.3) $\phi(Q_1 \cup Q_2) = \max\{\phi(Q_1), \phi(Q_2)\}$ for all $Q_1, Q_2 \in \mathcal{M}_X$
>   (semi–additivity).

The number $\phi(Q)$ is called the *measure of noncompactness of the set Q.*

The following properties can easily be deduced from the axioms in Definition 4.2.

Proposition 4.3. *Let ϕ be a measure of noncompactness on a complete metric space (X, d). Then ϕ has the following properties*

$$Q \subset \tilde{Q} \text{ implies } \phi(Q) \leq \phi(\tilde{Q}) \quad \text{(monotonicity);} \tag{4.1}$$

$$\phi(Q_1 \cap Q_2) \leq \min\{\phi(Q_1), \phi(Q_2)\} \text{ for all } Q_1, Q_2 \in \mathcal{M}_X. \tag{4.2}$$

$$\text{If } Q \text{ is finite then } \phi(Q) = 0 \text{ (non–singularity).} \tag{4.3}$$

Generalised Cantor's intersection property
If (Q_n) is a decreasing sequence of nonempty sets in $\mathcal{M}_X^c$ and $\lim_{n\to\infty} \phi(Q_n) = 0$, then the intersection
$$Q_\infty = \bigcap Q_n \neq \emptyset$$
is compact. $\tag{4.4}$

Remark 4.4. If X is a Banach space then a measure of noncompactness ϕ may have some additional properties related to the linear structure of a normed space, for instance

$$\phi(\lambda Q) = |\lambda| \phi(Q) \text{ for any scalar } \lambda \text{ and all } Q \in \mathcal{M}_X \quad \text{(homogeneity)} \tag{4.5}$$

$$\phi(Q_1 + Q_2) \leq \phi(Q_1) + \phi(Q_2) \text{ for all } Q_1, Q_2 \in \mathcal{M}_X \quad \text{(subadditivity)} \tag{4.6}$$

$$\phi(x + Q) = \phi(Q) \text{ for any } x \in X \text{ and all } Q \in \mathcal{M}_X \quad \text{(translation invariance).} \tag{4.7}$$

For every $Q_0 \in \mathcal{M}_X$ and for all $\varepsilon > 0$ there is $\delta > 0$ such that
$$|\phi(Q_0) - \phi(Q)| < \varepsilon \text{ for all } Q \in \mathcal{M}_X \text{ with } d_H(Q_0, Q) < \delta \tag{4.8}$$
(continuity)

$$\phi\left(co(Q)\right) = \phi(Q) \text{ for all } Q \in \mathcal{M}_X$$
(invariance under the passage to the convex hull). $\tag{4.9}$

The two most important measures of noncompactness are the Kuratowski and Hausdorff measures of noncompactness

First we define the measure of noncompactness introduced by Kuratowski in 1930.

Definition 4.5. Let (X, d) be a complete metric space. The function

$$\alpha : \mathcal{M}_X \to [0, \infty)$$

with

$$\alpha(Q) = \inf\left\{ \varepsilon > 0 : Q \subset \bigcup_{k=1}^n S_k, \ S_k \subset X, \ diam(S_k) < \varepsilon \ (k = 1, 2, \ldots, n \in \mathbb{N}) \right\}$$

is called the *Kuratowski measure of noncompactness (KMNC)*, and the real number $\alpha(Q)$ is called the *Kuratowski measure of noncompactness of Q.*

Now we define the Hausdorff or ball measure of noncompactness which was first introduced by Goldenštein, Goh'berg and Markus in 1957 [GGM1] and later studied by Goldenštein and Markus in 1965 [GGM2].

The definition of the Hausdorff measure of noncompactness is similar to that of the Kuratowski measure of noncompactness and the results are analogous.

Definition 4.6. Let (X, d) be a complete metric space. The function

$$\chi : \mathcal{M}_X \to [0, \infty)$$

with

$$\chi(Q) = \inf\left\{ \varepsilon > 0 : Q \subset \bigcup_{k=1}^n B_{r_k}(x_k), \ x_k \in X, \ r_k < \varepsilon \ (k = 1, 2, \ldots, n \in \mathbb{N}) \right\}$$

is called the *Hausdorff,* or *ball measure of noncompactness*, and the real number $\chi(Q)$ is called the *Hausdorff* or *ball measure of noncompactness of Q.*

Both the Kuratowski and the Hausdorff measure of noncompactness are a measure of noncompactness in the sense of Definition 4.2.

Theorem 4.7. *Let (X, d) be a complete metric space. Then the Kuratowski and Hausdorff measures measures of noncompactness α and χ are measures of noncompactness in the sense of Definition 4.2.*

In Banach spaces the functions α and χ satisfy some additional properties related to the linear structures of normed spaces. The statements of the following results for the Kuratowski measure of noncompactness are due to Darbo.

Theorem 4.8. *Let X be a normed space ψ denote the Kuratowski or Hausdorff measure of noncompactness, and $Q, Q_1, Q_2 \in \mathcal{M}_X$. Then we have*

$$\psi(Q_1 + Q_2) \leq \psi(Q_1) + \psi(Q_2) \qquad \text{(subadditivity)}, \tag{4.10}$$
$$\psi(Q + x) = \psi(Q) \text{ for each } x \in X \qquad \text{(translation invariance)}, \tag{4.11}$$
$$\psi(\lambda Q) = |\lambda| \psi(Q) \text{ for each scalar } \lambda \qquad \text{(homogeneity)}, \tag{4.12}$$

and

$$\psi(Q) = \psi(\text{co}(Q)) \qquad \text{(invariance under the passage to the convex hull)}. \tag{4.13}$$

Now we state and prove the Darbo–Sadovskiĭ theorem.

Theorem 4.9 (Darbo–Sadovskiĭ). *Let X be a Banach space, ϕ be a measure of noncompactness which is invariant under passage to the convex hull, $C \neq \emptyset$ be a bounded, closed and convex subset of X and $T : C \to C$ be a ϕ– condensing operator, that is, T is continuous and satisfies*

$$\phi(T(Q)) < \phi(Q) \text{ for all bounded non–precompact subsets } Q \text{ of } C. \tag{4.14}$$

Then T has a fixed point.

Proof. We choose a point $c \in C$ and denote by Σ the class of all closed and convex subsets K of C such that $c \in K$ and $T(K) \subset K$. Furthermore, we put

$$B = \bigcap_{K \in \Sigma} K \text{ and } A = \overline{\text{co}}\,(T(B) \cup \{c\}).$$

Obviously $\Sigma \neq \emptyset$, since $C \in \Sigma$, and $B \neq \emptyset$, since $c \in B$. We also have

$$T(B) = T\left(\bigcap_{K \in \Sigma} K\right) \subset \bigcap_{K \in \Sigma} T(K) \subset \bigcap_{K \in \Sigma} K = B,$$

and consequently $T : B \to B$.
Moreover, we have $B = A$. Indeed, since $c \in B$ and $T(B) \subset B$, it follows that $A \subset B$. This implies $T(A) \subset T(B) \subset A$, and so $A \in \Sigma$, and hence $B \subset A$.
Therefore the properties of ϕ now imply

$$\phi(B) = \phi(C) = \phi\left(\overline{\text{co}}\,(T(B) \cup \{c\})\right) = \phi\left(\text{co}\,(T(B) \cup \{c\})\right) = \phi\left(T(B) \cup \{c\}\right)$$
$$= \max\left\{\phi(T(B)), \phi(\{c\})\right\} = \phi(T(B)).$$

Since T is ϕ–condensing, it follows that $\phi(B) = 0$, and so B is compact. Obviously B is also convex. Thus it follows from Schauder's fixed point theorem, Theorem 3.5, that the operator $T : C \to C$ has a fixed point.
□

The following example will show that the theorem of Darbo and Sadovskiĭ fails to be true, if we assume that T is a k–contractive operator with constant $k = 1$, that is, if we replace the the condensing condition (4.14) by the condition

$$\phi(T(Q)) \leq \phi(Q) \text{ for all bounded } Q \text{ of } C. \tag{4.15}$$

Example 4.10. Let $\overline{B}_{\ell_2}$ be the closed unit ball in ℓ_2. We define the operator $T : \overline{B}_{\ell_2} \to \overline{B}_{\ell_2}$ by

$$T(x) = T\left((x_k)_{k=1}^\infty\right) = \left(\sqrt{1 - \|x\|_2}, x_1, x_2, \dots\right)$$

Then we can write $T = D + S$ where D is the one dimensional mapping

$$D(x) = D\left((x_k)_{k=1}^\infty\right) = \sqrt{1 - \|x\|_2}\,e^{(1)} \text{ and } S(x) = (0, x_1, x_2, \dots)$$

is an isometry. Hence T is a well–defined, continuous operator, and for every bounded subset Q of $\overline{B}_{\ell_2}$, we have

$$\alpha(T(Q)) \leq \alpha(D(Q) + S(Q)) \leq \alpha(D(Q)) + \alpha(S(Q)) = 0 + \alpha(Q).$$

So T is a k–set–contractive operator with constant $k = 1$. But T has no fixed points. If T had a fixed point $x \in \overline{B}_{\ell_2}$, then we would have $x_k = x_{k+1}$ for all $k \in \mathbb{N}$. Since $x \in \ell_2$, this implies $x_k = 0$ for all $k \in \mathbb{N}$, and then $T(x) = \sqrt{1 - \|x\|_2}\,e^{(0)} = e^{(0)} = (0, 0, 0, \dots)$, a contradiction.

5. Edelstein's results

For a function $f : X \to X$ on a complete metric space (X, d) which satisfies the condition

$$d(f(x), f(y)) < \lambda\, d(x, y) \text{ for all } x, y \in X \text{ with } x \neq y, \tag{5.1}$$

where $0 \leq \lambda < 1$, the Banach contraction principle yields the existence and uniqueness of fixed points.

If we take $\lambda = 1$ in the condition in (5.1) then we we obtain a contractive map, that is, a map which satisfies the condition

$$d(f(x), f(y)) < d(x, y) \text{ for all } x, y \in X \text{ with } x \neq y. \tag{5.2}$$

In 1962, Edelstein [20] published a paper in which he studied the fixed points of contractive maps using the next condition and assumption

The condition in (5.2) together with the assumption of the existence of $x \in X$ such that the iterative sequence $(f^n(x))$ contains a convergent subsequence $(f^{n_k}(x))$ in X, that is,

$$\text{there exists } x \in X \text{ such that } \{f^n(x)\} \supset \{f^{n_k}(x)\} \text{ such that } \lim_{k \to \infty} f^{n_k}(x) \in X, \tag{5.3}$$

provides the existence of a fixed point of f.

Theorem 5.1 (Edelstein [20]). *Let X be a metric space and $f : X \to X$ be a contractive map that satisfies the condition in (5.3). Then $u = \lim_{k \to \infty} f^{n_k} x$ is the unique fixed point of f.*

Proof. Let $\Delta = \{(x, x) : x \in X\}$, $Y = (X \times X) \setminus \Delta$, and $r : Y \to \mathbb{R}$ be the map defined by

$$r(x, y) = \frac{d(f(x), f(y))}{d(x, y)}. \tag{5.4}$$

The function r is continuous on Y, and there exists a neighborhood U of points $(u, f(u))$ such that $(x, y) \in U$ implies

$$0 \leq r(x, y) < R < 1. \tag{5.5}$$

Let $B_1 = B_\rho^1(u)$ and $B_2 = B_\rho^2(f(u))$ be the open balls with centers in u and $f(u)$, and radius ρ such that

$$\rho < \frac{1}{3}\, d(u, f(u)) \tag{5.6}$$

and $B_1 \times B_2 \subset U$.

It follows from (5.3) that there exists a natural number N such that $k > N$ implies $f^{n_k}(x) \in B_1$, and (5.2)

implies $f^{n_k+1}(x) \in B_2$.
For $k > N$, (5.6) implies

$$d(f^{n_k}(x), f^{n_k+1}(x)) > \rho, \tag{5.7}$$

and it follows from (5.4) and (5.5) that

$$d(f^{n_k+1}(x), f^{n_k+2}(x)) < Rd(f^{n_k}(x), f^{n_k+1}(x)). \tag{5.8}$$

Hence, (5.8) implies for $l > j > N$

$$\begin{aligned}
d(f^{n_l}(x), f^{n_l+1}(x)) &\le d(f^{n_{l-1}+1}(x), f^{n_{l-1}+2}(x)) \\
&< Rd(f^{n_{l-1}}(x), f^{n_{l-1}+1}(x)) \le \dots \\
&< R^{l-j}d(f^{n_j}(x), f^{n_j+1}(x)) \to 0 \ (l \to \infty),
\end{aligned}$$

which is a contradiction to (5.7). This we must have $f(u) = u$.
We assume that $v \ne u$ is also a fixed point of the function f. Then we have

$$d(f(u), f(v)) = d(u, v),$$

which is a contradiction to (5.2). □

The condition in (5.3) is always satisfied for a compact space. Therefore we have

Theorem 5.2 (Edelstein [20])**.** *Let (X, d) be a compact metric space and $f : X \to X$ be a map. We assume*

$$d(f(x), f(y)) < d(x, y) \text{ for all } x, y \in X \text{ with } x \ne y.$$

Then the function f has a unique fixed point.

We obtain the following result on the iteration sequence from Theorem 5.1.

Theorem 5.3 (Edelstein [20])**.**
We assume that the conditions of Theorem 5.1 are satisfied. If the sequence $(f^n(p))$ for $p \in X$ contains a convergent subsequence $(f^{n_k}(p))$ then its limit $u = \lim_{n \to \infty} f^n(p)$ in X exists and u is a fixed point of f.

Proof. By Theorem 5.1, we have $u = \lim_{k \to \infty} f^{n_k}(p)$. For given $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that $k > n_0$ implies $d(u, f^{n_k}(p)) < \delta$. If $m = n_k + l > n_k$, then we have

$$d(u, f^m(p)) = d(f^l(u), f^{n_k+l}(p)) < d(u, f^{n_k}(p)) < \delta. \qquad \square$$

6. Rakotch's results

The problem of defining a family of functions $F = \{\alpha(x, y)\}$ which satisfy the conditions $0 \le \alpha < 1$ and $\sup \alpha(x, y) = 1$ such that Banach's theorem is satisfied when the constant α is replaced by $\alpha(x, y) \in F$ was suggested by *H. Hanani*, and *Rakotch* published a result related to this problem in 1962 [43] In this subsection, we present some results from the mentioned paper.

Definition 6.1. Let (X, d) be a metric space. We denote by F_1 the family of all functions $\alpha(x, y)$ which satisfy the following conditions:

(1) $\alpha(x, y) = \alpha(d(x, y))$, that is, α depends only on the distance of x and y.

(2) $0 \le \alpha(d) < 1$ for all $d > 0$.

(3) $\alpha(d)$ is a monotone decreasing function of d.

Theorem 6.2. *Let (X, d) be a metric space, $f : X \to X$ be a contractive map, $M \subset X$ and $x_0 \in M$ such that*

$$d(x, x_0) - d(f(x), f(x_0)) \geq 2d(x_0, f(x_0)) \text{ for all } x \in X \setminus M, \tag{6.1}$$

and let $f(M)$ be a subset of a compact subset of X. Then there exists a unique fixed point of f.

Proof. We assume $f(x_0) \neq x_0$ and put $x_n = f^n(x_0)$ for $n = 1, 2, \ldots$, that is,

$$x_{n+1} = f(x_n) \text{ for } n = 0, 1, \ldots. \tag{6.2}$$

By Edelstein's theorem (Theorem 5.1), it suffices to show that $x_n \in M$ for each n.
Since f is a contractive map, the sequence $(d(x_n, x_{n+1}))$ is not increasing. Hence $f(x_0) \neq x_0$ implies

$$d(x_n, x_{n+1}) < d(x_0, x_1) \text{ for } n = 1, 2, \ldots. \tag{6.3}$$

We obtain from the triangle inequality

$$d(x_0, x_n) \leq d(x_0, x_1) + d(x_1, x_{n+1}) + d(x_n, x_{n+1}).$$

Now (6.2) and (6.3) yield

$$d(x_0, x_n) - d(f(x_0), f(x_n)) < 2d(x_0, f(x_0)),$$

and (6.1) implies $x_n \in M$ for all n. $\qquad\qquad\square$

Corollary 6.3. *Let f be a contractive map for which there exists a point $x_0 \in X$ such that for all $x \in X$*

$$d(f(x), f(x_0)) \leq \alpha(x, x_0)d(x, x_0), \tag{6.4}$$

where $\alpha(x, y) = \alpha(d(x, y)) \in F_1$. If $B_r(x_0)$ is the open ball in X, where

$$r = \frac{2d(x_0, f(x_0))}{1 - \alpha(2d(x_0, f(x_0)))},$$

and $f(B_r(x_0))$ is a subset of a compact subset of X, then the function f has a unique fixed point.

Proof. If we put $M = B(x_0, r)$ in Theorem 6.2, then by (6.4), the monotony of $\alpha(d)$ and $r \geq 2d(x_0, fx_0)$, the condition $d(x, x_0) \geq r$ implies

$$
\begin{aligned}
d(x, x_0) - d(f(x), f(x_0)) &\geq d(x, x_0) - \alpha(d(x, x_0))d(x, x_0) \\
&= [1 - \alpha(d(x, x_0))]d(x, x_0) \geq [1 - \alpha(r)]r \\
&\geq [1 - \alpha(2d(x_0, f(x_0)))]r = 2d(x_0, f(x_0)),
\end{aligned}
$$

that is, we have (6.1). $\qquad\qquad\square$

Theorem 6.4. *Let $f : X \to X$ be a contractive map on a complete metric space. We assume that there exist $M \subset X$ and a point $x_0 \in M$ such that*

$$d(x, x_0) - d(f(x), f(x_0)) \geq 2d(x_0, f(x_0)) \text{for each } x \in X \setminus M, \tag{6.5}$$

$$d(f(x), f(y)) \leq \alpha(x, y)d(x, y) \text{ for all } x, y \in M, \tag{6.6}$$

where

$$\alpha(x, y) = \alpha(d(x, y)) \in F_1.$$

Then the function f has a unique fixed point.

