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On some Banach lattice-valued operators: A Survey

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Abstract

In 1928, at the International Mathematical Congress held in Bologna (Italy), Frigyes Riesz introduced the notion of vector lattice on function spaces and, talked about linear operators that preserve the join operation, nowadays known in the literature as Riesz homomorphisms (see [32]). In this survey we review the behaviors of some non-linear join-preserving Riesz space-valued functions, and we show how existing addition dependent results can be proved in these environments mutatis mutandis. (We kindly refer the reader to the papers [1, 2, 3, 4, 6, 7, 8, 9, 10, 5] for more information.)

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1. Motivations, historical background and introduction

1.1. The motivations

By splitting Mathematics into two, the group of addition-related environments and the group of addition-free environments, we then ask the question to know whether there are addition-dependent environments and if any, what results they contain that can be proved in addition-free environments, the proofs being carried out mutatis-mutandis.

The collection of the present results aims to provide some answers to the above series of questions in the affirmative. In fact, we consider mappings whose target sets are lattices and show how existing addition-dependent results can be proved similarly in lattice environments. In the early 90's we substituted with the

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lattice join operation, the addition in the definition of measure as well as in the Lebesgue integral to obtain lattice-dependent operators which behave similarly as their counterparts in Measure Theory (sometimes under restraints), in the sense that existing major theorems in Measure Theory are also proved with the addition replaced by the join (or supremum). It is worth also to turn our interest to what make these two groups of environments different from each other, yet similar can their results be. The next targetted environment is the famous Cauchy functional equation. Replacing the addition by lattice operations the Hyers-Ulam stability problem can be posed. In this case also, the various solutions obtained for such problem are the same as their counterparts in the literature. Furthermore, on group structure separation theorem can also be proved when the target set is a lattice [10]. To illustrate the divergence of the above two groups, there are characterizations of various properties of measurable functions [3], as well as the characterization of an arbitrary infinite σ-algebra to be equinumerous with a power set [4].

2. Historical backgrounds and notations

2.1. About the convergence of function sequences

Augustin Louis Cauchy in 1821 published a faulty proof of the false statement that the pointwise limit of a sequence of continuous functions is always continuous. Joseph Fourier and Niels Henrik Abel found counter examples in the context of Fourier series. Dirichlet then analyzed Cauchy's proof and found the mistake: the notion of pointwise convergence had to be replaced by uniform convergence.
The concept of uniform convergence was probably first used by Christoph Gudermann. Later his pupil Karl Weierstrass coined the term *gleichmäßig konvergent* (German: uniform convergence) which he used in his 1841 paper *Zur Theorie der Potenzreihen*, published in 1894. Independently a similar concept was used by Philipp Ludwig von Seidel and George Gabriel Stokes but without having any major impact on further development. G.H. Hardy compares the three definitions in his paper *Sir George Stokes and the concept of uniform convergence* and remarks: *Weierstrass's discovery was the earliest, and he alone fully realized its far-reaching importance as one of the fundamental ideas of analysis.* For more materials about these facts we refer to [33] or
http://en.wikipedia.org/wiki/Uniform_convergence.

Ever since many other types of convergence have been brought to light. We can list some few of them: discrete and equal convergence introduced by Á. Császár and M. Laczkovich in 1975 (cf. [14, 15, 16]), topologically speaking the weak and strong convergence, the latest being at the origin of the so-called *Banach spaces*, which are very broad and interesting classes of functions, indeed.

2.2. Riesz spaces

A vector space over the field of real line endowed with a partial ordering is called a *Riesz space* if the following clauses are met:

1. the algebraic structure of the vector space and the ordering are compatible, i.e. the ordering is translation invariant and positive homogenious (referred to as *a vector lattice*),
2. every finite subset of the space has a least upper bound called the supremum.

It can be seen that a vector lattice is a Riesz space if and only if every pair of elements in the space has an infimum (cf. [11, Aliprantis and Burkinshaw, Lemma 1.2]). The next very important properties enjoyed by Riesz spaces are:

a. Every Riesz space is a distributive lattice.
b. The positive cone of any Riesz space is generating, i.e. every element of the space can be expressed as the differerence of two elements of the positive cone. (For more see [28].)

This last point means that working on the positive cone of a Riesz space is just as working on the whole space.

The notion of vector lattice was introduced by Frigyes Riesz on function spaces at the International Mathematical Congress in Bologna (1928), which was publised two years later (cf. [32]). Around the mid-thirties Riesz was relayed by Hans Freudenthal (cf. [20]) and L.V. Kantorovich (cf. [24, 25]) by simultaneously laying the strict axiomatic foundation of the theory of Riesz spaces. This new concept has grown very rapidly in the 1940s and early 50s, thanks to Japanese and Russian schools which were created to cultivate this young theory. (Cf. [11, Aliprantis and Burkinshaw] for more historical background.) At the earliest stages rather algebraic aspect of the theory was studied. The analytical aspect started with a series of articles by W.A.J. Luxemburg and A.C. Zaanen which can be found in the book by Aliprantis and Burkinshaw, reference [89]. Another aspect of the theory of Riesz spaces is topological (cf. [19, Fremlin]). We would also like to stress the important place supremum preserving linear operators (so-called Riesz homomorphisms) occupy in the literature.

2.3. **Notations.**

- ⋆ $\mathbb{N}$ denotes the set of positive integers.
- ⋆ $\mathbb{R}$ denotes the set of real numbers.
- ⋆ $\mathbb{R}_+$ denotes the set of non-negative real numbers.
- ⋆ $\chi(B)$ stands for the characteristic function of the set B.
- ⋆ $|B|$ designates the cardinality of the set B.
- ⋆ $\bigvee$ and $\vee$ (respectively, $\bigwedge$ and $\wedge$) stand for the maximum (respectively the minimum) operator.
- ⋆ $\mathcal{P} := \mathcal{P}_{<\infty} \cup \mathcal{P}_{\infty}$ will denote the set of all optimal measures defined on measurable space $(\Omega, \mathcal{F})$, with both Ω and $\mathcal{F}$ being infinite sets, where $\mathcal{P}_{<\infty}$ (resp. $\mathcal{P}_{\infty}$) denotes the set of all optimal measures whose generating systems are finite (resp. countably infinite).
- ⋆ For every $A \in \mathcal{F}$, we write $\overline{A}$ for the complement of A.
- ⋆ $A \subset B$ means set A is a proper subset of set B.
- ⋆ $A \subseteq B$ means set A is a subset of set B.
- ⋆ The power set of set A will be denoted by $\mathbb{P}(A)$ or 2^A.

We would like to note that our approach of dealing with Riesz spaces seems new. The results we present here are selected from [1, 2, 3, 4, 6, 8, 9, 7, 10, 5] and they all fall outside the scope of Riesz homomorphisms.

3. Optimal measures and the structure theorem

By replacing the addition in the definition of (probability) measure by the supremum we expect to obtain a non-additive set function which behaves almost like a (probability) measure. To this end normalizing properties and the continuity from below are necessary to have similar effects as in the case of measure.

3.1. Optimal measure

Definition 3.1 ([1], Definition 0.1)**.** A set function $p : \mathcal{F} \to [0, 1]$ will be called *optimal measure* if it satisfies the following three axioms:

Axiom 1. $p(\Omega) = 1$ and $p(\varnothing) = 0$.

Axiom 2. $p(B \cup E) = p(B) \vee p(E)$ for all measurable sets B and E.

Axiom 3. p is continuous from above, i.e. whenever $(E_n) \subset \mathcal{F}$ is a decreasing sequence, then $p\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} p(E_n) = \bigwedge_{n=1}^{\infty} p(E_n)$.

The triple $(\Omega, \mathcal{F}, p)$ will be referred to as an *optimal measure space*. For all measurable sets B and C with $B \subset C$, the identity

$$p\left(C \backslash B\right) = p\left(C\right) - p\left(B\right) + \min\left\{p\left(C \backslash B\right), p\left(B\right)\right\} \tag{3.1}$$

holds, and especially for all $B \in \mathcal{F}$,

$$p\left(\overline{B}\right) = 1 - p\left(B\right) + \min\left\{p\left(B\right), p\left(\overline{B}\right)\right\}.$$

In fact, it is obvious (via Axiom 2) that,

$$\begin{aligned} p\left(B\right) + p\left(C \backslash B\right) &= \max\left\{p\left(C \backslash B\right), p\left(B\right)\right\} + \min\left\{p\left(C \backslash B\right), p\left(B\right)\right\} \\ &= p\left(C\right) + \min\left\{p\left(C \backslash B\right), p\left(B\right)\right\}. \end{aligned}$$

Lemma 3.2 ([1], Lemma 0.1)**.** *Let $(B_n) \subset \mathcal{F}$ be any sequence tending increasingly to a measurable set B, and p an optimal measure. Then $\lim_{n\to\infty} p\left(B_n\right) = p\left(B\right)$.*

Proof. The lemma will be proved if we show that for some $n_0 \in \mathbb{N}$, the identity $p\left(B\right) = p\left(B_n\right)$ holds true whenever $n \geq n_0$. Assume that for every $n \in \mathbb{N}$, $p\left(B\right) \neq p\left(B_n\right)$, which is equivalent to $p\left(B_n\right) < p\left(B\right)$, for all $n \in \mathbb{N}$. This inequality, however, implies that $p\left(B\right) = p\left(B \backslash B_n\right)$ for each $n \in \mathbb{N}$. But since sequence $(B \backslash B_n)$ tends decreasingly to $\varnothing$, we must have that $p\left(B\right) = 0$, a contradiction which proves the lemma. $\square$

It is clear that every optimal measure p is monotonic and σ-subadditive.

The following example was given in [1], Example 3.1 and its check was left as an exercise.

Example 3.3. The function $\Phi : 2^{\mathbb{N}} \to [0, 1]$ defined by $\Phi\left(A\right) = \frac{1}{\min A}$ is an optimal measure (where $\min \emptyset = \infty$ by convention).

Proof. The normalization properties are obvious. We show that Φ is a join homomorphism. In fact, let $A, B \in 2^{\mathbb{N}}$ be arbitrary. Then as $\min(A \cup B) = \min\left\{\min A; \min B\right\}$ it ensues that

$$\Phi(A \cup B) = \frac{1}{\min\left\{\min A; \min B\right\}} = \frac{1}{\min A} \vee \frac{1}{\min B} = \Phi(A) \vee \Phi(B).$$

To check the continuity from above, pick arbitrarily a sequence $(A_n) \subset 2^{\mathbb{N}}$ which tends decreasingly to some subset A of $\mathbb{N}$. Then for all natural numbers n and from the trivial identity $A_n = A \cup (A_n \setminus A)$ we have $\Phi(A_n) = \Phi(A) \vee \Phi(A_n \setminus A)$. But since sequence $(A_n \setminus A)_{n\in\mathbb{N}}$ tends deacreasingly to the empty set, it follows that $\lim_{n\to\infty} \min(A_n \setminus A) = \infty$ which yields

$$\lim_{n\to\infty} \Phi(A_n \setminus A) = \lim_{n\to\infty} \frac{1}{\min(A_n \setminus A)} = 0.$$

Consequently,

$$\lim_{n\to\infty} \Phi(A_n) = \bigwedge_{n=1}^{\infty} \Phi(A_n) = \Phi(A) \vee \left(\bigwedge_{n=1}^{\infty} \Phi(A_n \setminus A)\right) = \Phi(A).$$

$\square$

Example 3.4 ([1], Example 0.1)**.** Let $(\Omega, \mathcal{F})$ be a measurable space, $(\omega_n) \subset \Omega$ be a fixed sequence, and $(\alpha_n) \subset [0, 1]$ a given sequence tending decreasingly to zero. The function $p : \mathcal{F} \to [0, 1]$, defined by

$$p\left(B\right) = \max\left\{\alpha_n : \omega_n \in B\right\} \tag{3.2}$$

is an optimal measure.

Moreover, if $\Omega = [0, 1]$ and $\mathcal{F}$ is a σ-algebra of $[0, 1]$ containing the Borel sets, then every optimal measure defined on $\mathcal{F}$ can be obtained as in (3.2).

Proof of the moreover part. We first prove that if $B \in \mathcal{F}$ and $p(B) = c > 0$, then there is an $x \in B$ which satisfies $p(\{x\}) = c$. To do this let us show that there exists a nested sequence of intervals $I_0 \supset I_1 \supset I_2 \supset \ldots$ such that $|I_n| = 2^{-n}$ and $p(B \cap I_n) = c$, for every $n \in \mathbb{N} \cup \{0\}$. In fact, let $I_0 = [0, 1]$. If I_n has been defined then let $I_n = E \cup H$, where E and H are non-overlapping intervals with $|E| = |H| = 2^{-n-1}$. Obviously, we may choose $I_{n+1} = E$ or H. By the continuity from above we have $p\left(\bigcap_{n=1}^{\infty}(B \cap I_n)\right) = c > 0$. In particular, $B \cap \left(\bigcap_{n=1}^{\infty} I_n\right) \neq \varnothing$. This implies that $B \cap \left(\bigcap_{n=1}^{\infty} I_n\right) = \{x\}$ and $p(\{x\}) = c$. Fix $c > 0$. Then the set $\{x : p(\{x\}) \geq c\}$ is finite. Assume in the contrary that there is an infinite sequence $(x_k) \subset [0, 1]$ such that $p(\{x_k\}) \geq c$, $k \in \mathbb{N}$. Thus denoting $B_k = \{x_k, x_{k+1}, \ldots\}$, it is clear that $\bigcap_{k=1}^{\infty} B_k = \varnothing$; but this contradicts the fact that $p(B_k) \geq c$. Consequently, the set $E_n = \left\{x : p(\{x\}) \geq n^{-1}\right\}$ is finite for all $n \in \mathbb{N}$. Hence there is a sequence $(x_n) \subset [0, 1]$ such that $p(\{x_n\}) \downarrow 0$ (as $n \to \infty$) and every point $x \in [0, 1]$ with $p(\{x\}) \geq 0$ is contained in (x_n). Therefore, for all $B \in \mathcal{F}$, $p(B) = \max\{\alpha_n : x_n \in B\}$ which is just the above optimal measure. □

3.2. The structure of optimal measures

By a p-atom we mean a measurable set $H, p(H) > 0$ such that whenever $B \in \mathcal{F}$ and $B \subset H$, then $p(B) = p(H)$ or $p(B) = 0$.

Definition 3.5 ([2], Definition 1.1)**.** A p-atom H is decomposable if there exists a subatom $B \subset H$ such that $p(B) = p(H) = p(H \backslash B)$. If no such subatom exists, we shall say that H is indecomposable.