Proof. We assume $f(x_0) \neq x_0$ and define the sequence (x_n) by $x_n = f^n(x_0)$ for $n = 1, 2, \ldots$. As in Theorem 6.2, we have by (6.5)

$$d(x_n, x_{n+1}) < d(x_0, x_1) \text{ for } n = 1, 2, \ldots \tag{6.7}$$

and $x_n \in M$ for each n.

We are going to prove that the sequence (x_n) is bounded. It follows from (6.6) and the definition of the sequence (x_n) that

$$d(x_1, x_{n+1}) = d(f(x_0), f(x_n)) \leq \alpha(d(x_0, x_n))d(x_0, x_n), \tag{6.8}$$

and, by the triangle inequality, we have

$$d(x_0, x_n) \leq d(x_0, x_1) + d(x_1, x_{n+1}) + d(x_n, x_{n+1}).$$

Hence (6.7) and (6.8) imply

$$[1 - \alpha(d(x_0, x_n))]d(x_0, x_n) < 2d(x_0, x_1).$$

If $d(x_0, x_n) \geq d_0$ for some given d_0, then we have by the monotony of α

$$\alpha(d(x_0, x_n)) \leq \alpha(d_0).$$

So we obtain

$$d(x_0, x_n) < \frac{2d(x_0, x_1)}{1 - \alpha(d(x_0, x_n))} \leq \frac{2d(x_0, x_1)}{1 - \alpha(d_0)} = C.$$

Hence we have for $R = \max\{d_0, C\}$

$$d(x_0, x_n) \leq R \text{ for } n = 1, 2, \ldots, \tag{6.9}$$

that is, the sequence (x_n) is bounded.

Let $p > 0$ be an arbitrary natural number. It follows from (6.6) that

$$d(x_{k+1}, x_{k+p+1}) \leq \alpha(x_k, x_{k+p})d(x_k, x_{k+p}),$$

that is,

$$d(x_n, x_{n+p}) \leq d(x_0, x_p) \prod_{k=0}^{n-1} \alpha(x_k, x_{k+p}).$$

Now (6.9) implies

$$d(x_n, x_{n+p}) \leq R \prod_{k=0}^{n-1} \alpha(x_k, x_{k+p}). \tag{6.10}$$

We prove that (x_n) is a Cauchy sequence. It is enough to show that, for every $\varepsilon > 0$, there exists N which depends only on ε (and not on p) such that, for all $p > 0$, we have $d(x_N, x_{N+p}) < \varepsilon$ (since the sequence $(d(x_n, x_{n+p}))$ is not increasing).

If $d(x_k, x_{k+p}) \geq \varepsilon$ for $k = 0, 1, \ldots, n-1$, then we obtain from (6.6) (because of the monotony of the function α)

$$\alpha(x_k, x_{k+p}) = \alpha(d(x_k, x_{k+p})) \leq \alpha(\varepsilon),$$

and then (6.10) implies

$$d(x_n, x_{n+p}) \leq R[\alpha(\varepsilon)]^n.$$

Since $\alpha(\varepsilon) < 1$ and $[\alpha(\varepsilon)]^n \to 0$ as $n \to \infty$, there exists a natural number N, independent of p, such that $d(x_N, x_{N+p}) < \varepsilon$ for each $p > 0$. Hence (x_n) is a Cauchy sequence.

Since X is a complete metric space, there exists $u \in X$ such $u = \lim_{n\to\infty} x_n$. Because of the continuity of the function f, u is a fixed point of f. $\square$

In particular, if $M = X$, we obtain the next corollary.

Corollary 6.5. *Let (X, d) be a complete metric space and*

$$d(f(x), f(y)) \leq \alpha(x, y) d(x, y) \text{ for all } x, y \in X, \tag{6.11}$$

where $\alpha(x, y) \in F_1$. Then the function f has a unique fixed point.

Remark 6.6. The preceding corollary and Theorem 6.4 are generalizations of Banach's fixed point theorem.

7. Boyd and Wong's nonlinear contraction

In this section, we present some results by Boyd and Wong [7] in 1969. In [7], Boyd and Wong studied fixed points for maps of the kind introduced in the next definition.

Definition 7.1. Let (X, d) be a metric space. A map $f : X \to X$ which satisfies the condition

$$d(f(x), f(y)) \leq \Psi(d(x, y)) \text{ for all } x, y \in X, \tag{7.1}$$

where Ψ is a function defined on the closure of the range of d, is called a Ψ *contraction*.
We denote the image of d by P and the closure of P by $\overline{P}$. Hence $P = \{d(x, y) : x, y \in X\}$.

Rakotch [43] proved that if $\Psi(t) = \alpha(t)t$, where α is a decreasing function with $\alpha(t) < 1$ for all $t > 0$, then the map f satisfying (7.1) has a unique fixed point u. It can be shown that if $\Psi(t) = \alpha(t)t$ and α is an increasing function with $\alpha(t) < 1$ for all $t \geq 0$, then the conclusion of Banach's theorem holds true. Boyd and Wong proved that it is enough to assume that $\Psi(t) < t$ for all $t > 0$ and Ψ is semicontinuous, and if a metric space is convex, then the last condition can be omitted.
We recall that a function $\varphi : X \to E$ $(E \subset \mathbb{R})$ is said to be *upper semi–continuous from the right at* $t_0 \in X$ if $t_n \to t_0+$ implies $\limsup_{n \to \infty} \varphi(t_n) \leq \varphi(t_0)$. A function $\varphi : X \mapsto E$ $(E \subset \mathbb{R})$ is said to be *upper semi–continuous from the right on X* if it is upper semi–continuous from the right at every $t \in X$.

Theorem 7.2. *Let (X, d) be a complete metric space and $f : X \to X$ be a map satisfying (7.1), where $\Psi : \overline{P} \mapsto [0, \infty)$ is upper semi–continuous from the right on $\overline{P}$ and satisfies $\Psi(t) < t$ for all $t \in \overline{P} \setminus \{0\}$. Then the function f has a unique fixed point x_0 and $f^n(x) \to x_0 \ (\to \infty)$ for each $x \in X$.*

Proof. Let $x \in X$ and

$$c_n = d(f^n(x), f^{n-1}(x)) \text{ for } n = 1, 2, \dots. \tag{7.2}$$

Then, because of (7.1), the sequence (c_n) is monotone decreasing. We put $\lim_{n \to \infty} c_n = c \geq 0$, and prove $c = 0$. If $c > 0$, then we have

$$c_{n+1} \leq \Psi(c_n), \tag{7.3}$$

hence

$$c \leq \limsup_{t \to c+} \Psi(t) \leq \Psi(c) < c, \tag{7.4}$$

which is a contradiction.
We are going to prove that $(f^n(x))$ is a Cauchy sequence for each $x \in X$. Then the limit point of this sequence is the unique fixed point of the function f. We assume that $(f^n(x))$ is not a Cauchy sequence. Then there exist $\varepsilon > 0$ and sequences $(m(k))$ and $(n(k))$ of natural numbers with $m(k) > n(k) \geq k$ such that

$$d_k = d(f^{m(k)}(x), f^{n(k)}(x)) \geq \varepsilon \text{ for all } k = 1, 2, \dots. \tag{7.5}$$

We may assume that

$$d(f^{m(k)-1}(x), f^{n(k)}(x)) < \varepsilon, \tag{7.6}$$

and choose $m(k)$ as the smallest integer greater than $n(k)$ which satisfies (7.5). It follows from (7.2) that

$$d_k \leq d(f^{m(k)}(x), f^{m(k)-1}(x)) + d(f^{m(k)-1}(x), f^{n(k)}(x)) \leq c_m + \varepsilon \leq c_k + \varepsilon. \tag{7.7}$$

Hence $d_k \to \varepsilon$ as $k \to \infty$. Since

$$
\begin{aligned}
d_k = d(f^{m(k)}(x), f^{n(k)}(x)) &\leq\ d(f^{m(k)}(x), f^{m(k)+1}(x)) + \\
&\quad + d(f^{m(k)+1}(x), f^{n(k)+1}(x)) + d(f^{n(k)+1}(x), f^{n(k)}(x)) \\
&\leq\ 2c_k + \Psi(d(f^{m(k)}(x), f^{n(k)}(x))) = 2c_k + \Psi(d_k), \tag{7.8}
\end{aligned}
$$

letting $k \to \infty$ in (7.8), we obtain $\varepsilon \leq \Psi(\varepsilon)$. This is a contradiction, because we have $\Psi(\varepsilon) < \varepsilon$ for $\varepsilon > 0$. $\square$

The following example will show that the condition of the continuity of the function Ψ in Theorem 7.2 cannot be dropped, in general.

Example 7.3. Let $X = \{x_n = n\sqrt{2} + 2^n : n = 0, \pm 1, \pm 2, \dots\}$ have the metric $d(x, y) = |x - y|$. Then X is a closed subset of the real numbers, and so complete. We assume that for each $p \in P$ ($p \neq 0$), there exists a unique pair (x_n, x_m) such that $p = d(x_n, x_m)$. We assume that

$$
d(x_j, x_k) = d(x_m, x_n) \text{ for some integers } j, k, m, n \text{ with } j > k \text{ and } m > n.
$$

Then we obtain

$$
-(m - n - j + k)\sqrt{2} = 2^j - 2^k - 2^m + 2^n. \tag{7.9}
$$

Since the left hand side in (7.9) is irrational or equal to zero and the right hand side is rational, it follows that both sides are equal to zero. Hence we have for $m - n = j - k = s$

$$
2^{n+s} - 2^n = 2^{k+s} - 2^k, \tag{7.10}
$$

which is only possible for $n = k$. We define the functions f by $f(x_n) = x_{n-1}$ and Ψ on P by

$$
\Psi(p) = |x_{n-1} - x_{m-1}| \text{ if } p = |x_n - x_m|. \tag{7.11}
$$

We put $\Psi(p) = 0$ for $p \in \overline{P} \setminus P$.
Then we have $\Psi(t) < t$ for all $t \in \overline{P} \setminus \{0\}$ and

$$
d(f(x), f(y)) = \Psi(d(x, y)), \tag{7.12}
$$

but the function f has no fixed point.

Theorem 7.2 shows that it is not possible to extend the function Ψ from the set P to the set $\overline{P}$ such that it is upper semi–continuous from the right with $\Psi(t) < t$ for $t \in \overline{P} \setminus \{0\}$. This can directly be seen for the point $\sqrt{2} \in \overline{P} \setminus P$.

If the condition $\Psi(t) < t$ is replaced by $\Psi(t_0) = t_0$ for some value t_0, then Theorem 7.2 does not hold. This is shown in the next example.

Example 7.4. Let $X = (-\infty, -1] \cup [1, \infty)$ and $d(x, y) = |x - y|$ for all $x, y \in X$. Also let

$$
f_1(x) = \begin{cases} \dfrac{1}{2}(x + 1) & \text{for } x \geq 1 \\[2mm] \dfrac{1}{2}(x - 1) & \text{for } x \leq -1. \end{cases} \quad \text{and} \quad f_2(x) = -f_1(x).
$$

Now the functions f_1 and f_2 satisfy (7.1), if we define

$$
\Psi(t) = \begin{cases} \dfrac{1}{2} t & \text{for } t < 2 \\[2mm] \dfrac{1}{2} t + 1 & \text{for } t \geq 2. \end{cases}
$$

We know that the function Ψ satisfies all the conditions in Theorem 7.2, but $\Psi(2) = 2$. The function f_1 has two fixed points -1 and 1 and the function f_2 has no fixed points.

Theorem 7.2 is a generalization of Rakotch's theorem. This is shown in the next example.

Example 7.5. Let $X = [0,1] \bigcup \{2,3,4\dots\}$ be the complete metric space with its metric d defined by

$$d(x,y) = \begin{cases} |x-y| & \text{if } x,y \in [0,1] \\ x+y & \text{if at least one of } x,y \notin [0,1]. \end{cases}$$

We define the function $f : X \to X$ by

$$f(x) = \begin{cases} x - \dfrac{1}{2}\,x^2 & \text{for } x \in [0,1] \\ x - 1 & \text{for } x = 2,3,\dots. \end{cases}$$

If $x,y \in [0,1]$ for $x - y = t > 0$, then we have

$$d(f(x),f(y)) = (x-y)\left(1 - \frac{1}{2}(x+y)\right) \le t\left(1 - \frac{1}{2}t\right),$$

and if $x \in \{2,3,4,\dots\}$ and $x > y$, then we have

$$d(f(x),f(y)) = f(x) + f(y) < x - 1 + y = d(x,y) - 1.$$

We define the function Ψ by

$$\Psi(t) = \begin{cases} t - \dfrac{1}{2}\,t^2 & \text{for } 0 \le t \le 1 \\ t - 1 & \text{for } 1 < t < \infty. \end{cases}$$

The function Ψ is upper semi–continuous from the right on the set $[0,\infty)$, $\Psi(t) < t$ for all $t > 0$, and the condition in (7.1) is satisfied.
Since

$$\lim_{n\to\infty} \frac{d(f(n),0)}{d(n,0)} = 1,$$

there is no decreasing function α with $\alpha(t) < 1$ for all $t > 0$ which satisfies (6.11). Furthermore, since

$$\lim_{x\to 0} \frac{d(f(x),0)}{d(x,0)} = 1,$$

there is no increasing function α with $\alpha(t) < 1$ for all $t > 0$ which satisfies (6.11).

8. Theorem of Meir-Keeler

In 1969, Meir and Keeler [39] proved a very interesting theorem and showed that the conclusion of Banach's fixed point theorem can be extended to a more general class of contractions. In this subsection, we present some results of the paper mentioned.

Definition 8.1. Let (X,d) be a metric space. The function $f : X \mapsto X$ is said to be a *weakly uniformly strict contraction*, or a *Meir–Keeler contraction (MK contraction)* if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \le d(x,y) < \varepsilon + \delta \text{ implies } d(f(x),f(y)) < \varepsilon. \tag{8.1}$$

Theorem 8.2 (Meir and Keeler [39])**.** *Let (X,d) be a complete metric space and $f : X \to X$ be a function. If (8.1) is satisfied, then f has a unique fixed point u. Moreover, we have for each $x \in X$*

$$\lim_{n\to\infty} f^n(x) = u. \tag{8.2}$$

Proof. First we note that (8.1) implies that f is a contractive map, that is,

$$x \neq y \text{ implies } d(f(x), f(y)) < d(x, y). \tag{8.3}$$

Hence f is a continuous function and has at most one fixed point.

We note that if $(f^n(x))$ is a Cauchy sequence for each $x \in X$, then the function has a unique fixed point, and (8.2) is satisfied. This follows from the following consideration. Since X is a complete space, every Cauchy sequence $(f^n(x))$ has a limit $u(x)$. The continuity of f implies

$$f(u(x)) = f\left(\lim_{n \to \infty} f^n(x)\right) = \lim_{n \to \infty} f^{n+1}(x) = u(x).$$

Hence $u(x)$ is the unique fixed point of f.

The proof of the theorem will be complete if we show that the sequence $(f^n(x)) = (x_n)$ of iterations is a Cauchy sequence for each $x \in X$. Let $x \in X$ and $c_n = d(x_n, x_{n+1})$ for $n = 1, 2, \ldots$. It follows from (8.3) that (c_n) is a decreasing sequence. If $\lim_{n \to \infty} c_n = \varepsilon > 0$, then the implication in (8.1) is not true for c_{m+1}, where c_m is chosen such that $c_m < \varepsilon + \delta$. This implies $\lim_{n \to \infty} c_n = 0$.

We assume that there exists a sequence (x_n) which is not a Cauchy sequence. Then there exists $2\varepsilon > 0$ such that, for each $m_0 \in \mathbb{N}$, there exist $n, m \in \mathbb{N}$ with $n, m > m_0$ and $d(x_m, x_n) > 2\varepsilon$. It follows from (8.1) that there exists $\delta > 0$ such that

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \text{ implies } d(f(x), f(y)) < \varepsilon. \tag{8.4}$$

The implication in (8.4) remains true if we replace δ by $\delta' = \min\{\delta, \varepsilon\}$. Let $m_0 \in \mathbb{N}$ be such that $c_{m_0} < \delta'/3$, and let $m, n > m_0$ such that $m < n$ and $d(x_m, x_n) > 2\varepsilon$. We prove that there exists $j \in \{m, m+1, \ldots, n\}$ such that

$$\varepsilon + \frac{2\delta'}{3} < d(x_m, x_j) < \varepsilon + \delta'. \tag{8.5}$$

To prove (8.5), we note that $d(x_{n-1}, x_n) < \delta/3$. Since $d(x_m, x_n) > 2\varepsilon$ and $d(x_m, x_n) \leq d(x_m, x_{n-1}) + d(x_{n-1}, x_n)$, it follows that

$$\varepsilon + \frac{2\delta'}{3} < d(x_m, x_{n-1}). \tag{8.6}$$

Let k be the smallest natural number in $\{m, m+1, \ldots, n\}$; (clearly $m < k \leq n-1$) such that

$$\varepsilon + \frac{2\delta'}{3} < d(x_m, x_k) \tag{8.7}$$

holds. We prove $d(x_m, x_k) < \varepsilon + \delta'$. If we assume that this is not true, then we have

$$\varepsilon + \delta' \leq d(x_m, x_k) \leq d(x_m, x_{k-1}) + d(x_{k-1}, x_k) < d(x_m, x_{k-1}) + \frac{\delta'}{3},$$

that is,

$$\varepsilon + \frac{2\delta'}{3} < d(x_m, x_{k-1}). \tag{8.8}$$

This is a contradiction to the the minimality condition of k in the inequality in (8.7). Therefore the inequality in (8.5) must hold.