Lemma 3.6 ([2], Lemma 1.1)**.** *Any atom H can be expressed as the union of finitely many disjoint indecomposable subatoms of the same optimal measure as H.*

Proof. We say that a measurable set E is *good* if it an be expressed as the union of finitely many disjoint indecomposable subatoms. Let H be an atom and suppose that H is not good. Then H is decomposable. Set $H = B_1 \cup C_1$, where B_1 and C_1 are disjoint measurable sets with $p(B_1) = p(C_1) = p(H)$. Since H is not good, at least one of the two measurable sets B_1 and C_1 is not good; suppose, e.g. that B_1 is not good. Then B_1 is decomposable. Write $B_1 = B_2 \cup C_2$, where B_2 and C_2 are disjoint measurable sets with $p(B_2) = p(C_2) = p(H)$. Continuing this process for every $n \in \mathbb{N}$ we obtain two measurable sets B_n and C_n such that the C_n's are pairwise disjoint with $p(C_n) = p(H)$. This, however, is impossible since $E_n = \bigcup_{k=n}^{\infty} C_k$ tends decreasingly to the empty set and hence, by Axiom 3, $p(E_n) \to p(\varnothing)$ as $n \to \infty$, which contradicts that $p(E_n) \geq p(C_n) = p(H) > 0$, $n \in \mathbb{N}$. □

An immediate consequent of Lemma 3.6 is as follows.

Remark 3.7 ([2], Remark 1.1)*.* Let H be any indecomposable p-atom and E any measurable set, with $p(E) > 0$. Then, either $p(H) = p(H \backslash E)$ and $p(H \cap E) = 0$, or $p(H) = p(H \cap E)$ and $p(H \backslash E) = 0$.

The Structure Theorem ([2], Theorem 1.2) Let $(\Omega, \mathcal{F}, p)$ be an optimal measure space. Then there exists a collection $\mathcal{H}(p) = \{H_n : n \in J\}$ of disjoint indecomposable p-atoms, where J is some countable (i.e. finite or countably infinite) index set, such that for every measurable set $B \in \mathcal{F}$ with $p(B) > 0$ we have

$$p(B) = \max\{p(B \cap H_n) : n \in J\}. \tag{3.3}$$

Moreover, if J is countably infinite, then the only limit point of the set $\{p(H_n) : n \in J\}$ is 0.

The proof was derived from the following lemmas, which we shall recollect without their proofs.

Lemma 3.8 ([2], Lemma 1.3)**.** *Let $E \in \mathcal{F}$ be with $p(E) > 0$, and $B_k \in \mathcal{F}$, $B_k \subset E$ $(k \in J)$, where J is any countable index set. Then*

$$p\left(\bigcup_{k \in J} B_k\right) < p(E) \quad \textit{if and only if} \quad p(B_k) < p(E) \quad \textit{for all } k \in J. \tag{3.4}$$

Lemma 3.9 ([2], Lemma 1.4). *For every sequence $(B_n) \subset \mathcal{F}$ and every optimal measure p we have*

$$p\left(\bigcup_{n=1}^{\infty} B_n\right) = \max\left\{p\left(B_n\right) : n \in \mathbb{N}\right\}.$$

Lemma 3.10 ([2], Lemma 1.5). *Every measurable set $E \in \mathcal{F}$ with $p(E) > 0$ contains an atom $H \subset E$ such that $p(E) = p(H)$.*

Lemma 3.11 ([2], Lemma 1.6). *Let $\mathcal{H} = \{H_n : n \in J\}$ be as above. Then for every measurable set $B \in \mathcal{F}$ with $p(B) > 0$, the identity(6.4)*

$$p\left(B \backslash \bigcup_{n \in J}\left(B \cap H_n\right)\right) = 0 \tag{3.5}$$

holds.

We are now in the position to prove the Structure Theorem.

Proof of the Structure Theorem. Let $\mathcal{G}$ be a set of pairwise disjoint atoms. It is clear that the collection of all such $\mathcal{G}$, denoted by Γ, is partially ordered by the set inclusion and every subset of Γ has an upper bound. Then, the Zorn lemma entails that Γ contains a maximal element, which we shall denote by $\mathcal{G}^*$. As we have done above, one can easily verify that the set

$$\left\{K \in \mathcal{G}^* : p(K) > n^{-1}\right\}$$

is finite. Hence $\mathcal{G}^* = \{K_j : j \in \nabla\}$, where ∇ is a countable index set. It is obvious that $p(K_j) \to 0$ as $j \to \infty$, whenever ∇ is a countably infinite set. Consequently, it ensues, via Lemma 3.6, that each atom $K_j \in \mathcal{G}^*$ can be expressed as the union of finitely many disjoint indecomposable subatoms of the same optimal measure as K_j. Finally, let us list these indecomposable atoms occurring in the decompositions of the elements of $\mathcal{G}^*$ as follows: $\mathcal{H} = \{H_n : n \in J\}$, where J is a countable index set. Now, via Lemma 3.9, the identity (3.5) and Axiom 2, one can easily observe that (3.3) holds for every set $B \in \mathcal{F}$, with $p(B) > 0$. It is also obvious that 0 is the only limit point of the set $\{p(H_n) : n \in J\}$ whenever J is a countably infinite set. This ends the proof of the theorem. $\square$

To end the section, we need to point out that an elementary proof was given to the Structure Theorem in [17].

4. Lebesgue's type integral in lattice environments

In comparison with the mathematical expectation or Lebesgue integral, we define a non-linear functional (first for non-negative measurable simple functions and secondly for non-negative measurable functions) which provide us with many well-known results in measure theory. Their proofs are carried out similarly.

4.1. Optimal average

In the whole section we shall be dealing with an arbitrary but fixed optimal measure space $(\Omega, \mathcal{F}, p)$.
Let

$$s = \sum_{i=1}^{n} b_i \chi\left(B_i\right)$$

be an arbitrary non-negative measurable simple function, where
$\{B_i : i = 1, \ldots, n\} \subset \mathcal{F}$ is a partition of Ω. Then the so-called optimal average of s is defined by

Definition 4.1 ([1], Definition 1.1)**.** The quantity

$$\bigg\backslash_{\Omega} s\,dp := \bigvee_{i=1}^{n} b_i p\left(B_i\right)$$

will be called optimal average of s, and for $E \in \mathcal{F}$

$$\bigg\backslash_{B} s\chi\left(E\right) dp := \bigvee_{i=1}^{n} b_i p\left(E \cap B_i\right)$$

as the optimal average of s on E, where $\chi\left(E\right)$ is the indicator function of the measurable set E. These quantities will be sometimes denoted respectively by $I\left(s\right)$ and $I_E\left(s\right)$.

As it is well-known, a measurable simple function can have many decompositions. The question thus arises (just as in the case of Lebesgue integral) whether or not the optimal average of a simple function depends on its decompositions. The following result gives a satisfactory answer to this question, making the definition of optimal average as deep as the Lebesgue integral is.

Theorem 4.2 ([1], Theorem 1.0)**.** *Let*

$$\sum_{i=1}^{n} b_i \chi\left(B_i\right) \quad \text{and} \quad \sum_{k=1}^{m} c_k \chi\left(C_k\right)$$

be two decompositions of a measurable simple function $s \geq 0$*, where* $\{B_i : i = 1, \ldots, n\}$ *and* $\{C_k : k = 1, \ldots, m\} \subset \mathcal{F}$ *are partitions of* Ω*. Then*

$$\max\left\{b_i p\left(B_i\right) : i = 1, \ldots, n\right\} = \max\left\{c_k p\left(C_k\right) : k = 1, \ldots, m\right\}.$$

Proof. Since $B_i = \bigcup_{k=1}^{m}\left(B_i \cap C_k\right)$ and $C_k = \bigcup_{i=1}^{n}\left(B_i \cap C_k\right)$, Axiom 2 of optimal measure implies that

$$p\left(B_i\right) = \max\left\{p\left(B_i \cap C_k\right) : k = 1, \ldots, m\right\} \text{ and } p\left(C_k\right) = \max\left\{p\left(B_i \cap C_k\right) : i = 1, \ldots, n\right\}$$

Thus

$$\max\left\{c_k p\left(C_k\right) : k = 1, \ldots, m\right\} = \max\left\{\max\left\{c_k p\left(B_i \cap C_k\right) : i = 1, \ldots, n\right\} : k = 1, \ldots, m\right\}$$

and

$$\max\left\{b_i p\left(B_i\right) : i = 1, \ldots, n\right\} = \max\left\{\max\left\{b_i p\left(B_i \cap C_k\right) : k = 1, \ldots, m\right\} : i = 1, \ldots, n\right\}.$$

Clearly, if $B_i \cap C_k \neq \varnothing$, then $b_i = c_k$, or if $B_i \cap C_k = \varnothing$, then $p\left(B_i \cap C_k\right) = 0$. Thus, by the associativity and the commutativity, we obtain

$$\max\left\{b_i p\left(B_i\right) : i = 1, \ldots, n\right\} = \max\left\{c_k p\left(C_k\right) : k = 1, \ldots, m\right\}.$$

This completes the proof. □

Proposition 4.3 ([1], Proposition 2.0)**.** *Let* $f \geq 0$ *be any bounded measurable function. Then*

$$\sup_{s \leq f} \bigg\backslash_{\Omega} s\,dp = \inf_{\overline{s} \geq f} \bigg\backslash_{\Omega} \overline{s} dp,$$

where s *and* $\overline{s}$ *denote non-negative measurable simple functions.*

Proof. Let f be a measurable function such that $0 \leq f \leq b$ on Ω, where b is some constant. Let $E_k = \left(kbn^{-1} \leq f \leq (k+1)\, bn^{-1}\right)$, $k = 1, \ldots, n$. Clearly, $\{E_k : k = 1, \ldots, n\} \subset \mathcal{F}$ is a partition of Ω. Define the following measurable simple functions:

$$s_n = bn^{-1} \sum_{k=0}^{n} k\chi\left(E_k\right), \quad \overline{s}_n = bn^{-1} \sum_{k=0}^{n} (k+1)\, \chi\left(E_k\right).$$

Obviously, $s_n \leq f \leq \overline{s}_n$. Then we can easily observe that

$$\sup_{s \leq f} \bigvee_{\Omega} s\, dp \geq \bigvee_{\Omega} s_n dp = n^{-1} b \max\left\{kp\left(E_k\right) : k = 0, \ldots, n\right\}$$

and

$$\inf_{\overline{s} \geq f} \bigvee_{\Omega} \overline{s} dp \leq \bigvee_{\Omega} \overline{s}_n dp = n^{-1} b \max\left\{(k+1)\, p\left(E_k\right) : k = 0, \ldots, n\right\}.$$

Hence

$$0 \leq \inf_{\overline{s} \geq f} \bigvee_{\Omega} \overline{s} dp - \sup_{s \leq f} \bigvee_{\Omega} s\, dp \leq bn^{-1}.$$

The result follows by letting $n \to \infty$ in this last inequality. $\square$

Definition 4.4 ([1], Definition 2.1)**.** The optimal average of a measurable function f is defined by

$$\bigvee_{\Omega} |f|\, dp = \sup \bigvee_{\Omega} s\, dp, \tag{4.1}$$

where the supremum is taken over all measurable simple functions $s \geq 0$ for which $s \leq |f|$. The optimal average of f on any given measurable set E is defined by $\bigvee_{E} |f|\, dp = \bigvee_{\Omega} \chi\left(E\right) |f|\, dp$.

For convenience reasons at times we shall write $A\, |f|$ for the optimal average of the measurable function f.

Proposition 4.5 ([1], Proposition 2.1)**.** *Let $f \geq 0$ and $g \geq 0$ be any measurable simple functions, $b \in \mathbb{R}_+$ and $B \in \mathcal{F}$ be arbitrary. Then*

1. $A\left(b\mathbf{1}\right) = b$.
2. $A\left(\chi\left(B\right)\right) = p\left(B\right)$.
3. $A\left(bf\right) = bAf$.
4. $A\left(f\chi\left(B\right)\right) = 0$ *if* $p\left(B\right) = 0$.
5. $Af \leq Ag$ *if* $f \leq g$.
6. $A\left(f+g\right) \leq Af + Ag$.
7. $A\left(f\chi\left(B\right)\right) = Af$ *if* $p\left(\overline{B}\right) = 0$.
8. $A\left(f \vee g\right) = Af \vee Ag$.

The almost everywhere notion in measure theory also makes sense in optimal measure theory.

Definition 4.6 ([1], Definition 2.2)**.** Let p be an optimal measure. A property is said to hold almost everywhere if the set of elements where it fails to hold is a set of optimal measure zero.

As an immediate consequent of the atomic structural behavior of optimal measures we can formulate the following.

Remark 4.7 ([2], Remark 2.1). If a function $f : \Omega \to \mathbb{R}$ is measurable, then it is constant almost everywhere on every indecomposable atom.

Proposition 4.8 ([2], Proposition 2.6)**.** *Let $p \in \mathcal{P}$ and f be any measurable function. Then*

$$\int_{\Omega} |f|\, dp = \sup \left\{ \int_{H_n} |f|\, dp : n \in J \right\},$$

where $\mathcal{H}(p) = \{H_n : n \in J\}$ is a p-generating countable system.
Moreover if $A|f| < \infty$, then $\int_{\Omega} |f|\, dp = \sup\{c_n \cdot p(H_n) : n \in J\}$, where $c_n = f(\omega)$ for almost all $\omega \in H_n$, $n \in J$.

Proposition 4.9 (**Optimal Markov Inequality** ([1], Proposition 2.2))**.** *Let $f \geq 0$ be any measurable function. Then for every number $x > 0$ we have*

$$xp(f \geq x) \leq Af.$$

Proposition 4.10 ([1], Proposition 3.4)**.** *Let $f \geq 0$ be any bounded measurable function. Then for every $\varepsilon > 0$ there is some $\delta > 0$ such that $\int_B f dp < \varepsilon$ whenever $B \in \mathcal{F}$, $p(B) < \delta$.*

Proof. By assumption $0 \leq f \leq b$ for some number $b > 0$. Then Proposition 4.5 entails, for the choice $0 < \delta < \varepsilon b^{-1}$, that $\int_B f dp \leq bp(B) < \delta b < \varepsilon$. $\square$

In the example below we shall show that Proposition 4.10 does not hold for unbounded measurable functions.

Example 4.11 ([1], Example 3.2)**.** Consider the measurable space $\left(\mathbb{N}, 2^{\mathbb{N}}\right)$. Define the set function $p : 2^{\mathbb{N}} \to [0, 1]$ by $p(B) = \dfrac{1}{\min B}$. It is known from Example 3.3 that p is an optimal measure. Consider the following measurable function $f(\omega) = \omega$, $\omega \in \mathbb{N}$. Clearly, $Af \geq 1$. Let $s = \sum_{j=1}^{n} b_j \chi(B_j)$ be a measurable simple function with $0 \leq s \leq f$. Denote $\omega_j = \min B_j$ for $j = 1, \ldots, n$. Then $p(B_j) = \dfrac{1}{\omega_j}$ and $b_j \leq \omega_j$ for all $j = 1, \ldots, n$. Thus $\int_{\Omega} s\, dp \leq 1$, and hence $\int_{\Omega} f \leq 1$. Consequently, $\int_{\Omega} f = 1$. On the one hand, there is no $\delta > 0$ such that $p(E) < \delta$ implies that $\int_E f dp < 1$. Indeed, $\int_{\{\omega\}} f dp = 1$ for every $\omega \in \mathbb{N}$, and $p(\{\omega\}) \to 0$ as $\omega \to \infty$.