Now

$$d(x_m, x_k) \leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{k+1}) + d(x_{k+1}, x_k),$$

(8.4) and (8.5) imply

$$d(x_m, x_j) \leq c_m + \varepsilon + c_k < \frac{\delta'}{3} + \varepsilon + \frac{\delta'}{3}.$$

This is a contradiction to (8.5). Hence (x_n) is a Cauchy sequence. $\qquad\square$

It is well known that the Meir-Keeler theorem generalizes Banach's contraction principle [2] and Edelstein's theorem [20].

Theorem 8.3 (Banach [2])**.** *Let (X, d) be a complete metric space and $f : X \to X$ be a contraction, that is, there exists $q \in [0, 1)$ such that*

$$d(f(x), f(y)) \leq q \cdot d(x, y) \text{ for all } x, y \in X. \tag{8.9}$$

Then f has a unique fixed point.

Proof. Let $\varepsilon > 0$ and $\delta = (1/q - 1)\varepsilon$. Then it follows from $d(x, y) < \varepsilon + \delta$ and $x \neq y$ that $d(f(x), f(y)) \leq qd(x, y) < q\varepsilon + q\delta = \varepsilon$. Hence the function f satisfies (8.1) and the proof follows from Theorem 8.2. □

Theorem 8.4 (Edelstein [20])**.** *Let (X, d) be a compact metric space and $f : X \to X$ be a map. We assume that*

$$d(f(x), f(y)) < d(x, y) \text{ for all } x, y \in X \text{ with } x \neq y.$$

Then the function f has a unique fixed point.

Proof. We assume that the function f does not satisfy the condition in (8.1). Then there exist $\varepsilon > 0$ and sequences (x_n) and (y_n) in X such that

$$d(x_n, y_n) < \epsilon + \frac{1}{n} \text{ and } d(f(x_n), f(y_n)) \geq \epsilon. \tag{8.10}$$

Since X is a compact set, there exist subsequences (x_{n_k}) and (y_{m_k}) of the sequences (x_n) and (y_n), which converge to some $x_0 \in X$ and some $y_0 \in X$, respectively. The continuity of the function f implies

$$d(x_0, y_0) \leq \varepsilon \leq d(f(x_0), f(y_0)) < d(x_0, y_0).$$

This is a contradiction, and consequently the function f must satisfy the condition in (8.1). Now the proof follows from Theorem 8.2. □

Rakotch [43], and Boyd and Wong [7] assumed that, among other conditions, the following inequalities are satisfied:

$$d(f(x), f(y)) \leq \psi(d(x, y)) \text{ and } \psi(t) < t \text{ for all } t \neq 0. \tag{8.11}$$

The next example shows that the Meir–Keeler theorem holds even if the condition in (8.11) is not satisfied.

Example 8.5. Let $X = [0, 1] \cup \{3, 4, 6, 7, \ldots, 3n, 3n + 1, \ldots\}$ be endowed with the Euclidean metric and the function f be defined by

$$f(x) = \begin{cases} \dfrac{x}{2} & \text{for } 0 \leq x \leq 1 \\ 0 & \text{for } x = 3n \\ 1 - \dfrac{1}{n + 2} & \text{for } x = 3n + 1. \end{cases}$$

Then the function f satisfies (8.1), and it follows from

$$d(f(x), f(y)) \leq \psi(d(x, y)) \text{ for all } x, y \in X \tag{8.12}$$

that $\psi(1) = 1$.

9. Theorems by Kannan, Chatterje and Zamfirescu

The first result is by Kannan [30] in 1968.

Theorem 9.1. *If (X, d) is a complete metric space, $0 \leq q < 1/2$ and $f : X \to X$ be a map such that*

$$d(f(x), f(y)) \leq q[d(x, f(x)) + d(y, f(y))] \text{ for all } x, y \in X, \tag{9.1}$$

then f has a unique fixed point, that is, there exists one and only one $u \in X$ such that $f(u) = u$.

Proof. (Joseph and Kwack [29]) Let

$$c = \inf\{d(x, f(x)) : x \in X\}.$$

Then we have $c \geq 0$. If $c > 0$, then $c(1-q)/q > c$ implies the existence of $x \in X$ such $d(x, f(x)) < c(1-q)/q$. Now we have

$$d(f(x), f^2(x)) \leq \frac{q}{1-q}d(x, f(x)) < c,$$

which is a contradiction, and so $c = 0$. Hence there exists a sequence (x_n) in X such that $\lim_n d(x_n, f(x_n)) = 0$. It follows from

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, f(x_m)) + d(f(x_m), f(x_n)) + d(x_n, f(x_n)) \\ &\leq (1+q)[d(x_m, f(x_m)) + d(x_n, f(x_n))], \end{aligned}$$

that (x_n) is a Cauchy sequence. So there exists $p \in X$ such that $\lim_{n \to \infty} x_n = p$. It follows that $\lim_{n \to \infty} f(x_n) = p$.
We prove $f(p) = p$. It follows from

$$\begin{aligned} d(p, f(p)) &\leq d(p, f(x_n)) + d(f(x_n), f(p)) \\ &\leq d(p, f(x_n)) + q[d(x_n, f(x_n)) + d(p, f(p))], \end{aligned}$$

as $n \to \infty$ that

$$d(p, f(p)) \leq qd(p, f(p)),$$

and so $p = f(p)$. Now (9.1) implies that the map f has a unique fixed point. □
  Banach's condition (2.1) and Kannan's (9.1) condition are independent. The condition in (2.1) implies the continuity of the map f, but this is not the case for the condition in (9.1). This follows from the following example.

Example 9.2. Let $X = [0, 1]$ and $f(x)$ be defined by

$$f(x) = \begin{cases} \dfrac{x}{4} & \text{for } x \in [0, 1/2) \\[2mm] \dfrac{x}{5} & \text{for } x \in [1/2, 1]. \end{cases}$$

The map f is discontinuous at the point $x = 1/2$ and so the condition in (2.1) is not satisfied, but the condition in (9.1) is satisfied for $q = 4/9$.

Example 9.3. Let $X = [0, 1]$ and $f(x) = x/3$ for $x \in [0, 1]$. Clearly, the condition in (2.1) is satisfied, but the condition in (9.1) is not satisfied (we may take $x = 1/3$ and $y = 0$).

  The next theorem was proved by Chatterje [12] in 1972.

Theorem 9.4. *If (X, d) is a complete metric, $0 \leq q < 1/2$ and $f : X \to X$ is a map which satisfies the condition*

$$d(f(x), f(y)) \leq q[d(x, f(y)) + d(y, f(x))] \text{ for all } x, y \in X,$$

then the function f has a unique fixed point.

  Proof (Fisher [23]). Let $x \in X$. Then we have

$$\begin{aligned} d(f^n(x), f^{n+1}(x)) &\leq q[d(f^{n-1}(x), f^{n+1}(x)) + d(f^n(x), f^n(x))] \\ &= qd(f^{n-1}(x), f^{n+1}(x)) \\ &\leq q[d(f^{n-1}(x), f^n(x)) + d(f^n(x), f^{n+1}(x))], \end{aligned}$$

hence

$$d(f^n(x), f^{n+1}(x)) \le \frac{q}{1-q} d(f^{n-1}(x), f^n(x))$$

$$\le \left(\frac{q}{1-q}\right)^2 d(f^{n-2}(x), f^{n-1}(x))$$

$$\le \left(\frac{q}{1-q}\right)^n d(x, f(x)).$$

So we obtain

$$d(f^n(x), f^{n+r}(x)) \le d(f^n(x), f^{n+1}(x)) + \cdots + d(f^{n+r-1}x, f^{n+r}(x))$$

$$\le \left[\left(\frac{q}{1-q}\right)^n + \cdots + \left(\frac{q}{1-q}\right)^{n+r-1}\right] d(x, f(x))$$

$$\le \left(\frac{q}{1-q}\right)^n \frac{1}{1-q} d(x, f(x)).$$

Since $q(1-q)^{-1} < 1$, it follows that $(f^n(x))$ is a Cauchy sequence in X. Since X is a complete metric space, there exists $z \in X$ such that $z = \lim_n f^n(x)$.
Now we have

$$d(z, f(z)) \le d(z, f^n(x)) + d(f^n(x), f(z))$$

$$\le d(z, f^n(x)) + q[d(f^{n-1}(x), f(z)) + d(f^n(x), z)].$$

Letting $n \to \infty$, we obtain

$$d(z, f(z)) \le qd(z, f(z)),$$

and since $q < 1/2$, we have

$$f(z) = z.$$

Hence z is a fixed point of the function f.
We assume that the function f has one more fixed point $z' \in X$. Then we have

$$d(z, z') = d(f(z), f(z'))$$

$$\le q[d(z, f(z')) + d(z', f(z))]$$

$$= 2qd(z, z').$$

Since $q < 1/2$, it follows that $z = z'$, that is, the fixed point of the function f is unique. $\square$

In 1972, Zamfirescu [54] connected Banach's, Kannan's and Chatterje's theorems.

Theorem 9.5 (Zamfirescu [54]). *Let (X, d) be a complete metric space and $f : X \to X$ be a map for which there exist real numbers $0 \le \alpha < 1$, $0 \le \beta < 1$ and $\gamma < 1/2$ such that, for each $x, y \in X$, at least one of the following conditions is satisfied:*

    (z_1) $d(f(x), f(y)) \le \alpha d(x, y);$
    (z_2) $d(f(x), f(y)) \le \beta[d(x, f(x)) + d(y, f(y))];$
    (z_3) $d(f(x), f(y)) \le \gamma[d(x, f(y)) + d(y, f(x))].$

Then the function f has a unique fixed point.

Proof. Let $x, y \in X$. Then at least one of the conditions (z_1), (z_2) or (z_3) is satisfied. If (z_2) is satisfied, then we have

$$d(f(x), f(y)) \le \beta[d(x, f(x)) + d(y, f(y))]$$

$$\leq \beta\{d(x, f(x)) + [d(y,x) + d(x, f(x)) + d(f(x), f(y))]\}.$$

This implies

$$(1 - \beta)d(f(x), f(y)) \leq 2\,\beta d(x, f(x)) + \beta d(x,y),$$

that is,

$$d(f(x), f(y)) \leq \frac{2\beta}{1 - \beta}\,d(x, f(x)) + \frac{\beta}{1 - \beta}\,d(x,y).$$

Similarly, if (z_3) is satisfied, we get the following estimate

$$d(f(x), f(y)) \leq \gamma[d(x, f(y)) + d(y, f(x))] \leq$$
$$\leq \gamma[d(x, f(x)) + d(f(x), f(y)) + d(y,x) + d(x, f(x))] \leq$$
$$\leq \gamma[2d(x, f(x)) + d(f(x), f(y)) + d(x,y)].$$

Hence we have

$$d(f(x), f(y)) \leq \frac{2\gamma}{1 - \gamma}\,d(x, f(x)) + \frac{\gamma}{1 - \gamma}\,d(x,y).$$

We put

$$\lambda = \max\left\{\alpha, \frac{\beta}{1 - \beta}, \frac{\gamma}{1 - \gamma}\right\}.$$

Then we have $0 \leq \lambda < 1$, and if (z_2) or (z_3) is satisfied for each $x, y \in X$, then

$$d(f(x), f(y)) \leq 2\lambda \cdot d(x, f(x)) + \lambda \cdot d(x,y). \tag{9.2}$$

In a similar way, it can be shown that if (z_2) or (z_3) is satisfied, then

$$d(f(x), f(y)) \leq 2\lambda \cdot d(x, f(y)) + \lambda \cdot d(x,y). \tag{9.3}$$

Obviously, (9.2) and (9.3) follow from (z_1).

It follows from (9.2) that the function f has at least one fixed point. Now we prove the existence of a fixed point of f. Let $x_0 \in X$ and

$$x_n = f^n(x_0) \text{ for } n = 1, 2, \ldots$$

be the Picard iteration of f.

If $x = x_n$ and $y = x_{n-1}$ are two successive approximations, then it follows from (9.3) that

$$d(x_{n+1}, x_n) \leq \lambda \cdot d(x_n, x_{n-1}).$$

So $(x_n)_{n=0}^\infty$ is a Cauchy sequence, and consequently convergent. Let $u \in X$ be its limit. Then we have

$$\lim_{n \to \infty} d(x_{n+1}, x_n) = 0.$$

By the triangle inequality and (9.2), it follows that

$$d(u, f(u)) \leq d(u, x_{n+1}) + d(f(x_n), f(u))$$
$$\leq d(u, x_{n+1}) + \lambda \cdot d(u, x_n) + 2\lambda d(x_n, f(x_n)),$$

and letting $n \to \infty$, we obtain $d(u, f(u)) = 0$, hence $f(u) = u$. $\qquad\square$

Remark 9.6. If a function f satisfies the condition in Theorem 9.4, we write $f \in (Z)$, in particular, if f satisfies one of the conditions in (z_i) for $i = 1, 2, 3$ in this theorem, then we write $f \in (z_i)$ for $i = 1, 2, 3$.

We consider the conditions (Z'): there exist nonnegative functions a, b and c satisfying the following condition

$$\sup_{x,y \in X} (a(x,y) + 2b(x,y) + 2c(x,y)) \leq \lambda < 1,$$

such that, for each $x, y \in X$,

$$d(f(x), f(y)) \leq a(x, y)d(x, y) + b(x, y)(d(x, f(x)) + d(y, f(y)))$$
$$+ c(x, y)(d(x, f(y)) + d(y, f(x)));$$

and (Z''): There exists a constant h with $0 \leq h < 1$ such that, for all $x, y \in X$,

$$d(f(x), f(y)) \leq h \max\Big\{ d(x, y), \frac{d(x, f(x)) + d(y, f(y))}{2},$$
$$\frac{d(x, f(y)) + d(y, f(x))}{2} \Big\}. \tag{9.4}$$

It can be proved ([49]) that the conditions in (Z), (Z') and (Z'') equivalent.

We show that (Z) implies (Z').

If the function f and $x, y \in X$ satisfy (z_1), then we define $a(x, y) = \alpha$ and $b = c = 0$. If for $x, y \in X$, for which the function f satisfies (z_2), we define $b(x, y) = \beta$ and $a = c = 0$, and similarly, in the case of (z_3), we define $c(x, y) = \gamma$ and $a = b = 0$.

We show that (Z') implies (Z'').

We put

$$M(x, y) = \max\Big\{ d(x, y), \frac{d(x, f(x)) + d(y, f(y))}{2},$$
$$\frac{d(x, f(y)) + d(y, f(x))}{2} \Big\}. \tag{9.5}$$

Let $f \in (Z')$. Then we have

$$d(f(x), f(y)) \leq [a(x, y) + 2b(x, y) + 2c(x, y)]M(x, y) \leq \lambda M(x, y),$$

and $f \in (Z'')$.

We show that (Z'') implies (Z).

For each $x, y \in X$, for which $M(x, y) = d(x, y)$, the function f satisfies (z_1) with $\alpha = h$. If $M(x, y) = [d(x, f(x)) + d(y, f(y))]/2$, then the function f satisfies (z_2) with $\beta = h/2$, and the function f satisfies (z_3) with $\gamma = h/2$, if $M(x, y) = [d(x, f(y)) + d(y, f(x))]/2$. $\quad\square$

10. Ćirić's generalized contraction

In [13], *Ćirić* generalized the well-known contractive condition and introduced a concept of a generalized contraction defined as follows.

Definition 10.1 (Ćirić [13])**.** Let (X, d) be a metric space. A mapping $f : X \to X$ is a λ–generalized contraction if, for all $x, y \in X$, there exist some nonnegative numbers $q(x, y)$, $r(x, y)$, $s(x, y)$ and $t(x, y)$ such that

$$\sup_{x, y \in X} \{q(x, y) + r(x, y) + s(x, y) + 2t(x, y)\} = \lambda < 1,$$

and for all $x, y \in X$,

$$d(f(x), f(y)) \leq q(x, y)d(x, y) + r(x, y)d(x, f(x)) + s(x, y)d(y, f(y))$$
$$+ t(x, y)(d(x, f(y)) + d(y, f(x))). \tag{10.1}$$

Obviously, this condition is equivalent to the fact that there exists a constant $h \in (0, 1)$ such that, for all $x, y \in X$,

$$d(f(x), f(y)) \leq h \max\Big\{ d(x, y),\ d(x, f(x)), d(y, f(y)),$$

$$\left. \frac{d(x, f(y)) + d(y, f(x))}{2} \right\}.$$ (10.2)

The next example shows that (10.1) indeed generalizes (2.1).

Example 10.2. Let $X = [0, 2] \subseteq \mathbb{R}$ and

$$f(x) = \begin{cases} \dfrac{x}{9} & \text{for } 0 \le x \le 1 \\[2mm] \dfrac{x}{10} & \text{for } 1 < x \le 2. \end{cases}$$

The map f does not satisfy (2.1) since, for $x = 999/1000$ and $y = 1001/1000$,

$$d(f(x), f(y)) = \frac{981}{90000} > 5 \cdot \frac{180}{90000} = 5d(x, y).$$

But (10.1) holds for $q(x, y) = 1/10$, $r(x, y) = s(x, y) = 1/4$ and $t(x, y) = 1/6$ for all $x, y \in X$.