4.2. The corresponding Radon-Nikodym Theorem in lattice environments

Definition 4.12 ([2], Definition 2.1)**.** By a quasi-optimal measure we a set function $q : \mathcal{F} \to \mathbb{R}_+$ satisfying Axioms 1-3, with the hypothesis $q(\Omega) = 1$ in Axiom 1 being replaced by the hypothesis $0 < q(\Omega) < \infty$.

Proposition 4.13 ([2], Proposition 2.1)**.** *If $f \geq 0$ is a bounded measurable function, then the set function $q_f : \mathcal{F} \to \mathbb{R}_+$,*

$$q_f(E) = \int_E f dp,$$

is a quasi-optimal measure.

Definition 4.14 ([2], Definition 2.2)**.** We shall say that a quasi-optimal measure q is absolutely continuous relative to p (abbreviated $q \ll p$) if $q(B) = 0$ whenever $p(B) = 0$, $B \in \mathcal{F}$.

Proposition 4.15 ([2], Proposition 2.2)**.** *Let q be a quasi-optimal measure. Then $q \ll p$ if and only if for every $\varepsilon > 0$ there is some $\delta > 0$ such that $q(B) < \varepsilon$ whenever $p(B) < \delta$, $B \in \mathcal{F}$.*

The proof of Proposition 4.15 is similarly done as in the case of measure theory.

Lemma 4.16 ([2], Lemma 2.3)**.** *Let q be a quasi-optimal measure and $\mathcal{H}(p)$ be a p-generating system. If $q \ll p$, then*

$$\mathcal{H}(q) = \{H \in \mathcal{H}(p) : q(H) > 0\}$$

is a q-generating system.

Remark 4.17 ([3], Remark 2.1). Let $p, q \in \mathcal{P}, \mathcal{H}(p) = \{H_n : n \in J\}$ be a p-generating countable system and f any measurable function. Suppose that $q \ll p$ and $q(H) \leq p(H)$ for every $H \in \mathcal{H}(p)$. Then $\bigvee_{\Omega} |f|\, dq \leq \bigvee_{\Omega} |f|\, dp$, provided that $\bigvee_{\Omega} |f|\, dp < \infty$.

This remark is immediate from Lemma 4.16 and Proposition 4.8.

Theorem 4.18 (**Optimal Radon-Nikodym** ([2], Theorem 2.4))**.** *Let q be a quasi-optimal measure such that $q \ll p$. Then there exists a unique measurable function $f \geq 0$ such that for every measurable set $B \in \mathcal{F}$,*

$$q(B) = \bigvee_{B} f dp.$$

This measurable function, explicitly given in (4.2), will be called Optimal Radon-Nikodym derivative and denoted by $\dfrac{dq}{dp}$.

Proof. Let $\mathcal{H}(p) = \{H_n : n \in J\}$ be a p-generating countable system. Define the following non-negative measurable function

$$f = \max\left\{\frac{q(H_n)}{p(H_n)} \cdot \chi(H_n) : n \in J\right\}. \tag{4.2}$$

Fix an index $n \in J$ and let $B \in \mathcal{F}$, $p(B) > 0$. Then Remark 3.7 and the absolute continuity property imply that

$$\frac{q(H_n)}{p(H_n)} p(B \cap H_n) = \begin{cases} 0 & \text{if } p(B \cap H_n) = 0 \\ q(B \cap H_n), & \text{otherwise.} \end{cases}$$

Hence, by a simple calculation, one can observe that

$$\bigvee_{B} f dp = \max\{q(B \cap H_n) : n \in J\}.$$

Consequently, Lemma 4.16 yields

$$\bigvee_{B} f dp = \begin{cases} \max\{q(B \cap H_n) : q(H_n) > 0,\, n \in J\} & \text{if } q(B) > 0 \\ 0, & \text{otherwise,} \end{cases}$$

and thus (4.2) holds.

Let us show that the decomposition (4.2) is unique. In fact, there exist two measurable functions $f \geq 0$ and $g \geq 0$ satisfying (4.2) . Then for each set $B \in \mathcal{F}$, we have:

$$\bigvee_{B} f dp = \bigvee_{B} g dp.$$

Put $E_1 = (f < g)$ and $E_2 = (g < f)$. Obviously, E_1 and $E_2 \in \mathcal{F}$. If the inequality $p(E_1) > 0$ should hold, it would follow that

$$\mathop{\Big\backslash}\limits_{E_1} gdp = \mathop{\Big\backslash}\limits_{E_1} fdp < \mathop{\Big\backslash}\limits_{E_1} gdp,$$

which is impossible. This contradiction yields $p(E_1) = 0$. We can similarly show that $p(E_2) = 0$. These last two equalities imply that $p(f \neq g) = 0$, i.e. the decomposition (4.2) is unique. The theorem is thus proved. □

5. Counterparts in lattice environments of well-known convergence theorems

5.1. Some convergence with respect to individual optimal measures

In this subsection we shall explore in lattice environments the counterparts of the monotone convergence theorem, the Fatou's lemma and the dominated convergence theorem well-known in Measure Theory. The results are related to an arbitrarily fixed optimal measure space $(\Omega, \mathcal{F}, p)$, unless otherwise stated.

Theorem 5.1 (**Optimal monotone convergence**, ([1], Theorem 3.1)**.**

1. *If (f_n) is an increasing sequence of non-negative measurable functions, then*

$$\lim_{n\to\infty} \mathop{\Big\backslash}\limits_{\Omega} f_n dp = \mathop{\Big\backslash}\limits_{\Omega} \left(\lim_{n\to\infty} f_n\right) dp.$$

2. *If (g_n) is a decreasing sequence of non-negative measurable functions with $g_1 \le b$ for some $b \in (0, \infty)$, then*

$$\lim_{n\to\infty} \mathop{\Big\backslash}\limits_{\Omega} g_n dp = \mathop{\Big\backslash}\limits_{\Omega} \left(\lim_{n\to\infty} g_n\right) dp.$$

The following example shows why the optimal monotone convergence theorem fails to hold for all decreasing sequences of measurable functions.

Example 5.2 ([1], Example 3.1)**.** Let $\left(\mathbb{N}, 2^{\mathbb{N}}, p\right)$ be the optimal measure space we considered in Example 4.11. Define the following measurable function

$$g_n(\omega) = \begin{cases} 0 & \text{if } \omega < n \\ \omega & \text{if } \omega \ge n. \end{cases}$$

Obviously, sequence $(g_n)_{n\in\mathbb{N}}$ tends decreasingly to zero as $n \to \infty$. It will be enough to show that $\mathop{\Big\backslash}\limits_{\mathbb{N}} g_n\, dp = 1$ for all $n \in \mathbb{N}$. In fact, it is clear by definition that $(g_n < n) = \{1, \ldots, n-1\}$ and $(g_n \ge n) = \{n, n+1, \ldots\}$, and so $\mathbb{N} = (g_n < n) \cup (g_n \ge n)$ for every fixed natural number $n \in \mathbb{N}$. We also know by definition that g_n assumes the value 0 on $\{1, \ldots, n-1\}$ and the value n on $\{n, n+1, \ldots\}$, for every fixed natural number $n \in \mathbb{N}$. Hence, by the considered optimal measure we trivially have

$$\mathop{\Big\backslash}\limits_{\mathbb{N}} g_n\, dp = \mathop{\Big\backslash}\limits_{\{n, n+1, \ldots\}} g_n\, dp = np(\{n, n+1, \ldots\}) = \frac{n}{\min(\{n, n+1, \ldots\})} = 1.$$

Lemma 5.3 (**Optimal Fatou** ([1], Lemma 3.2))**.** *If $(f_n)_{n\in\mathbb{N}}$ and $(h_n)_{n\in\mathbb{N}}$ are sequences of non-negative measurable functions, then for every optimal measure p, we have that:*

1. $\mathop{\Big\backslash}\limits_{\Omega} \left(\liminf\limits_{n\to\infty} f_n\right) dp \le \liminf\limits_{n\to\infty} \mathop{\Big\backslash}\limits_{\Omega} f_n dp$;

2. $\limsup\limits_{n\to\infty} \mathop{\Big\backslash}\limits_{\Omega} h_n dp \le \mathop{\Big\backslash}\limits_{\Omega} \left(\limsup\limits_{n\to\infty} h_n\right) dp$ *, whenever $(h_n)_{n\in\mathbb{N}}$ is a uniformly bounded sequence.*

Theorem 5.4 (**Optimal Dominated Convergenc** ([1], Theorem 3.3))**.** *Let $(f_n)_{n\in\mathbb{N}}$ be a uniformly bounded sequence of non-negative measurable functions. Then $A\left(\lim\limits_{n\to\infty} f_n\right) = Af$, where $\lim\limits_{n\to\infty} f_n = f$ almost everywhere.*

6. Banach lattice induced by optimal measures

Throughout this section we shall deal with an arbitrary but fixed optimal measure space $(\Omega, \mathcal{F}, p)$, i.e. $(\Omega, \mathcal{F})$ is a measurable space and p an optimal measure.

6.1. The counterpart of the L^p-spaces ($p \in [1, \infty]$) in lattice environments

Definition 6.1. Let $f : \Omega \to \mathbb{R} \cup \{-\infty, \infty\}$ be any measurable function. We shall say that f belongs to:

1. $\mathcal{A}^\infty$ if $p\left(|f| \le b\right) = 1$ for some constant $b \in (0, \infty)$.

2. $\mathcal{A}^\alpha$ if $\bigvee_{\Omega} |f|^\alpha \, dp < \infty$, $\alpha \in [1, \infty)$.

For any $\alpha \in [1, \infty]$, the space $\mathcal{A}_\alpha$ endowed with the norm $\|\cdot\|_\alpha$, defined by

$$\|f\|_{\mathcal{A}^\alpha} := \begin{cases} \inf\{b \in (0, \infty) : p\left(|f| \le b\right) = 1\}, & \text{if } f \in \mathcal{A}_\infty,\ \alpha = \infty \\ \sqrt[\alpha]{\bigvee_{\Omega} |f|^\alpha \, dp}, & \text{if } f \in \mathcal{A}_\alpha,\ \alpha \in [1, \infty) \end{cases}$$

As in the case of L^p-spaces ($p \in [1, \infty]$) in Measure Theory, it can be similarly seen that $\|\cdot\|_\alpha$ is a semi-norm for every $\alpha \in [1, \infty]$.

Lemma 6.2 ([1], Lemma 4.1)**.**

1. *$A\,|fg| \le \|f\|_{\mathcal{A}^\alpha} \|g\|_{\mathcal{A}^\infty}$ whenever $f \in \mathcal{A}^1$ and $g \in \mathcal{A}^\infty$.*

2. *Let α and $\beta \in (1, \infty)$ be such that $\alpha^{-1} + \beta^{-1}$. Then $A\,|fg| \le \|f\|_{\mathcal{A}^\alpha} \|g\|_{\mathcal{A}^\beta}$ (called the optimal Hölder inequality), whenever $f \in \mathcal{A}^\alpha$ and $g \in \mathcal{A}^\beta$.*

3. *$\|f + g\|_{\mathcal{A}^\alpha} \le \|f\|_{\mathcal{A}^\alpha} + \|g\|_{\mathcal{A}^\alpha}$ (called the optimal Minkowski inequality) whenever $f \in \mathcal{A}^\alpha$ and $g \in \mathcal{A}^\alpha$, with $\alpha \in [1, \infty]$.*

Theorem 6.3 ([1], Theorem 4.2)**.** *For each number $\alpha \in [1, \infty]$, $\mathcal{A}^\alpha$ is a Banach space (i.e. every Cauchy sequence in $\mathcal{A}^\alpha$ converges to a measurable function in $\mathcal{A}^\alpha$-norm).*

6.2. Orlicz-space and its dual in lattice environments

Let Φ be a *convex Young function*, i.e.

$$\Phi(x) = \int_0^x \varphi(t)\, dt,\ x \in \mathbb{R}_+,$$

where $\varphi : (0, \infty) \to (0, \infty)$ is a right-continuous and increasing function such that $\varphi(0) \ge 0$ and $\varphi(\infty) = \infty$.

The conjugate Young functions are defined as follows:

For $t \in (0, \infty)$ put $\psi(t) := \sup\{x > 0 : \varphi(x) < t\}$ and let $\psi(0) = 0$. It can be easily checked that ψ satisfies all the conditions imposed on φ and we trivially have $\psi(\varphi(x)) \le x \le \psi(\varphi(x) + 0)$, whenever $x \in (0, \infty)$.

The convex Young function

$$\Psi(x) := \int_0^x \psi(t)\, dt,\ \ x \in [0, \infty),$$

is said to be *conjugate* to Φ and the pair (Φ, Ψ) is referred to as *mutually conjugate convex Young functions.*

Every pair (Φ, Ψ) of mutually conjugate convex Young functions satisfies the fundamental Young inequality

$$xy \le \Phi(x) + \Psi(y) \tag{6.1}$$

for all $x,\ y \in [0,\ \infty)$, and the Young equality

$$xy = \Phi(x) + \Psi(y) \tag{6.2}$$

if and only if $y \in [\varphi(x),\ \varphi(x+0)]$ or $x \in [\psi(y),\ \psi(y+0)]$. (For more about convex Young functions, see [26].)

We extend some basic results about the Orlicz L^{Φ} space in Measure Theory to the framework of Optimal Measure Theory, by generalizing the space $\mathcal{A}^{\alpha}$ to the space $\mathcal{A}^{\Phi}$, where Φ is a convex Young function. In the image of the dual space of the Orlicz L^{Φ} space some set of non-linear functionals $F : \mathcal{A}^{\Phi} \to [0,\ \infty]$, (called the laud space of $\mathcal{A}^{\Phi}$), is studied.

Definition 6.4 ([7], Definition 2.1)**.** We say that a measurable function f belongs to $\mathcal{A}^{\Phi}$ if there is a constant $c \in (0,\ \infty)$ such that

$$\int_{\Omega} \Phi\left(\frac{|f|}{c}\right) dp \le 1. \tag{6.3}$$

In the image of the Luxemburg norm define on $\mathcal{A}^{\Phi}$ the operator $\|\cdot\|_{\mathcal{A}^{\Phi}}$ by

$$\|f\|_{\mathcal{A}^{\Phi}} = \inf\left\{c \in (0,\ \infty) : \int_{\Omega} \Phi\left(\frac{|f|}{c}\right) dp \le 1\right\}, \tag{6.4}$$

and $\|f\|_{\mathcal{A}^{\Phi}} = \infty$ if there is no $c \in (0,\ \infty)$ such that (6.3) holds.

Note that if $\Phi(t) = \dfrac{t^{1+\alpha}}{1+\alpha}$, $t \in [0,\ \infty)$ and $\alpha \in (0,\ \infty)$, then $\mathcal{A}^{\Phi} = \mathcal{A}^{1+\alpha}$.

Theorem 6.5 ([7], Theorem 2.2)**.** *Let $\Phi : [0,\ \infty) \to [0,\ \infty)$ be any function and f a non-negative finite measurable function. Then the inequality*

$$\Phi\left(\int_{\Omega} f\,dp\right) \le \int_{\Omega} \Phi(f)\,dp$$

holds, and is referred to as the Optimal Jensen inequality, provided that Φ is a convex Young function. Furthermore, the inequality is reversed if Φ is a concave Young function.