Example 10.3. Let $X = [0, 10] \subset \mathbb{R}$ and $f(x) = 3/4$ for each $x \in X$. For $x = 0$ and $y = 8$, the function f satisfies (9.1) with $q < 3$. But the condition in (10.1) is satisfied on all of X with $q(x, y) = 3/4$ and $r(x, y) = s(x, y) = t(x, y) = 1/20$.

Definition 10.4. Let (X, d) be a metric space, $f : X \to X$ be a map, and $x \in X$. An f–orbit of the element x is the set $O(x; f)$ defined by

$$O(x; f) = \{f^n(x) : n \in \mathbb{N}_0\}.$$

If f is given, then the usual notation is $O(x)$. Furthermore, for all $n \in \mathbb{N}$, we define the set

$$O(x, n) = \{x, f(x), f^2(x), \ldots, f^n(x)\}.$$

The space X is said to be an f–*orbitally complete metric space* if any Cauchy sequence in $O(x; f)$ for $x \in X$ converges in X.

Obviously, every complete metric space is f–orbitally complete, but the converse implication does not hold, in general. It is clear from the proof of Banach's theorem that it is enough to assume that (X, d) is f–orbitally complete instead of complete. The same remark applies for λ–generalized contractions, as is stated in the following theorem.

Theorem 10.5 (Ćirić [13])**.** *If $f : X \mapsto X$ is a λ–generalized contraction on an f–orbitally complete metric space X, then, for any $x \in X$, the iterative sequence $(f^n(x))$ converges to the unique fixed point u of f, and*

$$d(f^n(x), u) \le \frac{\lambda^n}{1 - \lambda} \cdot d(x, f(x)).$$

Proof. For an arbitrary $x \in X$, we define the sequence (x_n) by $x_0 = x$ and $x_n = f(x_{n-1})$ for $n \in \mathbb{N}$. Then we obtain from (10.1)

$$\begin{aligned} d(x_n, x_{n+1}) &= d(f(x_{n-1}), f(x_n)) \le q(x_{n-1}, x_n)d(x_{n-1}, x_n) \\ &\quad + r(x_{n-1}, x_n)d(x_{n-1}, f(x_{n-1})) + s(x_{n-1}, x_n)d(x_n, f(x_n)) \\ &\quad + t(x_{n-1}, x_n)(d(x_{n-1}, f(x_n)) + d(x_n, f(x_{n-1}))) \\ &= q(x_{n-1}, x_n)d(x_{n-1}, x_n) + r(x_{n-1}, x_n)d(x_{n-1}, x_n) \\ &\quad + s(x_{n-1}, x_n)d(x_n, x_{n+1}) + t(x_{n-1}, x_n)d(x_{n-1}, x_{n+1}), \end{aligned}$$

and moreover

$$d(x_n, x_{n+1}) \le (q(x_{n-1}, x_n) + r(x_{n-1}, x_n))d(x_{n-1}, x_n)$$

$$+ s(x_{n-1}, x_n)d(x_n, x_{n+1})$$
$$+ t(x_{n-1}, x_n)(d(x_{n-1}, x_n) + d(x_n, x_{n+1})),$$

so

$$d(x_n, x_{n+1}) \leq \frac{q(x_{n-1}, x_n) + r(x_{n-1}, x_n) + t(x_{n-1}, x_n)}{1 - s(x_{n-1}, x_n) - t(x_{n-1}, x_n)} d(x_{n-1}, x_n). \tag{10.3}$$

Because of

$$q(x, y) + r(x, y) + t(x, y) + \lambda s(x, y) + \lambda t(x, y) \leq \lambda,$$

we get

$$\frac{q(x, y) + r(x, y) + t(x, y)}{1 - s(x, y) - t(x, y)} \leq \lambda \text{ for all } x, y \in X$$

and, combined with (10.3), it follows that

$$d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n). \tag{10.4}$$

We remark that (10.4) allows us to consider f as a contraction under special assumptions, and

$$d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n) \leq \cdots \leq \lambda^n d(x, f(x)).$$

Obviously, we have for all $m \geq n$

$$d(x_n, x_m) \leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \leq \sum_{k=n}^{m-1} \lambda^k d(x, f(x)),$$

hence

$$d(x_n, x_{n+p}) \leq \frac{\lambda^n}{1 - \lambda} d(x, f(x)). \tag{10.5}$$

implies that (x_n) is a Cauchy sequence in $O(x)$. Let $z \in X$ denote its limit. It remains to show $f(z) = z$ by estimating $d(f(z), f(x_n))$.

$$
\begin{aligned}
d(f(z), f(x_n)) &\leq q(z, x_n)d(z, x_n) + r(z, x_n)(d(z, x_{n+1}) + d(x_{n+1}, f(z))) \\
&\quad + s(z, x_n)d(x_n, x_{n+1}) + t(z, x_n)(d(z, x_{n+1}) + d(f(z), x_n)) \\
&\leq \lambda d(z, x_n) + (r(z, x_n) + t(z, x_n))d(z, x_{n+1}) \\
&\quad + r(z, x_n)d(f(x_n), f(z)) + s(z, x_n)d(x_n, x_{n+1}) \\
&\quad + t(z, x_n)(d(f(z), f(x_n)) + d(f(x_n), x_n)) \\
&\leq d(z, x_n) + \lambda d(z, x_{n+1}) \\
&\quad + (r(z, x_n) + t(z, x_n))d(f(z), f(x_n)) + \lambda d(x_n, x_{n+1}) \\
&\leq \lambda(d(z, x_n) + d(z, x_{n+1}) + d(x_n, x_{n+1})) + \lambda d(f(z), f(x_n)).
\end{aligned}
$$

Thus we have

$$d(f(z), f(x_n)) \leq \frac{\lambda}{1 - \lambda} \left[d(z, x_n) + d(z, x_{n+1}) + d(x_n, x_{n+1}) \right].$$

that is, z is a fixed point of the function f. The uniqueness easily follows from (10.1) and the estimation inequality is implied by (10.5). □

The contractive condition (10.1) for generalized contractions implies many others, thus Theorem 10.5 has numerous consequences among which we will state two analogous to Corollaries 2.8 and 2.10 of Banach's theorem.

Theorem 10.6. *If $f : X \to X$ is a map of an f–orbitally complete metric space (X, d) such that, for some $k \in \mathbb{N}$, f^k is a λ–generalized contraction for all $x \in X$, then the iterative sequence $(f^n(x))$ converges to a unique fixed point z of f, and*

$$d(f^n(x), z) \leq (\lambda')^n \rho(x, f(x)), \text{ where } \lambda' = \lambda^{1/k}$$

$$\text{and } \rho(x, f(x)) = \max\{\lambda^{-1} d(f^r(x), f^{r+k}(x)) : r = 0, 1, \ldots, k - 1\}.$$

Proof. The existence of a unique fixed point directly follows from Theorem 10.5. It remains to estimate $d(f^n(x), z)$ for each $n \in \mathbb{N}$. Since $n = mk + r$ for $m = [n/k]$ and $0 \leq r < k$, we have

$$\begin{aligned} d(f^n(x), z) = d(f^{mk}(f^r(x)), z) &\leq \frac{\lambda^m}{1 - \lambda} d(f^r(x), f^k(f^r(x))) \\ &= (\lambda^{1/k})^{mk+r-r} d(f^r(x), f^{k+r}(x)) \\ &\leq (\lambda^{1/k})^{mk+r-k} d(f^r(x), f^{r+k}(x)) \\ &= (\lambda^{1/k})^n \lambda^{-1} d(f^r(x), f^{r+k}(x)), \end{aligned}$$

hence

$$d(f^n(x), z) \leq (\lambda^{1/k})^n \max\{\lambda^{-1} d(f^r(x), f^{r+k}(x)) : r = 0, 1, \ldots k - 1\}. \qquad \square$$

As in the case of Banach's theorem, we may consider some local properties of Theorem 10.5.

Theorem 10.7. *Let $f : B \to X$ be a map of an f–orbitally complete metric space (X, d), where $B = B_r(x_0) = \{x \in X : d(x_0, x) \leq r\}$ for some $x_0 \in X$ and $r > 0$. If f is a λ–generalized contraction on B and*

$$d(x_0, f(x_0)) \leq (1 - \lambda) \cdot r, \tag{10.6}$$

then the sequence $(f^n(x_0))$ converges to a unique fixed point z of f in B and

$$d(f^n(x_0), z) \leq \lambda^n \cdot r \text{ for } \lambda = \sup_{x,y \in B} [q(x, y) + r(x, y) + 2t(x, y)].$$

Proof It is clear that $x_n \in B$ for all $n \in \mathbb{N}$, due to (10.6) and mathematical induction. Analogously as in the proof of Theorem 10.5, it follows that $(f^n(x_0))$ is a Cauchy sequence in B and its limit is a fixed point of f. Inequality (10.1) guarantees uniqueness. $\qquad \square$

11. The Reich and Hardy–Rogers theorems

In 1971, Reich [44] proved the following theorem which generalizes Banach's and Kannan's theorems. (We note that for $a = b = 0$, we obtain Banach's theorem, Theorem 2.2, and for $a = b$ and $c = 0$, we obtain Kannan's theorem, Theorem 9.1).

Theorem 11.1 (Reich [44]). *Let (X, d) be a complete metric space and $f : X \to X$ be a map for which there exists nonnegative numbers a, b and c with $a + b + c < 1$ such that for all $x, y \in X$,*

$$d(f(x), f(y)) \leq ad(x, f(x)) + bd(y, f(y)) + cd(x, y). \tag{11.1}$$

Then the map f has a unique fixed point.

Proof. Let $x \in X$. We consider the sequence $(f^n(x))$. If we put $x = f^n(x)$ and $y = f^{n-1}(x)$ in (11.1), then we have for all $n \geq 1$

$$\begin{aligned} d(f(f^n(x)), f(f^{n-1}(x))) \leq \\ ad(f^n(x), f(f^n(x))) + bd(f^{n-1}(x), f(f^{n-1}(x))) + cd(f^n(x), f^{n-1}(x)). \end{aligned}$$

Hence, we obtain

$$d(f^{n+1}(x)), f^n(x))) \leq pd(f^n(x), f^{n-1}(x)),$$

where $0 \leq p = (b+c)/(1-a) < 1$. It follows that

$$d(f^{n+1}(x)), f^n(x))) \leq p^n d(x, f(x)),$$

and for every $m > n$,

$$d(f^m(x)), f^n(x))) \leq \frac{p^n}{1-p} \cdot d(x, f(x)).$$

Thus $(f^n(x))$ is a Cauchy sequence, and there exists $z \in X$ with $z = \lim_{n\to\infty} f^n(x)$.
We are going to show $f(z) = z$. It suffices to show $\lim_{n\to\infty} f^{n+1}(x) = f(z)$. When we choose $x = f^n(x)$ and $y = z$ in (11.1), then we have for all $n \geq 1$

$$\begin{aligned}
d(f^{n+1}(x)), f(z)) &\leq ad(f^n(x), f^{n+1}(x)) + bd(z, f(z)) + cd(f^n(x), z) \\
&\leq ad(f^n(x), f^{n+1}(x)) + bd(f^{n+1}(x), f(z)) + bd(f^{n+1}(x), z) + cd(f^n(x), z) \\
&\leq ap^n d(x, f(x)) + bd(f^{n+1}(x), f(z)) + bd(f^{n+1}(x), z) + cd(f^n(x), z).
\end{aligned}$$

Thus we obtain for $n \to \infty$

$$d(f^{n+1}(x)), f(z)) \leq \frac{ap^n d(x, f(x)) + bd(f^{n+1}(x), z) + cd(f^n(x), z)}{1-b} \to 0.$$

We are going to show that the map f has a unique fixed point.
If we assume that $x, y \in X$ with $x \neq y$ are fixed points of the map f, then we have

$$d(x, y) = d(f(x), f(y)) \leq ad(x, f(x)) + bd(y, f(y)) + cd(x, y) = cd(x, y),$$

which implies $x = y$. □

Example 11.2. Let $X = [0, 1]$ have the natural metric and the map $f : X \to X$ be defined by $f(x) = x/3$ for $0 \leq x < 1$ and $f(1) = 1/6$. Then the map f does not satisfy Banach's condition, since it is not continuous; neither does it satisfy Kannan's condition, since

$$d(f(0), f(1/3)) = \frac{1}{2} \left[d(0, f(0)) + d(1/3, f(1/3)) \right].$$

But the map f satisfies the condition in(11.1), for instance, for $a = 1/6$, $b = 1/9$ and $c = 1/3$.

Corollary 11.3 (Reich [44])**.** *Let (X, d) be a complete metric space and $f_n : X \to X$ for $n = 1, 2, \ldots$ be a sequence of maps satisfying the condition in (11.1) with the same constants a, b and c and with the fixed points $u_n \in X$. We define the map $f : X \to X$ by $f(x) = \lim_{n\to\infty} f_n(x)$ for $x \in X$. Then the map f has a unique fixed point $u \in X$ and $u = \lim u_{n\to\infty} = u$.*

    Proof. Since the metric d is a continuous function, it follows that the function f satisfies the condition in (11.1), and therefore has a unique fixed point $u \in X$. We note that

$$\begin{aligned}
d(u_n, u) = d(f_n(u_n), f(u)) &\leq d(f_n(u_n), f_n(u)) + d(f_n(u), f(u)) \\
&\leq ad(u_n, f_n(u_n)) + bd(u, f_n(u)) + cd(u_n, u) + d(f_n(u), f(u)).
\end{aligned}$$

Hence we have

$$d(u_n, u) \leq \frac{(b+1)d(f_n(u), f(u))}{1-c} \to 0 \ (n \to \infty). \qquad □$$

Hardy and Rogers [25] improved some of Reich's results [44] including the following theorem.

Theorem 11.4 (Hardy and Rogers [25])**.**
Let (X, d) be metric space and $f : X \to X$ be a map such that for all $x, y \in X$,

$$
\begin{aligned}
d(f(x), f(y)) \leq {} & ad(x, f(x)) + bd(y, f(y)) \\
& + cd(x, f(y)) + ed(y, f(x)) + hd(x, y),
\end{aligned}
\tag{11.2}
$$

where $a, b, c, e, h \geq 0$ and $\alpha = a + b + c + e + h$.

(i) *If (X, d) is a complete metric space and $\alpha < 1$, then the map f has a unique fixed point.*

(ii) *If (X, d) is compact, f is continuous and the condition in (11.2) is replaced by*

$$
\begin{aligned}
d(f(x), f(y)) < {} & ad(x, f(x)) + bd(y, f(y)) \\
& + cd(x, f(y)) + ed(y, f(x)) + hd(x, y),
\end{aligned}
\tag{11.3}
$$

for all $x \neq y$, and $\alpha = 1$, then f has a unique fixed point.

The following lemma is essential in the proof of this theorem, but for the reader's convenience, we state it separately.

Lemma 11.5. *We assume that (11.2) is satisfied and $\alpha < 1$. Then there exists $\beta < 1$ such that*

$$
d(f(x), f^2(x)) \leq \beta d(x, f(x)).
\tag{11.4}
$$

If $\alpha = 1$ and (11.3) is satisfied, then

$$
x \neq f(x) \text{ implies } d(f(x), f^2(x)) \leq \beta d(x, f(x)).
\tag{11.5}
$$

Proof. In the first case, for $\alpha < 1$, we put $y = f(x)$, and observe

$$
d(f(x), f^2(x)) \leq \frac{a + h}{1 - b} \cdot d(x, f(x)) + \frac{c}{1 - b} \cdot d(x, f^2(x)).
\tag{11.6}
$$

which, along with $d(f(x), f^2(x)) \geq d(f^2(x), x) - d(f(x), x)$ and (11.6), leads to

$$
d(f^2(x), x) - d(f(x), x) \leq \frac{a + h}{1 - b} \cdot d(x, f(x)) + \frac{c}{1 - b} \cdot d(x, f^2(x)),
\tag{11.7}
$$

that is,

$$
d(f^2(x), x) \leq \frac{1 + a + h - b}{1 - b - c} \cdot d(x, f(x)).
\tag{11.8}
$$

By (11.9), inserting (11.8) in (11.6), we obtain

$$
d(f(x), f^2(x)) \leq \frac{a + c + h}{1 - b - c} \cdot d(x, f(x)).
\tag{11.9}
$$

and replacing a and c by b and e (which is permitted because of the symmetry of the metric d), we get

$$
d(f(x), f^2(x)) \leq \frac{b + e + h}{1 - a - e} \cdot d(x, f(x)).
\tag{11.10}
$$

If we put

$$
\beta = \min\left\{ \frac{a + c + h}{1 - b - c}, \frac{b + e + h}{1 - a - e} \right\},
\tag{11.11}
$$

then (11.4) is satisfied.
The remainder of the lemma is shown analogously.   □

Proof of Theorem 11.4. To prove Part (i), we first observe that, by (11.4), for all $m > n$,

$$d(f^m(x), f^n(x)) \leq d(f^m(x), f^{m-1}(x)) + \cdots + d(f^{n+1}(x), f^n(x))$$
$$\leq \beta^n(l + \beta + \cdots + \beta^{m-n})d(x, f(x))$$
$$\leq \frac{\beta^n}{1 - \beta} \cdot d(x, f(x)).$$

Hence $(f^n(x))$ is a Cauchy sequence and $z \in X$ is its limit. It remains to show $f(z) = z$. This follows directly from $\lim_{n\to\infty} f^{n+1}(x) = f(z)$.
he following inequality holds by (11.2)

$$d(z, f(z)) \leq d(f^{n+1}(x), f(z)) + d(f^{n+1}(x), z) \tag{11.12}$$
$$\leq ad(f^n(x), f^{n+1}(x)) + bd(z, f(z)) \tag{11.13}$$
$$+ cd(f^n(x), f(z)) + (e + 1)d(f^{n+1}(x), z) + hd(f^n(x), z). \tag{11.14}$$

Letting $n \to \infty$ in (11.12), we obtain

$$d(z, f(z)) \leq (b + c)d(z, f(z)),$$

and $b + c < 1$ implies $z = f(z)$. The uniqueness clearly follows from (11.2).
We note that, under the assumptions in (ii), there is some $y \in X$ such that

$$\inf\{d(x, f(x)) : x \in X\} = d(y, f(y)).$$

Because of (11.5), it follows that $y = f(y)$. The uniqueness is shown as previously discussed. □

12. Ćirić's quasi-contraction

In 1971, Ćirić [14] used a concept of generalized contraction to replace the linear combination of distances in (10.1) by their maximum, and defined a new class of contractive mappings called *quasi–contractions*.

Definition 12.1 (Ćirić [14]). A map $f : X \to X$ of a metric space (X, d) is a quasi-contraction if there exists some λ with $0 < \lambda < 1$ such that

$$d(f(x), f(y)) \leq \lambda \cdot \max\{d(x, y), d(x, f(x)), d(y, f(y)), d(x, f(y)), d(y, f(x))\} \tag{12.1}$$

for all $x, y \in X$.

Obviously if a mapping f satisfies condition(2.1), then (12.1) also holds. An example presented by Ćirić shows that the converse implication is not true, in general.

Example 12.2 (Ćirić [14]). Let

$$M_1 = \left\{ \frac{m}{n} : m = 0, 3^k, \, n = 3k + 1, \, k \in \mathbb{N}_0 \right\}$$
$$M_2 = \left\{ \frac{m}{n} : m = 3^k, \, n = 3k + 2, \, k \in \mathbb{N}_0 \right\}$$

and $M = M_1 \cup M_2$ be the metric space with the usual metric $d(x, y) = |x - y|$ for all $x, y \in M$. The map $f : X \to X$ defined by

$$f(x) = \begin{cases} \dfrac{3x}{5} & (x \in M_1) \\ \dfrac{x}{8} & (x \in M_2) \end{cases}$$

is a quasi-contraction for $\lambda = 3/5$, but not a contraction ($x = 1, y = 1/2$).

Example 12.3. Let $X = [0, 3] \cup [4, 5]$ have the natural metric and the map $f : X \to X$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in [0, 3] \\ 3 & \text{if } x \in [4, 5]. \end{cases}$$

Then, for each $x \in [4, 5]$, we have $d(x, f(x)) \le 2$ and $d(f(x), f^2(x)) = 3$. So we have $d(f(x), f^2(x)) > d(x, f(x))$. We show that the function f satisfies the condition in (12.1).

Let $x \in [0, 3]$ and $y \in [4, 5]$. Then we have $d(f(x), f(y)) = 3$ and $d(y, f(x)) \ge 4$. Hence it follows that $d(f(x), f(y)) = (3/4)4 \le (3/4) \max\{d(x, f(y)), d(y, f(x))\}$.

Thus the function f satisfies (12.1) for $\lambda = 3/4$ and all $x, y \in X$.

Example 12.4. Let $f(x) = 0$ for all $0 \le x < 1$ and $f(1) = 1/2$. Then the function f satisfies (12.1) but not (10.1) [49]. We note that

$$d\left(f\left(\frac{1}{2}\right), f(1)\right) = \frac{1}{2} = \frac{d\left(\frac{1}{2}, f(1)\right) + d\left(1, f(\frac{1}{2})\right)}{2},$$

$$d\left(\frac{1}{2}, 1\right) = d\left(\frac{1}{2}, f\left(\frac{1}{2}\right)\right) = d(1, f(1)) = \frac{1}{2},$$

$$d(f(x), f(y)) = 0 \text{ for all } x \ne y \text{ and } x, y \ne 1,$$

$$d(f(x), f(1)) = \frac{1}{2} \le \frac{3}{4} \cdot d(1, f(x)) = \frac{3}{4} \text{ for } x \ne 1.$$

Theorem 12.5 (Ćirić [14]). *If $f : X \to X$ is a quasi–contraction on an f–orbitally complete metric space (X, d), then f has a unique fixed point z in X, and the iterative sequence $(f^n(x))$ converges to z for any $x \in X$. Moreover, we have*

$$d(f^n(x), z) \le \frac{\lambda^n}{1 - \lambda} d(x, f(x)).$$

Proof. We put $\alpha(x, n) = \text{diam}(O(x, n))$, and $\alpha(x) = \text{diam}(O(x))$ where diam denotes a diameter of a set. Then we have

$$\alpha(f(x), n - 1) = \text{diam}(\{f(x), f^2(x), \ldots, f^n(x)\}) \le \lambda \alpha(x, n). \tag{12.2}$$

Obviously, if $\alpha(f(x), n - 1) = d(f^j(x), f^k(x))$ for $1 \le j < k \le n$, then (12.1) yields

$$\begin{aligned} \alpha(f(x), n - 1) &= d(f(f^{j-1}(x)), f(f^{k-1}(x))) \\ &\le \lambda \max\{d(f^{j-1}(x), f^{k-1}(x)), d(f^{j-1}(x), f^j(x)), d(f^{k-1}(x), f^k(x)), \\ &\qquad\qquad d(f^{j-1}(x), f^k(x)), d(f^{k-1}(x), f^j(x))\} \\ &\le \lambda \text{diam}(\{f^{j-1}(x), f^j(x), \ldots, f^k(x)\}) \\ &\le \lambda \text{diam}(\{x, f(x), \ldots, f^n(x)\}) \\ &= \lambda \alpha(x, n), \end{aligned}$$

and (12.2) holds.

Furthermore, we obtain from (12.2),

$$\alpha(x, n) = d(x, f^k(x)) \text{ for some } k \le n. \tag{12.3}$$

It follows from (12.2), (12.3) and the triangle inequality that

$$\begin{aligned} \alpha(x, n) = d(x, f^k(x)) &\le d(x, f(x)) + d(f(x), f^k(x)) \\ &\le d(x, f(x)) + \alpha(f(x), n - 1) \\ &\le d(x, f(x)) + \lambda \alpha(x, n), \end{aligned}$$

and

$$\alpha(x,n) \leq \frac{1}{1-\lambda} \cdot d(x,f(x)). \tag{12.4}$$

Since $\lim_{n\to\infty} \alpha(x,n) = \alpha(x)$, (12.4) implies

$$\alpha(x) \leq \frac{1}{1-\lambda} \cdot d(x,f(x)), \tag{12.5}$$

so the f–orbit of x has a finite diameter.

We write $\beta_n(x)$ for the diameter of $\alpha(f^n(x))$.

The sequence $(\beta_n(x))$ is non–increasing and bounded, so there exists $\lim_{n\to\infty} \beta_n(x) = \beta(x)$ and $\beta(x) \leq \beta_n(x)$ for all $n \in \mathbb{N}$.

Letting $n \to \infty$ in (12.2), we obtain

$$\alpha(f(x)) \leq \lambda \alpha(x), \tag{12.6}$$

hence

$$\beta_{n+1}(x) = \alpha(f(f^n(x))) \leq \lambda\alpha(f^n(x)) = \lambda\beta_n(x) \ (n \in \mathbb{N})$$

and

$$\beta(x) \leq \lambda\beta(x),$$

so $\beta(x) = 0$ and $(f^n(x))$ is a Cauchy sequence in X.

Let $z = \lim_{n\to\infty} f^n(x)$. Because of (12.1), we have

$$d(f(u),f(f^n(x))) \leq$$
$$\lambda\max\left\{d(u,f^n(x)),d(u,f(u)),d(f^n(x),f^{n+1}(x)),d(u,f^{n+1}(x)),d(f^n(x),f(u))\right\},$$

hence

$$d(f(u),u) \leq \lambda\,d(u,f(u)),$$

that is, $f(u) = u$. The uniqueness also follows from (12.1).

We obtain from (12.6), $\alpha(f^n(x)) \leq \lambda^n\alpha(x)$ and combined with (12.5)

$$\alpha(f^n(x)) \leq \frac{\lambda^n}{1-\lambda}d(x,f(x)).$$

If $n,m \in \mathbb{N}$ and $m \geq n$, then

$$d(f^n(x),f^m(x)) \leq \alpha(f^n(x)) \leq \frac{\lambda^n}{1-\lambda}d(x,f(x)),$$

and when $m \to \infty$, then

$$d(f^n(x),z) \leq \frac{\lambda^n}{1-\lambda}d(x,f(x)). \qquad \square$$

13. Caristi's Theorem

There are many extensions of Banach's contraction principle, one of the most studied ones is that by Caristi [11], 1976. Caristi's theorem [11] may be motivated by the following consideration. If (X,d) is a metric space and $T : X \to X$ is a contraction with a Lipschitz constant $k \in [0,1)$, then we have

$$d(x,T(x)) = \frac{1}{1-k} \cdot d(x,T(x)) - \frac{k}{1-k} \cdot d(x,T(x))$$
$$\leq \frac{1}{1-k} \cdot d(x,T(x)) - \frac{1}{1-k} \cdot d(T(x),T(T(x)))$$
$$= \phi(x) - \phi(T(x)),$$

for all $x \in X$, where $\phi(x) = (1-k)^{-1}d(x,T(x))$.

It is well known that Caristi's theorem (or the Caristi–Kirk, or the Caristi– Kirk–Browder theorem) is equivalent to Ekeland's variation principle [19] which is very important because of its numerous applications. The ordinal proof of the Caristi–Kirk theorem is rather complicated and, in the literature, there are several different proofs of that theorem.

We mention that the map $\varphi : X \to \mathbb{E}$ ($\mathbb{E} \subset \mathbb{R}$) is lower semicontinuous at $x \in X$ if, for every sequence (x_n), it follows from $\lim_{n\to\infty} x_n = x$ that $\varphi(x) \leq \liminf_{n\to\infty} \varphi(x_n)$. The map $\varphi : X \to \mathbb{E}$ is lower semicontinuous on X if it is lower semicontinuous at every $x \in X$.

Theorem 13.1 (Caristi [11])**.** *Let (X,d) be a complete metric space, $T : X \to X$ and $\phi : X \to [0,\infty)$ be lower semicontinuous such that*

$$d(x,T(x)) \leq \phi(x) - \phi(T(x)) \text{ for all } x \in X. \tag{13.1}$$

Then T has a fixed point.

Proof (Ćirić [15]). For each $x \in X$, we put

$$P(x) = \big\{y \in X : d(x,y) \leq \phi(x) - \phi(y)\big\},$$
$$\alpha(x) = \inf \big\{\phi(y) : y \in P(x)\big\}.$$

Since $x \in P(x)$, $P(x)$ is a nonempty set and $0 \leq \alpha(x) \leq \phi(x)$.

Let $x \in X$. We define the sequence (x_n) in X such that $x_1 = x$, and if $x_1, x_2, \ldots, x_n$ are already defined then we define $x_{n+1} \in P(x_n)$ such that $\phi(x_{n+1}) \leq \alpha(x_n) + 1/n$. Hence the sequence (x_n) satisfies the following conditions:

$$d(x_n, x_{n+1}) \leq \phi(x_n) - \phi(x_{n+1});$$
$$\alpha(x_n) \leq \phi(x_{n+1}) \leq \alpha(x_n) + 1/n. \tag{13.2}$$

Since $(\phi(x_n))$ is a decreasing sequence of real numbers, there exists $\alpha \geq 0$ such that

$$\alpha = \lim_{n\to\infty} \phi(x_n) = \lim_{n\to\infty} \alpha(x_n). \tag{13.3}$$

Let $k \in \mathbb{N}$. It follows from (13.2) that there exists N_k such that $\phi(x_n) < \alpha + 1/k$ for every $n \geq N_k$. Hence the monotonicity of the sequence $(\phi(x_n))$ for $m \geq n \geq N_k$ implies $\alpha \leq \phi(x_m) \leq \phi(x_n < \alpha + 1/k$, that is,

$$\phi(x_n) - \phi(x_m) < 1/k \text{ for each } m \geq n \geq N_k. \tag{13.4}$$

We have from the triangle inequality and the inequality in (13.2)

$$d(x_n, x_m) \leq \sum_{s=n}^{m-1} d(x_s, x_{s+1}) \leq \phi(x_n) - \phi(x_m). \tag{13.5}$$

Now (13.4) implies

$$d(x_n, x_m) < 1/k \text{ for each } m \geq n \geq N_k.$$

Since (x_n) is a Cauchy sequence and X is a complete metric space the sequence converges to some $u \in X$. Since ϕ is lower semicontinuous, we obtain from (13.5) that

$$\phi(u) \leq \liminf_{m\to\infty} \phi(x_m) \leq \liminf_{m\to\infty} [\phi(x_n) - d(x_n, x_m)] = \phi(x_n) - d(x_n, u),$$

and so

$$d(x_n, u) \leq \phi(x_n) - \phi(u).$$

Hence we have $u \in P(x_n)$ for all $n \in \mathbb{N}$ and $\alpha(x_n) \leq \phi(u)$. Now (13.3) implies $\alpha \leq \phi(u)$. On the other hand, since ϕ is lower semicontinuous, (13.3) implies $\phi(u) \leq \liminf_{n\to\infty} \phi(x_n) = \alpha$. Hence we have $\phi(u) = \alpha$. Since $u \in P(x_n)$ for each $n \in \mathbb{N}$, (13.1) implies $Tu \in P(u)$, that is,

$$
\begin{aligned}
d(x_n, Tu) &\leq d(x_n, u) + d(u, Tu) \\
&\leq \phi(x_n) - \phi(u) + \phi(u) - \phi(Tu) \\
&= \phi(x_n) - \phi(Tu).
\end{aligned}
$$

Hence we have $Tu \in P(x_n)$ for each $n \in \mathbb{N}$. It follows that

$$\phi(Tu) \geq \alpha(x_n) \text{ for each } n \in \mathbb{N}.$$

Now (13.3 implies

$$\phi(Tu) \geq \alpha.$$

Since (13.1) implies $\phi(Tu) \leq \phi(u)$ and $\phi(u) = \alpha$, we have

$$\phi(u) = \alpha \leq \phi(Tu) \leq \phi(u),$$

and so $\phi(Tu) = \phi(u)$. Now (13.1) implies

$$d(u, Tu) \leq \phi(u) - \phi(Tu) = 0,$$

that is, $Tu = u$. $\square$

Theorem 13.2 (Ekeland [19], 1972). *Let $\phi : X \to R$ be an upper semicontinuous function on the complete metric space (X, d). If ϕ is bounded above then there exists $u \in X$ such that*

$$\phi(u) < \phi(x) + d(u, x) \text{ for } x \in X \text{ with } x \neq u. \tag{13.6}$$

Proof (Ćirić [15]). We are going to show that u from the proof of Theorem 13.1 is the desired point. Using the same notations for $x \neq u$ we have to prove $x \notin P(u)$. We suppose that this is not the case, that is, for some $v \neq u$, we have $v \in P(u)$. Then $0 < d(u, v) \leq \phi(u) - \phi(v)$ implies $\phi(v) < \phi(u) = \alpha$. Since

$$
\begin{aligned}
d(x_n, v) &\leq d(x_n, u) + d(u, v) \\
&\leq \phi(x_n) - \phi(u) + \phi(u) - \phi(v) \\
&= \phi(x_n) - \phi(v),
\end{aligned}
$$

it follows that $v \in P(x_n)$. Hence we have

$$\alpha(x_n) \leq \phi(v) \text{ for all } n \in \mathbb{N}.$$

We obtain for $n \to \infty$

$$\alpha \leq \phi(v),$$

which is a contradiction to $\phi(v) < \alpha = \phi(u)$. Hence we have $x \notin P(u)$ for $x \in X$ with $x \neq u$, and so

$$x \neq u \text{ implies } d(u, x) > \phi(u) - \phi(x). \qquad \square$$

Proof (Brézis and Browder [8]). By Theorem 13.2 there exists $u \in X$ which satisfies the condition in (13.6). It follows that $Tu = u$, for $Tu \neq u$ would imply $\phi(Tu) - \phi(u) > -d(u, Tu)$, which contradicts (13.1). We note that Theorem 13.2 can be proved by Theorem 13.1. Indeed, if we assume that the conclusion of Theorem 13.2 is not true, then, for each $x \in X$, there exists $y \in X$ with $y \neq x$ such that $\phi(y) - \phi(x) \leq -d(x, y)$. Hence we may define a map $T : X \to X$ which satisfies (13.1), but does not have a fixed point. $\square$

We are going to present a proof of Caristi's theorem given by Kirk and Saliga [31]. First we prove a result by Brézis and Browder [8], the well–known *Brézis –Browder [8] principle of ordering*.

Let $(X, \leq)$ be a partially ordered set. We denote $S(x) = \{y \in X : x \leq y\}$ for $x \in X$. A sequence (x_n) in X is said to be *increasing* if $x_n \leq x_{n+1}$ for each $n \in \mathbb{N}$.

Theorem 13.3 (Brézis and Browder [8])**.** *Let the function* $\phi : X \to \mathbb{R}$ *satisfy the following conditions:*

(1) $x \leq y$ implies $\phi(x) \leq \phi(y)$;

(2) for every increasing sequence (x_n) in X with $\phi(x_n) \leq C < \infty$ for each $n \in \mathbb{N}$, there exists $y \in X$ such that $x_n \leq y$ for each $n \in \mathbb{N}$;

(3) for each $x \in X$ there exists $u \in X$ such that $x \leq u$ and $\phi(x) < \phi(u)$.

Then $\phi(S(x))$ is a bounded set for each $x \in X$.