We prepare the ground for the proof of Theorem 6.5.

Let $J \subset \mathbb{N}$ be an index set. Then the weighted supremum of a sequence $(b_n)_{n\in J} \subset [0,\ \infty)$ is defined by $\sup\limits_{n\in J} b_n\alpha_n$, where $(\alpha_n)_{n\in J} \subset [0,\ 1]$ is a prescribed sequence with 0 as its unique limit point if the index set is infinite (in symbol $|J| = \infty$).

Remark 6.6 ([7], Remark 3.1)*.* For all $d \in \mathbb{R}$, $c \in (0,\ \infty)$ and $(b_n)_{n\in J} \subset [0,\ \infty)$, where J is an index set, then

$$\sup_{n\in J}(d + cb_n) = d + c\sup_{n\in J} b_n.$$

Remark 6.6 is obvious.

Lemma 6.7 ([7], Lemma 3.2)**.** *Let $J \subset \mathbb{N}$ be an index set and $\Phi : [0,\ \infty) \to [0,\ \infty)$ be any function. Consider two sequences $(b_n)_{n\in J} \subset [0,\ \infty)$ and $(\alpha_n)_{n\in J} \subset [0,\ 1]$ possessing 0 as its unique limit point if $|J| = \infty$. Then*

$$\Phi\left(\sup_{n\in J} b_n\alpha_n\right) \le \sup_{n\in J} \Phi(b_n)\,\alpha_n$$

provided that Φ is a convex Young function. Furthermore, the inequality is reversed if Φ is a concave Young function.

The Proof of Theorem 6.5. We note that the proof follows from the conjunction of both Proposition 2.1 in [3] and the above Lemma 6.7. □

Definition 6.8 ([7], Definition 2.3)**.** Let $\mathcal{A}_+^\Phi := \left\{f \in \mathcal{A}^\Phi : f \geq 0\right\}$. We say that a functional $F : \mathcal{A}_+^\Phi \to [0, \infty]$ belongs to $\widetilde{\mathcal{A}^\Phi}$ if the following conditions hold true simultaneously:

1. For all $f,\, h \in \mathcal{A}_+^\Phi$, and $\alpha,\, \beta \in [0, \infty)$ we have

$$F(\alpha f \vee \beta h) = \alpha F(f) \vee \beta F(h).$$

2. F is continuous from below, i.e. if $(f_n)_{n\in\mathbb{N}} \subset \mathcal{A}_+^\Phi$ is an increasing sequence, then

$$\lim_{n\to\infty} F(f_n) = F\left(\lim_{n\to\infty} f_n\right).$$

3. There is some constant $C > 0$ for which

$$F(f) \leq C \|f\|_{\mathcal{A}^\Phi}, \text{ whenever } f \in \mathcal{A}_+^\Phi.$$

We extend Definition 6.8 to the entire $\mathcal{A}^\Phi$ space as follows.

Definition 6.9 ([7], Definition 2.4)**.** A functional $F \circ |\cdot| : \mathcal{A}^\Phi \to [0, \infty]$ is said to belong to $\widetilde{\mathcal{A}^\Phi}$ if the following conditions hold true simultaneously:

1. For all $f,\, h \in \mathcal{A}^\Phi$, and $\alpha,\, \beta \in [0, \infty)$ we have

$$F(\alpha |f| \vee \beta |h|) = \alpha F(|f|) \vee \beta F(|h|).$$

2. F is non-negatively continuous from below, i.e. if $(f_n)_{n\in\mathbb{N}} \subset \mathcal{A}^\Phi$ is a non-negative increasing sequence, then

$$\lim_{n\to\infty} F(f_n) = F\left(\lim_{n\to\infty} f_n\right).$$

3. There is some constant $C > 0$ for which

$$F(|f|) \leq C \|f\|_{\mathcal{A}^\Phi}, \text{ whenever } f \in \mathcal{A}^\Phi.$$

The set $\widetilde{\mathcal{A}^\Phi}$ will thus be referred to as the "*laud*" *space* of $\mathcal{A}^\Phi$, in contrast with the "dual" space of L^Φ in Measure Theory.

The counterpart of Proposition IX-2-2 in the appendix of [30] can be stated as follows.

Theorem 6.10 ([7], Theorem 2.5)**.** *The following assertions hold.*

1. *The mapping* $\|\cdot\|_{\mathcal{A}^\Phi} : \mathcal{A}^\Phi \to [0, \infty)$ *defined by* (6.4) *is a norm.*
2. $\mathcal{A}^\Phi \subset \mathcal{A}^1$*, i.e. there exist some constant* $\delta > 0$ *such that*

$$\delta \|f\|_{\mathcal{A}^1} \leq \|f\|_{\mathcal{A}^\Phi},$$

whenever $f \in \mathcal{A}^\Phi$.
3. $\mathcal{A}^\Phi$ *is a Banach space, i.e. every Cauchy sequence in* $\mathcal{A}^\Phi$ *converges to a measurable function in* $\mathcal{A}^\Phi$*-norm.*
4. *If* $f \in \mathcal{A}^\Phi$ *and* $h \in \mathcal{A}^\Psi$*, then*

$$\|fh\|_{\mathcal{A}^1} \leq 2 \|f\|_{\mathcal{A}^\Phi} \cdot \|h\|_{\mathcal{A}^\Psi},$$

which shall be referred to as the Optimal Hölder Inequality.

5. *Given any $h \in \mathcal{A}^{\Psi}$, the mapping $F_h \circ |\cdot| : \mathcal{A}^{\Phi} \to [0, \infty)$ defined by*

$$F_h(|f|) = \bigvee_{\Omega} |fh|\, dp,$$

belongs to the laud space of $\mathcal{A}^{\Phi}$. Moreover, letting $\mathfrak{M}$ stand for the set of all measurable functions defined on $(\Omega, \mathcal{F})$, the quantity

$$\|h\|^{*}_{\mathcal{A}^{\Phi}} := \sup_{f \in \mathcal{A}^{\Phi} \setminus \{0\}} \frac{F_h(|f|)}{\|f\|_{\mathcal{A}^{\Phi}}} = \sup \left\{ F_h(|f|) : f \in \mathfrak{M}, \quad \bigvee_{\Omega} \Phi(|f|)\, dp \le 1 \right\} \tag{6.5}$$

defines a norm on the space $\mathcal{A}^{\Psi}$ which is equivalent to the norm $\|\cdot\|_{\mathcal{A}^{\Psi}}$, more precisely

$$\lambda \|h\|_{\mathcal{A}^{\Psi}} \le \|h\|^{*}_{\mathcal{A}^{\Phi}} \le 2\|h\|_{\mathcal{A}^{\Psi}},$$

for some constant $\lambda \in (0, 2]$ and all $h \in \mathcal{A}^{\Psi}$.

6. *If $F \circ |\cdot| : \mathcal{A}^{\Phi} \to [0, \infty)$ is a mapping belonging to $\widetilde{\mathcal{A}^{\Phi}}$, then there is an $h \in \mathcal{A}^{\Psi}$ with $\|h\|_{\mathcal{A}^{\Psi}} \le C$ (the constant C being as in Definition 6.9) such that for all $f \in \mathcal{A}^{\Phi}$,*

$$F(|f|) = \bigvee_{\Omega} |fh|\, dp.$$

Before tackling the proof of Theorem 6.10 (which goes down the line of the proof given in [30] for Proposition IX-2-2), some essential results need to be mentioned with the proofs.