Proof. For $a \in X$, let
$$p(a) = \sup_{b \in S(a)} \phi(b).$$

We are going to show $p(x) = +\infty$ for each $x \in X$. We assume that $p(x) < \infty$ for some $x \in X$. We define a sequence (x_n) by induction such that $x_1 = x$, $x_{n+1} \in S(x_n)$ and $p(x_n) \leq \phi(x_{n+1}) + (1/n)$ for each $n \in \mathbb{N}$. Since $\phi(x_{n+1}) \leq p(x) < \infty$, the condition in (2) implies that there exists $y \in X$ such that $x_n \leq y$ for each n. It follows from the condition in (3) that there exists $u \in X$ such that $y \leq u$ and $\phi(y) < \phi(u)$. Since $x_n \leq u$, we have $\phi(u) \leq p(x_n)$ for all n. Furthermore, we have $x_{n+1} \leq y$, so $\phi(x_{n+1}) \leq \phi(y)$, and so

$$\phi(u) \leq p(x_n) \leq \phi(x_{n+1}) + (1/n) \leq \phi(y) + (1/n) \text{ for all } n \in \mathbb{N},$$

hence $\phi(u) \leq \phi(y)$, which is a contradiction. □

Theorem 13.4. *Let $(X, \preceq)$ be a partially ordered set, $x \in X$ and $S(x) = \{y \in X : x \preceq y\}$. We assume that the map $\psi : X \to \mathbb{R}$ satisfies the following conditions:*

(a) $x \preceq y$ with $x \neq y$ implies $\psi(x) < \psi(y)$;

(b) for each increasing sequence (x_n) in X, for which $\psi(x_n) \leq C < \infty$ for each $n \in \mathbb{N}$, there exists $y \in X$ such that $x_n \preceq y$ for each $n \in \mathbb{N}$;

(c) for each $x \in X$, the set $\psi(S(x))$ is bounded above.

Then, for each $x \in X$ there exists $x' \in S(x)$ such that x' is maximal in X, that is, $\{x'\} = S(x')$.

Proof. We apply Theorem 13.3 to the set $X = S(x)$; since the conditions in (1) and (2) of Theorem 13.3 are satisfied, and the conclusion of the theorem does not hold, it follows that the condition in (3) is not satisfied for some $x' \in S(x)$. Hence we have $S(x') = \{x'\}$. □

We remark that the map $\varphi : X \to \mathbb{R}$ is lower semicontinuous from above if $x_n \in X$ for $n = 1, 2, \dots$, $\lim_{n \to \infty} x_n = x$ and $(\varphi(x_n)) \downarrow r$ imply $\varphi(x) \leq r$.

Theorem 13.5 (Kirk and Saliga [31])**.**
We assume that (X, d) is a complete metric space and $T : X \to X$ is an arbitrary map such that we have for each $x \in X$
$$d(x, T(x)) \leq \varphi(x) - \varphi(T(x)), \tag{13.7}$$

where the map $\varphi : X \to \mathbb{R}$ is bounded above and lower semicontinuous. Then the map T has a fixed point in X.

Proof. We introduce *Brondsted's partial order* $\preceq$ on X as follows: For each $x, y \in X$, we have

$$x \preceq y \text{ if and only if } d(x, y) \leq \varphi(x) - \varphi(y),$$

and let $\psi = -\varphi$. Then the condition in (a) of Theorem 13.4 is satisfied, and the condition in (c) follows from the fact that the map φ is bounded below. To show the condition in (b), we assume that (x_n) is an

increasing sequence in $(X, \preceq)$ such that $\psi(x_n) \leq C < \infty$ for each n. Then $(\varphi(x_n))$ is a decreasing sequence in $\mathbb{R}$, and there exists $r \in \mathbb{R}$ such that $\lim_{n\to\infty} \varphi(x_n) = r$. Since $(\varphi(x_n))$ is a decreasing sequence, we have for each $m > n$

$$\lim_{n,m\to\infty} d(x_n, x_m) \leq \lim_{n,m\to\infty} [\varphi(x_n) - \varphi(x_m)] = 0.$$

Hence (x_n) is a Cauchy sequence in X. It follows that there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. From $\varphi(x_n) \downarrow r$ and $\varphi(x) \leq r$, it follows that

$$\begin{aligned} d(x_n, x) &\leq \lim_m d(x_n, x_m) \leq \lim_m [\varphi(x_n) - \varphi(x_m)] \\ &= \varphi(x_n) - r \leq \varphi(x_n) - \varphi(x). \end{aligned}$$

Hence x is an upper bound for the sequence (x_n) in $(X, \preceq)$ and so we have proved the condition in (b). Now it follows by Theorem 13.4 that $(X, \preceq)$ has a maximal element x'. Since (13.7) implies $x' \preceq T(x')$, we have $T(x') = x'$. □

Siegel [52] proved in 1977 in an original way, a generalized version of Caristi's theorem. Here we present some of his results [52].

Let (X, d) be a complete metric space, $\phi : X \to \mathbb{R}^+$, the set of nonnegative real numbers, and $g : X \to X$ be a not necessarily continuous map such that $d(x, g(x)) \leq \phi(x) - \phi(g(x))$ for all $x \in X$.

If a sequence of functions f_i for $i \leq 1 < \infty$ is given, then we define the product

$$\prod_{k=1}^{\infty} f_k x = \lim_{k\to\infty} f_k f_{k-1} \cdots f_1 x,$$

if the limit exists, and call it the *countable decomposition of the given sequence of functions.*

Definition 13.6. Let $\Phi = \{f : f : X \to X \text{ and } d(x, f(x)) \leq \phi(x) - \phi(f(x))\}$. We put $\Phi_g = \{f : f \in \Phi \text{ and } \phi(f) \leq \phi(g)\}$.

Lemma 13.7. *Let ϕ be an upper semi continuous function, and (x_i) be a sequence in X such that $d(x_i, x_{i+1}) \leq \phi(x_i) - \phi(x_{i+1})$ for each i. Then there exists $\overline{x} \in X$ such that $\overline{x} = \lim_{i\to\infty} x_i$ and $d(x_i, \overline{x}) \leq \phi(x_i) - \phi(\overline{x})$ for each i.*

Proof. Since the sequence $(\phi(x_i))_i$ is not increasing and bounded below by zero, and since $d(x_i, x_j) \leq \phi(x_i) - \phi(x_j)$ for $i \leq j$, (x_i) is a Cauchy sequence in X. Let $\overline{x} = \lim x_i$. Since ϕ is an upper semicontinuous function, it follows

$$d(x_i, \overline{x}) = \lim_{j\to\infty} d(x_i, x_j) \leq \phi(x_i) - \lim_{j\to\infty} \phi(x_j) \leq \phi(x_i) - \phi(\overline{x}). \qquad \square$$

Lemma 13.8. *The sets Φ and Φ_g are closed by the composition of functions and if ϕ is an upper semicontinuous function then the sets Φ and Φ_g are closed by the countable composition of sequences of functions.*

Proof. We prove that the sets Φ and Φ_g are closed by the composition of functions. If $f_1, f_2 \in \Phi$, then we have

$$\begin{aligned} d(x, f_2 f_1(x)) &\leq d(x, f_1(x)) + d(f_1(x), f_2 f_1(x)) \\ &\leq (\phi(x)\phi(f_1(x))) + (\phi(f_1(x)) - \phi(f_2 f_1(x))) \\ &= \phi(x) - \phi(f_2 f_1(x)). \end{aligned}$$

Hence we have $f_2 f_1 \in \Phi$. If $f_1 \in \Phi_g$, then $\phi(f_1(x)) - \phi(f_2(f_1(x))) \geq 0$ implies $\phi(f_2 f_1) \leq \phi(g)$, and so $f_2 f_1 \in \Phi_g$.

The remainder of the proof follows from the fact that, for each $x \in X$, the sequence $(x_i) = (f_i f_{i-1} \cdots f_1)(x)$ satisfies the conditions of Lemma 13.7. □

Definition 13.9. We introduce the following notations:

(1) For $A \subset X$ the *diameter of A* is defined as

$$\delta(A) = \sup_{x_i, x_j \in A} (d(x_i, x_j)).$$

(2) $r(A) = \inf_{x \in A}(\phi(x));$

(3) Let $\Phi' \subseteq \Phi$. For each $x \in X$, we put $S_x = \{fx : f \in \Phi'\}$.

Lemma 13.10. *We have $\delta(S_x) \leq 2(\phi(x) - r(S_x))$.*

   Proof. We have

$$\begin{aligned}
d(f_1(x), f_2(x)) &\leq d(x, f_1(x)) + d(x, f_2(x)) \\
&\leq \phi(x) - \phi(f_1(x)) + \phi(x) - \phi(f_2(x)) \\
&\leq 2(\phi(x) - r(S_x)).\square
\end{aligned}$$

   The main result of Siegel's paper [52] is the following theorem.

Theorem 13.11 (Siegel [52], 1977). *. Let $\Phi' \subseteq \Phi$ be sets of functions closed by the composition of functions. Also let $x_0 \in X$.*

(a) *If the set Φ' is closed for the composition of a countable sequence of functions, then there exists $\overline{f} \in \Phi'$ such that $\overline{x} = \overline{f}(x_0)$ and $g(\overline{x}) = \overline{x}$ for all $g \in \Phi'$.*

(b) *If the elements of Φ' are continuous functions, then there exists a sequence of functions $f_i \in \Phi'$ and $\overline{x} = \lim_{i \to \infty} f_i f_{i-1} \cdots f_1(x_0)$ such that $g(\overline{x}) = \overline{x}$ for each $g \in \Phi'$.*

   Proof. Let (ε_i) be a sequence of positive real numbers converging to zero and $\varepsilon > 0$. Then there exists $f_1 \in \Phi'$ such that $\phi(f_1(x_0)) - r(S_{x_0}) < \varepsilon/2$. We put $x_1 = f_1(x_0)$. Since the set Φ' is closed under the composition of functions it follows that $S_{x_1} \subseteq S_{x_0}$ and

$$\delta(S_{x_1}) \leq 2(\phi(x_1) - r(S_{x_1})) \leq 2(\phi(f_1(x_0)) - r(S_{x_0})) < \varepsilon_1.$$

Continuing in this way, we obtain a sequence of function f_i such that $x_i = f_i(x_{i-1})$, $S_{x_{i+1}} \subseteq S_{x_i}$ and $\delta(S_{x_i}) < \varepsilon_i$.
We know from the condition in (a) that there exists $\overline{f} = \prod_{i=1}^{\infty} f_i \in \Phi'$. Let $\overline{x} = \overline{f}(x_0)$. Since $\overline{x} = \prod_{j=i+1}^{\infty} f_j(x_i)$, it follows that $\overline{x} \in S_{x_i}$ for all i. On the other hand, $\lim_{i \to \infty} \delta(S_{x_i}) = 0$ implies $\overline{x} = \cap_{i=0}^{\infty} S_{x_i}$.
Now we prove that $g(\overline{x}) = \overline{x}$ for each $g \in \Phi'$. This is a consequence of the fact that $g(\overline{x}) \in S_{x_i}$ for each i, and because of $g(\overline{x}) = g(\prod_{j=i+1}^{\infty} f_j(x_i))$.
We know from the condition in (b) that there exists

$$\overline{x} = \lim_{i \to \infty} f_i f_{i-1} \cdots f_1(x_0) = \lim_{i \to \infty} x_i.$$

Since $(x_j)_{j>i} \subseteq S_i$ for each i, it follows that $\overline{x} \in \overline{S_i}$, where $\overline{S_i}$ is the closure of S_i. Since $\delta(\overline{S_i}) = \delta(S_i)$ it follows that $\overline{x} = \cap_{i=0}^{\infty} \overline{S_i}$.
We are going to show $g(\overline{x}) = \overline{x}$ for each $g \in \Phi'$. We note that $g(x_i) \in S_{x_i}$ for each i. Since g is a continuous function, for each $\varepsilon > 0$, there exists an i_0 such that

$$\{x \in X : d(g(\overline{x}), x) < \varepsilon\} \bigcap S_{x_i} \neq \emptyset \text{ for all } i > i_0.$$

Hence if $i > i_0$, it follows that $d(g(\overline{x}), \overline{x}) < \varepsilon + \varepsilon_i$. Now $\varepsilon_i \to 0$ implies $d(g(\overline{x}), \overline{x}) \leq \varepsilon$, and since ε is arbitrary, we have $g(\overline{x}) = \overline{x}$. $\square$

Remark 13.12. In the previous theorem, in the condition in (b), we may take $\Phi' = \{g^n\}$, the set of continuous functions and their finite iterations. Then we have as in Banach's contraction theorem

$$\overline{x} = \lim_{n \to \infty} g^n(x_0).$$

14. A Theorem by Bollenbacher and Hicks

The following result is related to Caristi's theorem 13.1.

Theorem 14.1 (Eisenfeld and Lakshmikantham [22])**.**
Let (X, d) be a metric space and $f : X \to X$ be a map. Then there exists a map $\phi : X \mapsto [0, \infty)$ for which

$$d(x, f(x)) \le \phi(x) - \phi(f(x)) \text{ for } x \in X, \tag{14.1}$$

if and only if the series

$$\sum_{n=0}^{\infty} d(f^n(x), f^{n+1}(x)) \tag{14.2}$$

converges for each $x \in X$.

Proof. We assume that the condition in (14.1) is satisfied. We show that the series in (14.2) converges. This follows from the fact that, for each $n \in \mathbb{N}$,

$$\sum_{k=0}^{n} d(f^k(x), f^{k+1}(x)) = d(x, f(x)) + \cdots + d(f^{n-1}(x), f^n(x))$$
$$\le (\phi(x) - \phi(fx)) + \cdots + (\phi(f^{n-1}(x)) - \phi(f^n(x)))$$
$$= \phi(x) - \phi(f^n(x)) \le \phi(x).$$

If the series (14.2) converges for each $x \in X$, the we define a map $\phi : X \to [0, \infty)$ by

$$\phi(x) = \sum_{k=0}^{\infty} d(f^k(x), f^{k+1}(x)) \text{ for all } x \in X.$$

Clearly this map ϕ satisfies the condition in (14.1). $\qquad \square$

Let $x \in X$ and $O(x, \infty) = \{x, f(x), f^2(x), \dots\}$ be the orbit of x. The map $G : X \to [0, \infty)$ is said to be f–orbitally upper semicontinuous at x if, for each sequence (x_n) in $O(x, \infty)$, it follows from $\lim_{n \to \infty} x_n = u$ that $G(u) \le \liminf_{n \to \infty} G(x_n)$.

We note that if the condition in (14.1) is satisfied for each $y \in (x, \infty)$, then the series (14.2) converges for x, since the sequence of partial sums is nondecreasing and bounded by $\phi(x)$.

In 1988, Bollenbacher and Hicks [5] proved the following very interesting theorem, the corollaries of which include many generalizations of Banach's fixed point theorem.

Theorem 14.2 (Bollenbacher and Hicks [5])**.** *Let (X, d) be a metric space, and $f : X \to X$ and $\phi : X \to [0, \infty)$. We assume that there exists x such that*

$$d(y, f(y)) \le \phi(y) - \phi((f(y))) \text{ for each } y \in O(x, \infty), \tag{14.3}$$

and that each Cauchy sequence in $O(x, \infty)$ converges to some point in X. Then we have:

(1) $\lim_{n \to \infty} f^n(x) = \overline{x}$ exists;

(2) $d(f^n(x), \overline{x}) \le \phi(f^n(x))$;

(3) $f(\overline{x}) = \overline{x}$ if and only if $G(x) = d(x, f(x))$ is f–orbitally upper semicontinuous at x;

(4) $d(f^n(x), x) \le \phi(x)$ and $d(\overline{x}, x) \le \phi(x)$.

Proof. It follows from Theorem 14.1 that the series

$$\sum_{k=0}^{\infty} d(f^k(x), f^{k+1}(x))$$

converges.

We prove that $(f^n(x))$ is a Cauchy sequence. This follows since, for each $m > n$,

$$d(f^n(x), f^m(x)) \leq d(f^n(x), f^{n+1}(x)) + \cdots + d(f^{m-1}(x), f^m(x))$$

$$= \sum_{k=n}^{m-1} d(f^k(x), f^{k+1}(x)),$$

and from the fact that the series $\sum_{k=n}^{\infty} d(f^k(x), f^{k+1}(x))$ converges. Hence there exist $\overline{x} \in X$ such that the condition in (1) is satisfied. Now from

$$0 \leq d(f^n(x), f^m(x)) \leq \sum_{k=n}^{m-1} d(f^k(x), f^{k+1}(x))$$

$$\leq \sum_{k=n}^{m-1} [\phi(f^k(x)) - \phi(f^{k+1}(x))] = \phi(f^n(x)) - \phi(f^m(x)) \leq \phi(f^n(x)),$$

the condition in (2) follows as $m \to \infty$.

To prove the condition in (3), we assume that $x_n = f^n(x) \to \overline{x}$ as $(n \to \infty)$. If G is f–orbitally upper semicontinuous at x, then we have

$$0 \leq d(\overline{x}, f(\overline{x})) = G(\overline{x}) \leq \liminf_{n \to \infty} G(x_n) = \liminf_{n \to \infty} d(f^n(x), f^{n+1}(x)) = 0,$$

and so $f(\overline{x}) = \overline{x}$.

Now we assume $f(\overline{x}) = \overline{x}$ and that (x_n) is a sequence in $O(x, \infty)$ such that $\lim_{n \to \infty} x_n = \overline{x}$. Then we have

$$G(\overline{x}) = d(\overline{x}, f(\overline{x})) = 0 \leq \liminf_{n \to \infty} d(x_n, f(x_n)) = \liminf_{n \to \infty} G(x_n),$$

and so G is an f–orbitally upper semicontinuous function at x.

The condition in (4) follows from

$$d(x, f^n(x)) \leq d(x, f(x)) + d(f(x), f^2(x)) + \cdots + d(f^{n-1}(x), f^n(x))$$

$$\leq [\phi(x) - \phi(f(x))] + [\phi(f(x)) - \phi(f^2(x))] + \dots$$

$$+ [\phi(f^{n-1}(x)) - \phi(f^n(x))]$$

$$= \phi(x) - \phi(f^n(x)) \leq \phi(x),$$

and since as $n \to \infty$, we get $d(x, \overline{x}) \leq \phi(x)$. $\qquad \square$

Corollary 14.3. ([26]) *Let (X, d) be a complete metric space and $0 < k < 1$. We assume that, for $f : X \to X$, there exists x such that*

$$d(f(y), f^2(y)) \leq kd(y, f(y)) \text{ for each } y \in O(x, \infty). \tag{14.4}$$

Then we have

(1) $\lim_{n \to \infty} f^n(x) = \overline{x}$ *exists;*

(2) $d(f^n(x), \overline{x}) \leq k^n(1 - k)^{-1}d(x, f(x))$;

(3) $f(\overline{x}) = \overline{x}$ *if and only if* $G(x) = d(x, f(x))$ *is an* f*–orbitally upper semicontinuous function at* x;

(4) $d(f^n(x), x) \leq (1-k)^{-1}d(x, f(x))$ *and* $d(\overline{x}, x) \leq (1-k)^{-1}d(x, f(x))$.

Proof. Let $\phi(y) = (1-k)^{-1}d(y, f(y))$ for all $y \in O(x, \infty)$. If we take $y = f^n(x)$ in (14.4), then we get

$$d(f^{n+1}(x), f^{n+2}(x)) \leq kd(f^n(x), f^{n+1}(x)),$$

and so

$$d(f^n(x), f^{n+1}(x)) - kd(f^n(x), f^{n+1}(x)) \leq d(f^n(x), f^{n+1}(x)) - d(f^{n+1}(x), f^{n+2}(x)).$$

Hence we have

$$d(f^n(x), f^{n+1}(x)) \leq \frac{1}{(1-k)} \cdot [d(f^n(x), f^{n+1}(x)) - d(f^{n+1}(x), f^{n+2}(x))],$$

that is,

$$d(y, f(y)) \leq \phi(y) - \phi(f(y)).$$

Now the conditions in (1), (3) and (4) follow immediately from Theorem 14.2.
We remark that (14.4) implies $d(f^n(x), f^{n+1}(x)) \leq k^n d(x, fx)$, and Theorem 14.2 implies

$$d(f^n(x), \overline{x}) \leq \phi(f^n(x)) = \frac{1}{1-k} \cdot d(f^n(x), f^{n+1}(x)) \leq \frac{k^n}{1-k} \cdot d(x, f(x)),$$

hence (2). $\qquad \square$

Remark 14.4. We remark that it is not necessary for ϕ to be an upper semicontinuous function, but it is enough that the condition in (14.1) is satisfied only on $O(x, \infty)$ for some x. Furthermore, it can be easier to check that G is an upper semicontinuous function than to check this for the function ϕ. Even if ϕ is an upper semicontinuous function and (14.1) is satisfied for each $x \in X$, it is not necessary in Caristi's theorem that $f\overline{x} = \overline{x}$, but only $f(x_0) = x_0$ for some x_0 in X.

Example 14.5. Let $X = [0, 1]$ and $\phi(x) = x$ for all $x \in X$. We define the map f by

$$f(x) = \begin{cases} 0 & \text{for } x \in \left[0, \dfrac{1}{2}\right] \\ \dfrac{x}{2} + \dfrac{1}{4} & \text{for } x \in \left(\dfrac{1}{2}, 1\right]. \end{cases}$$

For each $x \in [0, 1/2]$, we have $d(x, f(x)) = d(x, 0) = x$ and $\phi(x) - \phi(f(x)) = \phi(x) - 0 = x - 0 = x$. If $x \in (1/2, 1]$, then $d(x, f(x)) = x/2 - 1/4 = \phi(x) - \phi(f(x))$. Hence we have $d(x, f(x)) = \phi(x) - \phi(f(x))$ for all $x \in X$. We note that 0 is the only fixed point of the function f. If $x > 1/2$, then $\lim f^n(x) = 1/2 \neq f(1/2) = 0$.

Example 14.6. Let $X = \{(x, y) : 0 \leq x, y \leq 1\}$, d be the usual metric on X and $f(x, y) = (x, 0)$ for all $(x, y) \in X$. Then $f(f(p)) = f(p)$ for all $p \in X$ and $0 = d(f(p), f^2(p)) \leq (1/2)d(p, f(p))$. As in Corollary 14.3, let $\phi(p) = 2d(p, f(p))$ and $d(p, f(p)) \leq \phi(p) - \phi(f(p))$. This example shows that, even if both maps f and ϕ are continuous, then f may have more fixed points than one.

Example 14.7. We define the map $f : [-1, 1] \to [-1, 1]$ by

$$f(x) = \begin{cases} -1 & \text{for } x < 0 \\ \dfrac{x}{4} & \text{for } x \geq 0. \end{cases}$$

We note that $d(f(x), f^2(x)) \leq (1/4)d(x, f(x))$ for all $x \in [-1, 1]$. As in Corollary 14.3, let $\phi(x) = (4/3)d(x, f(x))$ for all $x \in [-1, 1]$. If $x < 0$, then we have $\lim_{n \to \infty} f^n(x) = -1 = f(-1)$, and if $x > 0$, then we have $\lim_{n \to \infty} f^n(x) = 0 = f(0)$. Hence 0 and -1 are the only fixed points of the map f. In this example, f and ϕ are discontinuous functions, $\phi(x) = (4/3)d(x, f(x))$ is an upper semicontinuous function and $d(x, f(x)) \leq \phi(x) - \phi(f(x))$.

15. Mann iteration

The continuous function $f : [0, 1] \to [0, 1]$ with $f(x) = -x$ for $x \in [0, 1]$ has a unique fixed point 0. The Picard iteration sequence $(f^n(x_0))$ diverges for all initial values $x_0 \neq 0$.

The *Mann iterations* are more general than the Picard iterations, that is, the Picard iterations are special cases of the Mann iterations which Mann introduced in his paper [37] in 1953.

Let E be a convex compact subset of a Banach space X, and $T : E \to E$ be a continuous map. By Schauder's fixed point theorem [51], there exists at least one fixed point of the function T, that is, there exists $p \in E$ such that $T(p) = p$.

In 1953, Mann ([37]) studied the problem of constructing a sequence (x_n) in E which converges to a fixed point of T. Usually an arbitrary initial value $x_1 \in E$ is chosen, and then the sequence of successive iterations (x_n) of x_1 defined by

$$x_{n+1} = T(x_n) \text{ for } n = 1, 2, \ldots \tag{15.1}$$

is considered. If this sequences converges, then its limit is a fixed point of the function T.

Definition 15.1 (Dotson [18]). Let E be a vector space, C be a convex subset of E, $f : C \to C$ be a map and $x_1 \in C$. We assume that the infinite matrix $A = [a_{nj}]$ satisfies the conditions

(A_1) $\qquad a_{nj} \geq 0$ for all $j \leq n$ and $a_{nj} = 0$ for $j > n$;

(A_2) $\qquad \sum_{j=1}^{n} a_{nj} = 1$ for each $n \geq 1$;

(A_3) $\qquad \lim_{n \to \infty} a_{nj} = 0$ for each $j \geq 1$.

We define the sequence (x_n) by $x_{n+1} = f(v_n)$, where

$$v_n = \sum_{j=1}^{n} a_{nj} x_j.$$

The sequence (x_n) is called the *Mann iterative sequence*, or simply, *Mann iteration*, and usually denoted by $M(x_1, A, f)$.

Hence the matrix A in Definition 15.1 has the following form

$$A = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ a_{21} & a_{22} & 0 & \ldots & 0 & 0 \\ . & . & . & . & . & . \\ a_{n1} & a_{n2} & \ldots & a_{nn} & 0 & 0 \\ . & . & . & . & . & . \end{bmatrix}.$$

Theorem 15.2 ([37]). *If one of the sequences (x_n) or (v_n) is convergent, then they both converge. In this case, they converge to the same limit point which is a fixed point of the function T.*

Proof. Let $\lim_{n \to \infty} x_n = p$. Since A is a regular matrix, it follows that $\lim_{n \to \infty} v_n = p$. The continuity of the function T implies $\lim_{n \to \infty} T(v_n) = T(p)$, and from $T(v_n) = x_{n+1}$, it follows that $T(p) = p$. If we assume $\lim_{n \to \infty} v_n = q$, then $\lim_{n \to \infty} x_{n+1} = T(q)$, and the regularity of the matrix A implies $\lim_{n \to \infty} v_n = T(q)$. Hence we have $T(q) = q$. $\qquad \square$

If the sequences (x_n) and (v_n) are not convergent, then, since E is a compact set, each of the two sequences has at least two distinct accumulation points.

Let X and V be the sets of accumulation points of the sequences (x_n) and (v_n), respectively.

Theorem 15.3 ([37])**.** *If the matrix A satisfies the conditions in (A_1), (A_2) and (A_3) and*

$$\lim_{n\to\infty} \sum_{k=1}^{n} |a_{n+1,k} - a_{n,k}| = 0, \tag{15.2}$$

then X and V are closed and connected sets.

Proof. The set V is closed and compact, and by (15.2), $\lim_{n\to\infty}(v_{n+1} - v_n) = 0$. Hence the set V is connected. Since the function T is continuous and $X = T(V)$, it follows that X is a closed and connected set. □

Theorem 15.4 ([37])**.** *The set V is a subset of $\mathrm{co}(X)$, where $\mathrm{co}(X)$ denotes the convex hull of the set X.*

Proof. By Mazur's theorem [38], $\mathrm{co}(X)$ is a closed set. All but finitely many terms of the sequence (x_n) are elements of each open set that contains the set $\mathrm{co}(X)$. Hence for all sufficiently large n, v_n are arbitrarily close to the the set X. Hence the limit point of each convergent subsequence of the sequence (v_n) is an element of the set $\mathrm{co}(X)$. □

Example 15.5. Let A be the Cesàro matrix of order 1, that is,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & \ldots & & & \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \ldots & & & \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \ldots & & & \\ . & . & . & . & . & . & . & . \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \ldots & \frac{1}{n} & 0 & 0 & \ldots \\ . & . & . & . & . & . & . & . \end{bmatrix}.$$

The matrix A satisfies all the assumptions for a matrix in this subsection. In this case, the Mann method $M(x_1, A, T)$ is usually referred to as the *mean value method*, where the initial value is $x_1 \in E$ and

$$x_{n+1} = T(v_n) \text{ and } v_n = \frac{1}{n} \sum_{k=1}^{n} x_k \text{ for all } n = 1, 2, \ldots .$$

We note

$$v_{n+1} - v_n = \frac{n \sum_{k=1}^{n+1} x_k - (n+1) \sum_{k=1}^{n} x_k}{(n+1)n} = \frac{T(v_n) - v_n}{n+1}. \tag{15.3}$$

In many special problems, the iterative method $M(x_1, A, T)$ converges even when the method $T^n x_1$ diverges.

Example 15.6. Let $E = \{x \in \mathbb{R}^2 : \|x\| \le 1\}$, where $\|\cdot\|$ is the Euclidean norm. Furthermore, let A be the Cesàro matrix of order 1 and the function $T : E \to E$ be the rotation about the center by the angle $\pi/4$. Then the Picard iteration $T^n(x_1)$ does not converge for any $x_1 \in E \setminus \{0\}$. Using Mann's method the $M(x_1, A, T)$, the sequences (x_n) and (v_n) always converge (*on a spiral*) to the center, independently of the choice of the initial value x_1.

Definition 15.7 ([18])**.** The Mann iterative method $M(x_1, A, f)$ is called *normal Mann iterative method* if the matrix $A = [a_{nj}]$, besides the conditions (A_1), (A_2) and (A_3), also satisfies the next two conditions

$(A_4) \qquad a_{n+1,j} = (1 - a_{n+1,n+1})a_{nj} \text{ for } (j = 1, 2, \ldots, n; n = 1, 2, \ldots);$

(A_5) either $a_{nn} = 1$ for all n, or $a_{nn} < 1$ for all $n > 1$.

In his paper [18], Dotson proved the following theorem.

Theorem 15.8 (Dotson [18])**.** *The following statements are true:*

(a) *The Mann method $M(x_1, A, f)$ is normal if and only if the matrix $A = [a_{nj}]$ satisfies the conditions in (A_1), (A_2), (A_4), (A_5) and (A_3'), where*

$$\sum_{n=1}^{\infty} a_{nn} \text{ is a divergent series.} \tag{A_3'}$$

(b) *The matrices $A = [a_{nj}]$ (except for the identity matrix) in all normal Mann methods $M(x_1, A, f)$ are constructed as follows:*
Let $0 \le c_n < 1$ for all $n = 1, 2, \ldots$ and the series $\sum_{n=1}^{\infty} c_n$ be divergent. Then the matrix $A = (a_{nj})$ is defined by by

$$\begin{cases} a_{11} = 1, \ a_{1j} = 0 \text{ for } j > 1; \\ a_{n+1,n+1} = c_n \text{ for } n = 1, 2, \ldots \\ a_{n+1,j} = a_{jj} \prod_{i=j}^{n}(1 - c_i) \text{ for } j = 1, 2, \ldots, n \\ a_{n+1,j} = 0 \text{ for } j > n + 1 \text{ and } n = 1, 2, \ldots \end{cases}$$

(c) *The sequence (v_n) in the normal Mann method $M(x_1, A, f)$ satisfies*

$$v_{n+1} = (1 - c_n)v_n + c_n f(v_n) \text{ for } n = 1, 2, \ldots, \tag{15.4}$$

where

$$c_n = a_{n+1,n+1} \text{ for all } n. \tag{15.5}$$

Proof. The statement in (a) follows from the following well–known result on infinite products, namely, that if $0 \le c_n < 1$ for all n, then $\lim_{n \to \infty} \prod_{k=1}^{n}(1 - c_k) = 0$ if and only if the series $\sum_{k=1}^{\infty} c_k$ diverges.
To prove the statement in (b), we note that if the matrix A satisfies the conditions in $(A1)$–$(A5)$, then it satisfies the condition in (b). It can be proved that if the matrix A satisfies the conditions in (b), where $c_n = a_{n+1,n+1}$ for all $n \in \mathbb{N}$, then it satisfies the conditions in $(A1)$–$(A5)$.
The proof of (c) follows if we use the condition in $(A4)$ and the definitions of the sequences (v_n) and (x_n) in Mann's method $M(x_1, A, T)$. $\qquad\square$

Example 15.9. For each λ with $0 \le \lambda < 1$, let the infinite matrix $A_\lambda = (a_{nj})$ be defined by

$$\begin{cases} a_{n1} = \lambda^{n-1} \\ a_{nj} = \lambda^{n-j}(1 - \lambda) \text{ for } j = 2, 3, \ldots, n, \\ a_{nj} = 0 \text{ for } j > n \text{ and } n = 1, 2, 3, \ldots, \end{cases},$$

where, for $\lambda = 0$, we put $a_{nn} = 1$ for all n. Hence A_0 is the infinite identity matrix. It can be shown that for each λ with $0 \le \lambda < 1$, $M(x_1, A_\lambda, T)$ is a normal Mann method with $c_n = a_{n+1,n+1} = 1 - \lambda$ for all $n = 1, 2, 3 \ldots$. Hence the sequence (v_n) in the normal Mann method $M(x_1, A_\lambda, T)$ is defined by

$$v_{n+1} = \lambda v_n + (1 - \lambda)T(v_n) \text{ for all} n.$$

Let $S_\lambda = \lambda I + (1 - \lambda)T$ (where I is the identity map). Hence we have

$$v_{n+1} = S_\lambda(v_n) = S_\lambda^n(v_1) = S_\lambda^n(x_1) \text{ for all } n.$$

We note that $S_0 = T$ and, in this case, the sequence (v_n) is obtained by Picard's iteration $(T^n(x_1))$. The sequence $(S_{1/2}^n(x_1))$ of Picard's iterations of the map $S_{1/2} = (1/2)(I + T)$ was studied by Krasnoselskii [32] and Edelstein [21], and the sequence $(S_\lambda^n(x_1))$ of Picard's iterations of the map S_λ for $0 < \lambda < 1$ was studied by Schäfer [50], Browder and Petryshyn [9], and Opial [40].

In the literature, mainly the normal Mann iterative method is studied.

16. Continuous functions on $[a, b] \subset \mathbb{R}$

Now we consider the case when the Banach space is the real line $\mathbb{R}$, and the convex compact set E is a closed interval and A is the Cesàro matrix of order 1.

Theorem 16.1 (Mann [37])**.** *Let $T : [a, b] \to [a, b]$ be a continuous map which has a unique fixed point $p \in [a, b]$ and A be the Cesàro matrix of order 1. Then Mann's sequence $M(x_1, A, T)$ converges to p for each $x_1 \in [a, b]$.*

Proof. It follows from (15.3) that $v_{n+1} - v_n \to 0$ as $n \to \infty$. Since T is a continuous function and p is the unique fixed point of T, it follows that $T(x) - x > 0$ for $x < p$ and $T(x) - x < 0$ for $x > p$. Hence, for each $\delta > 0$, there exists $\varepsilon > 0$ such that $|x - p| \geq \delta$ implies $|T(x) - x| \geq \varepsilon$. It follows from (15.3) that

$$v_{n+1} = v_1 + \sum_{k=1}^{n} \frac{T(v_k) - v_k}{k+1}.$$

Now from our previous considerations, we have $\lim_{n \to \infty} v_n = p$, and by Theorem 15.2, we obtain $\lim_{n \to \infty} x_n = p$. $\square$

In higher dimensional spaces, results similar to that of Theorem 16.1 have not been obtained.

Remark 16.2. Reinermann [45] defined a summability matrix A as follows

$$a_{nk} = \begin{cases} c_k \prod_{j=k+1}^{n}(1 - c_j) & \text{for } k < n \\ c_n & \text{for } k = n \\ 0 & \text{for } k > n, \end{cases} \tag{16.1}$$

where the real sequence (c_n) satisfies the following conditions

(i) $c_0 = 1$,

(ii) $0 < c_n < 1$ for $n \geq 1$,

(iii) $\sum_{k=0}^{\infty} c_k$ diverges.

It can be proved that A is a regular matrix, and satisfies the following conditions

$$0 \leq a_{nk} \leq 1 \text{ for } n, k = 0, 1, 2, \ldots \tag{16.2}$$

$$\sum_{k=0}^{n} a_{nk} = 1 \text{ for } n = 0, 1, 2, \ldots \tag{16.3}$$

Reinermann also considered the condition $c_n = 1$, since he included the identity matrix in his considerations. Since the identity matrix is of no special interest, in all interesting applications, it is assumed that $c_n < 1$. Then he considered the iterative scheme $\overline{x}_0 = x_0 \in E$ and $x_{n+1} = \sum_{k=0}^{n} a_{nk} f(x_k)$, which can be written as

$$x_{n+1} = (1 - c_n)x_n + c_n f(x_n). \tag{16.4}$$

It is well known by Brower's fixed point theorem that a continuous map from $[a, b]$ to $[a, b]$ has at least one fixed point. Reinermann proved the following result.

Theorem 16.3 (Reinermann [45], 1969)**.** *Let $a, b \in \mathbb{R}$, $a < b$, $E = [a, b]$ and $f : E \to E$ be a continuous map with at most one fixed point. If the matrix A is defined by (16.1) and the sequence (c_n) satisfies the conditions (i)–(iii) and $\lim c_n = 0$, then the iterative scheme (16.4), for $x_0 \in [a, b]$, converges to the fixed point of f.*

Proof. Without loss of generality, we may assume $a = 0$ and $b = 1$. By Brouwer's fixed point theorem and our assumption, there exists a unique fixed point $x \in [0,1]$ of the function f. Now we have

$$\text{for all } y \in [0,1] \text{ with } y < x \text{ it follows that } f(y) - y > 0; \tag{16.5}$$
$$\text{for all } y \in [0,1] \text{ with } (y > x) \text{ it follows that } f(y) - y < 0. \tag{16.6}$$

If $x = 0$, then we obviously have (16.5). If $x > 0$ and if there exists $y_1 \in [0,1]$ with $y_1 < x$ such that $f(y_1) - y_1 \leq 0$, then $f(0) - 0 = f(0) \geq 0$ implies that there exists $z \in [0, y_1]$ such that $f(z) = z$. Now $z \neq x$, which is a contradiction to the uniqueness of the fixed point.

The case (16.6) is proved analogously.

There are two alternatives **I.** and **II.** for the sequence (x_n):

I. There exists $n_1 \in \mathbb{N}$ such $x_{n_1} = x$.

Then $x_n = x$ for all $n \geq n_1$ and the theorem is proved.

II. For each $n \in \mathbb{N}$, we have $x_n \neq x$.

In this case, we have the following three possibilities:

1. There exists $n_0 \in \mathbb{N}$ such that $x_n < x$ for all $n > n_0$. Then we have

    $$x_{n+1} - x_n = c_n(f(x_n) - x_n),$$

    and (16.5) implies that (x_n) is a monotone increasing sequence; so the sequence converges, since $x_n \leq 1$ for all n. By Theorem 15.2, and since the function f has only one fixed point $x \in [0,1]$, it follows that $\lim_n x_n = x$.

2. There exists $m_0 \in \mathbb{N}$ such that $x_n > x$ for all $n \geq m_0$. In this case, it follows by (16.6) that $\lim_{n \to \infty} x_n = x$, as in Case **1.**.

3. We assume that the possibilities **1.** and **2.** are not true. Let $\varepsilon > 0$ be given. We choose $n_0 \in \mathbb{N}$ such that

    $$|x_{n+1} - x_n| < \varepsilon \text{ for all } n \geq n_0.$$

    This is possible, since

    $$|x_{n+1} - x_n| \leq 2c_n \text{ and } \lim_{n \to \infty} c_n = 0.$$

    We are going to prove that there exists $n_1 \in \mathbb{N}$ with $n_1 \geq n_0$ such that $|x_{n_1} - x| < \varepsilon$, that is,

    $$\text{there exists } n_1 \geq n_0 \text{ such that } -\varepsilon < x_{n_1} - x < \varepsilon. \tag{16.7}$$

    If (16.7) is not true, then

    $$x_n \leq x - \varepsilon \text{ or } x_n \geq x + \varepsilon \text{ for each } n \geq n_0. \tag{16.8}$$

Now, if $x_{n_0} \leq x - \varepsilon$, then $x_n \leq x - \varepsilon$ for all $n \geq n_0$ (because of $|x_{n+1} - x_n| < \varepsilon$), that is, the condition in **1.** is satisfied. Analogously, if $x_{n_0} \geq x + \varepsilon$, then $x_n \geq x + \varepsilon$ for all $n \geq n_0$ (again because of $|x_{n+1} - x_n| < \varepsilon$), that is, the condition in **2.** is satisfied. Hence, in all cases, the conditions in **1.** or **2.** are satisfied. So we have shown (16.7).

We are going to prove that we have $|x_n - x| < \varepsilon$ for all $n \geq n_1$. This is true for $n = n_1$. If $n \geq n_1$ and if $|x_n - x| < \varepsilon$, then we have the following possibilities **A.** and **B.**:

A. $x - \varepsilon < x_n < x$. Then we have (a) or (b) for x_{n+1}:

  (a) $x_{n+1} < x$. In this case, $x_{n+1} - x_n = c_n(f(x_n) - x_n)$ and (16.5) imply $x_{n+1} - x_n > 0$, Hence we have

    $$|x_{n+1} - x| = x - x_{n+1} < x - x_n = |x_n - x| < \varepsilon.$$

(b) $x_{n+1} > x$. Now we have

$$|x_{n+1} - x| = x_{n+1} - x < x_{n+1} - x_n = |x_{n+1} - x_n| < \varepsilon.$$

B. $x < x_n < x + \varepsilon$. Now (16.6) implies the conclusion as in **A.**, that is, $|x_{n+1} - x| < \varepsilon$.
Hence $|x_{n+1} - x| < \varepsilon$. It follows by mathematical induction that $|x_n - x| < \varepsilon$ for all $n \geq n_1$, thus
$\lim_n x_n = x$. □

We note that if we put $c_n := 1/(n+1)$ for all n, then Theorem 16.3 implies Theorem 16.1.

In 1971, Franks and Marzec [24] showed that the condition of the uniqueness of the fixed point p in
Theorem 16.1 is not necessary.

We note that any continuous function $f : [0,1] \to [0,1]$ has at least one fixed point by Brouwer's fixed
point theorem.

Theorem 16.4 (Franks and Marzec ([24])). *Let $f : [0,1] \to [0,1]$ be a continuous function. Then the
iterative sequence*

$$x_{n+1} = f(\tilde{x}_n) \text{ for } n = 1, 2, \ldots \tag{16.9}$$

$$\tilde{x}_n = \sum_{k=1}^{n} \frac{x_k}{n} \text{ for } n = 1, 2, \ldots, \tag{16.10}$$

$$\tilde{x}_1 = x_1 \in [0,1], \tag{16.11}$$

converges to a fixed point of the function f in the interval $[0,1]$.

Proof. It follows from (16.9) and (16.10) that

$$\tilde{x}_{n+1} = \frac{f(\tilde{x}_n) - \tilde{x}_n}{n+1} + \tilde{x}_n \text{ for } n = 1, 2, \ldots. \tag{16.12}$$

Since $\tilde{x}_n$ and $f(\tilde{x}_n) \in [0,1]$ for all n, we have

$$|\tilde{x}_{n+1} - \tilde{x}_n| \leq \frac{1}{n+1} \text{ for } n = 1, 2, \ldots. \tag{16.13}$$

It suffices to prove that this sequence is convergent and its limit $\xi \in [0,1]$ is a fixed point of the function f.

1. We prove that the sequence $(\tilde{x}_n)$ is convergent. The sequence $(\tilde{x}_n)$ is in $[0,1]$, and so has at least one
accumulation point. We assume that the sequence $(\tilde{x}_n)$ has two distinct accumulation points ξ_1 and
ξ_2 with $\xi_1 < \xi_2$.

a. We are going to show that we have, from the assumption above, $f(x) = x$ for all $x \in (\xi_1, \xi_2)$. Let
$x^* \in (\xi_1, \xi_2)$. If $f(x^*) > x^*$, then, since f is a continuous function, there exists $\delta \in (0, (x^* - \xi_1)/2)$
such that $|x - x^*| < \delta$ implies $f(x) > x$. Hence $|\tilde{x}_n - x^*| < \delta$ implies $f(\tilde{x}_n) > \tilde{x}_n$. Thus we obtain
from (16.12) that

$$|\tilde{x}_n - x^*| < \delta \text{ implies } \tilde{x}_{n+1} > \tilde{x}_n. \tag{16.14}$$

By (16.13), there exists N such that

$$|\tilde{x}_{n+1} - \tilde{x}_n| < \delta \text{ for } n = N, N+1, \ldots. \tag{16.15}$$

Since $\xi_2 > x^*$ is an accumulation point of the sequence $(\tilde{x}_n)$, we can choose N such that $\tilde{x}_N > \tilde{x}^*$.
It follows from (16.14) and (16.15) that

$$\tilde{x}_n > x^* - \delta > \xi_1 \text{ for } n = N, N+1, \ldots.$$

Thus ξ_1 is not an accumulation point of the sequence $(\tilde{x}_n)$, which contradicts our assumption.
If $f(x^*) < x^*$, then, similarly as above, we obtain that ξ_2 is not an accumulation point of the
sequence $(\tilde{x}_n)$, which again is a contradiction. Hence $f(x^*) = x^*$ for each $x^* \in (\xi_1, \xi_2)$.

b. Let us prove that ξ_1 and ξ_2 are not accumulation points of the sequence $(\tilde{x}_n)$. We note that

$$\tilde{x}_n \notin (\xi_1, \xi_2) \text{ for } n = 1, 2, \dots. \tag{16.16}$$

If $f(\tilde{x}_n) = \tilde{x}_n$, then (16.12) implies $\tilde{x}_m = \tilde{x}_n$ for all $m > n$. So neither ξ_1 nor ξ_2 can be an accumulation point of the sequence $(\tilde{x}_n)$. Furthermore, (16.13) and (16.16) imply that there exists a natural number M such that $\tilde{x}_M \geq \xi_2$ for all $n > M$. Hence ξ is not an accumulation point of the sequence $(\tilde{x}_n)$. It follows from $\tilde{x}_M \leq \xi_1$ that $\tilde{x}_n < \xi_1 < \xi_2$ for all $n > M$. Hence ξ_2 is not an accumulation point of the sequence $(\tilde{x}_n)$. Consequently the sequence $(\tilde{x}_n)$ cannot have two distinct accumulation points, and so this sequence is convergent. We put $\lim_n \tilde{x}_n = \xi \in [0,1]$.

2. We show $f(\xi) = \xi$. We assume $f(\xi) > \xi$. Let

$$\varepsilon = \frac{f(\xi) - \xi}{2} > 0.$$

Since the sequence $(\tilde{x}_n)$ converges to ξ and the function f is continuous, there exists a natural number N such that $f(\tilde{x}_n) - \tilde{x}_n > \varepsilon$ for each $n > N$. It follows from (16.12) that

$$\tilde{x}_{n+1} - \tilde{x}_n = \frac{f(\tilde{x}_n) - \tilde{x}_n}{n+1} > \frac{\varepsilon}{n+1}.$$

Hence we have

$$\lim_{m \to \infty} (\tilde{x}_{N+m} - \tilde{x}_N) = \lim_{m \to \infty} \sum_{n=N}^{m-1} (\tilde{x}_{n+1} - \tilde{x}_n)$$

$$\geq \lim_{m \to \infty} \sum_{n=N}^{m-1} \frac{\varepsilon}{n+1} = \infty.$$

So $\tilde{x}_n \to \infty$ as $n \to \infty$, which contradicts the fact that $\tilde{x}_m \in [0,1]$ for all m. If $f(\xi) < \xi$, then it can be shown that $\tilde{x}_n \to -\infty$ as $n \to \infty$, which again is a contradiction. So we have $f(\xi) = \xi$. $\qquad \square$

Rhoades ([48], [47] and [16]), among other things, generalized many results presented in this section. He noted the importance of the condition in (15.2).

Let X be a normed space, E be a nonempty, closed, bounded and convex, subset of X and $f : E \to E$ be a map which has at least one fixed point in E, and let A be an infinite matrix. We consider the iterative scheme

$$\overline{x}_0 = x_0 \in E \tag{16.17}$$

$$\overline{x}_{n+1} = f(x_n) \text{ for } n = 0, 1, 2, \dots \tag{16.18}$$

$$x_n = \sum_{k=0}^{n} a_{nk} \overline{x}_k \text{ for } n = 1, 2, 3, \dots. \tag{16.19}$$

The question is which are the necessary and sufficient conditions for the matrix A such that the above iterative scheme converges to a fixed point of the function f?

Many results were obtained by the use of the iterative scheme of the form above (16.17)–(16.19) for various classes of infinite matrices.

An infinite matrix A is said to be regular if $x \in c$ and $x_n \to l$ as $n \to \infty$ implies $A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k \to l$ as $n \to \infty$. The matrix A is triangular if all entries below the main diagonal are equal to zero. We consider regular triangular matrices A which satisfy

$$0 \leq a_{nk} \leq 1 \text{ for all } n, k = 0, 1, 2, \dots \tag{16.20}$$

$$\sum_{k=0}^{n} a_{nk} = 1 \text{ for all } n = 0, 1, 2, \dots. \tag{16.21}$$

The conditions in (16.20) and (16.21) are necessary for $x_n, \overline{x}_n \in E$. The scheme (16.17)–(16.19) is a Mann method [37].

Barone proved in [4] (see (15.2)) that a necessary condition that a regular matrix A maps all bounded sequence into sequences with the property that the set of their accumulation points is connected is the following

$$\lim_n \sum_{k=0}^{\infty} |a_{nk} - a_{n-1,k}| = 0. \tag{16.22}$$

In [46], Rhoades made the following assumption.

Assumption Let $f : [a,b] \to [a,b]$ be a continuous function, A be a regular matrix which satisfies the conditions in (16.20)–(16.22). Then the iterative scheme defined by (16.17)–(16.19) converges to a fixed point of the function f.

In the next example, he showed that the assumption above does not hold if the condition (16.22) removed.

Example 16.5. Let A be be the identity matrix, $[a,b] = [0,1]$, $f(x) = 1 - x$ and $x_0 = 0$.

Rhoades showed that the statement above is true for the large class of weighted means matrices. (For the definition and properties of these matrices see [25, p. 57].)

The weighted means method is a triangular method of the matrix $A = (a_{nk})$ defined by $a_{nk} = p_k/P_n$, where $p_0 > 0$, $p_n \geq 0$ for $n > 0$, $P_n = \sum_{k=0}^{n} p_k$ and $P_n \to \infty$ as $n \to \infty$. Then the matrix A satisfies the condition in (16.22) if and only if $p_n/P_n \to 0$ as $n \to \infty$.

Theorem 16.6 (Rhoades [48]). *Let A be the matrix of a regular weighted means method which satisfies the condition in (16.22). Let $E = [a,b]$ and $f : E \mapsto E$ be a continuous map. Then the iterative scheme (16.17)– (16.19) converges to a fixed point of the function f.*

Proof. Without loss of generality, we may suppose that $[a,b] = [0,1]$. Every regular weighted means method satisfies the conditions in (16.20) and (16.21). By (16.19), we have

$$x_{n+1} = \frac{p_{n+1}}{P_{n+1}}(f(x_n) - x_n) + x_n \text{ for all } n. \tag{16.23}$$

Since $x_n, f(x_n) \in [0,1]$, it follows from (16.23) that

$$|x_{n+1} - x_n| \leq \frac{p_{n+1}}{P_{n+1}} \to 0 \ (n \to \infty).$$

Now, by the proof of Theorem 16.4, the sequence (x_n) is convergent.
We have to show that the sequence converges to a fixed point of the function f. Let $z = \lim_{n \to \infty} x_n$. Then we have $\lim_{n \to \infty} f(x_n) = f(z)$. It follows from $\overline{x}_{n+1} = f(x_n)$ for each $n \in \mathbb{N}$ that $\lim_{n \to \infty} \overline{x}_n = f(z)$. Since A is a regular matrix, we obtain $z = \lim_{n \to \infty} x_n = \lim_{n \to \infty} A_n(\overline{x}) = f(z)$. $\square$

Theorem 16.4 can be proved by taking $p_n = 1$ in Theorem 16.6.

Theorem 16.6 implies Theorem 16.3. Furthermore the mentioned iterative schemes were defined independently by Outlaw and Groetsch [41], and Dotson [18]. We note that Theorem 15.8 (that is, [18, Theorem 2]) characterizes the method in (16.1) and (i)–(iii).

If we choose $c_n = (n+1)^{-1}$, the previous statement is Theorem 16.1.

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