Remark 6.11 ([7], Remark 1.1). Let be given any optimal measure p with $\mathcal{H}(p) = \{H_n : n \in J\}$ its generating system and a measurable set $A \in \mathcal{F}$. Then $p(A) = 0$ if and only if $p(A \cap H) = 0$ for every $H \in \mathcal{H}(p)$.

Lemma 6.12 ([7], Lemma 3.5)**.** *Let y be a bounded measurable function and consider the quasi-optimal measure $q_y : \mathcal{F} \to [0, \infty)$,*

$$q_y(A) = \bigvee_{A} |y|\, dp.$$

Then $dq_y = |y|\, dp$ p-a.e. Moreover,

$$|y| = \max\left\{ \frac{q_y(H)}{p(H)} \cdot \chi_H : H \in \mathcal{H}(p),\, q_y(H) > 0 \right\}$$

on $\bigcup \mathcal{H}(p)$.

Remark 6.13 ([7], Remark 3.6). Given any convex Young function Φ, for every $f \in \mathcal{A}^{\Phi}$ we have

$$\|f\|_{\mathcal{A}^{\Phi}} \le \max\left\{ 1\,;\ \bigvee_{\Omega} \Phi(|f|)\, dp \right\}.$$

Remark 6.14 ([7], Remark 3.7). For every measurable function f we have that $\|f\|_{\mathcal{A}^{\Phi}} \le 1$ if and only if

$$\bigvee_{\Omega} \Phi(|f|)\, dp \le 1.$$

Remark 6.15 ([7], Remark 3.8). For any convex Young function Ψ and any measurable simple function of the form $h = b\chi_A$ where $A \in \mathcal{F}$ with $p(A) > 0$ we have

$$\|h\|_{\mathcal{A}^{\Psi}} = \frac{|b|}{\Psi^{-1}\left(\frac{1}{p(A)}\right)}.$$

Remarks 6.14 and 6.15 can be easily checked, so we shall omit their proofs.

The Proof of Theorem 6.10.

PART 1. Let f, h be any measurable functions. It is trivial that $\|f\|_{\mathcal{A}^\Phi} \geq 0$. We want to prove that if $\|f\|_{\mathcal{A}^\Phi} = 0$, then $p(|f| \neq 0) = 0$. In fact, suppose that $\|f\|_{\mathcal{A}^\Phi} = 0$ but $p(0 < |f| \leq \infty) = p(|f| \neq 0) > 0$. Then by Remark 6.11 a non-empty subset J_0 of the index set J exists such that $p(H_n \cap (0 < |f| \leq \infty)) > 0$, whenever $n \in J_0$ and $p(H_n \cap (0 < |f| \leq \infty)) = 0$ otherwise, where J is the index set of the generating system $\mathcal{H}(p) = \{H_n : n \in J\}$. Note that $\|f\|_{\mathcal{A}^\Phi} = \inf S$, where

$$S = \left\{ \delta > 0 : \bigvee_{\Omega} \Phi\left(\frac{|f|}{\delta}\right) dp \leq 1 \right\}.$$

From the assumption and the definition of the infimum there is a sequence $(\delta_k)_{k\in\mathbb{N}} \subset S$ such that $0 < \delta_k < \dfrac{1}{k}$ for all $k \in \mathbb{N}$. By applying the Optimal Jensen Inequality we can observe that

$$1 \geq \bigvee_{\Omega} \Phi\left(\frac{|f|}{\delta_k}\right) dp \geq \Phi\left(\bigvee_{\Omega} \frac{|f|}{\delta_k} dp \right).$$

Hence

$$\delta_k \Phi^{-1}(1) \geq \bigvee_{\Omega} |f|\, dp,$$

which implies, via Proposition 2.1 in [3], that

$$\sup_{n\in J_0} \bigvee_{H_n \cap (0<|f|\leq\infty)} |f|\, dp = \bigvee_{\Omega} |f|\, dp = 0. \tag{6.6}$$

Clearly, $p(|f| = \infty) = 0$, otherwise the left hand side of (6.6) would assume the value ∞, a contradiction. Then necessarily, $p(H_n \cap (0 < |f| < \infty)) = 0$ for every $n \in J_0$, which is impossible because of the assumption. By this absurdity we have thus proved that if $\|f\|_{\mathcal{A}^\Phi} = 0$, then $f = 0$, p-a.e. Note that its converse is obvious. We show the triangle inequality in the next step. In fact, via the monotonicity and the convexity, we observe that

$$\Phi\left(\frac{|f+h|}{\|f\|_{\mathcal{A}^\Phi} + \|h\|_{\mathcal{A}^\Phi}}\right) \leq \Phi\left(\frac{|f|+|h|}{\|f\|_{\mathcal{A}^\Phi} + \|h\|_{\mathcal{A}^\Phi}}\right) \leq$$
$$\leq \frac{\|f\|_{\mathcal{A}^\Phi}}{\|f\|_{\mathcal{A}^\Phi} + \|h\|_{\mathcal{A}^\Phi}} \Phi\left(\frac{|f|}{\|f\|_{\mathcal{A}^\Phi}}\right) + \frac{\|h\|_{\mathcal{A}^\Phi}}{\|f\|_{\mathcal{A}^\Phi} + \|h\|_{\mathcal{A}^\Phi}} \Phi\left(\frac{|h|}{\|h\|_{\mathcal{A}^\Phi}}\right).$$

Hence

$$\bigvee_{\Omega} \Phi\left(\frac{|f+h|}{\|f\|_{\mathcal{A}^\Phi} + \|h\|_{\mathcal{A}^\Phi}}\right) \leq \frac{\|f\|_{\mathcal{A}^\Phi}}{\|f\|_{\mathcal{A}^\Phi} + \|h\|_{\mathcal{A}^\Phi}} \bigvee_{\Omega} \Phi\left(\frac{|f|}{\|f\|_{\mathcal{A}^\Phi}}\right) dp +$$
$$+ \frac{\|h\|_{\mathcal{A}^\Phi}}{\|f\|_{\mathcal{A}^\Phi} + \|h\|_{\mathcal{A}^\Phi}} \bigvee_{\Omega} \Phi\left(\frac{|h|}{\|h\|_{\mathcal{A}^\Phi}}\right) dp \leq 1,$$

since

$$\bigvee_{\Omega} \Phi\left(\frac{|f|}{\|f\|_{\mathcal{A}^\Phi}}\right) dp \leq 1 \quad \text{and} \quad \bigvee_{\Omega} \Phi\left(\frac{|h|}{\|h\|_{\mathcal{A}^\Phi}}\right) dp \leq 1.$$

Consequently,

$$\|f+h\|_{\mathcal{A}^\Phi} \leq \|f\|_{\mathcal{A}^\Phi} + \|h\|_{\mathcal{A}^\Phi}.$$

We leave to the reader the verification of the homogeneity axiom.

Part 2. We prove that $\delta_1 \|f\|_{\mathcal{A}^1} \leq \|f\|_{\mathcal{A}^\Phi}$ for some constant $\delta_1 > 0$ and all $f \in \mathcal{A}^\Phi$. In fact, let $u_0 \in (0, \infty)$ such that $\varphi(u_0) > 0$ and $u_0 + (\varphi(u_0))^{-1} \geq 1$. Making use of the inequality here below (proved in [30] on page 198)

$$\Phi(x) \geq (x - u_0)^+ \varphi(u_0), \; x \in [0, \infty),$$

we have

$$1 \geq \int_\Omega \Phi\left(\frac{|f|}{\|f\|_{\mathcal{A}^\Phi}}\right) dp \geq \varphi(u_0) \int_\Omega \left(\frac{|f|}{\|f\|_{\mathcal{A}^\Phi}} - u_0\right)^+ dp$$

and hence by Remark 6.6,

$$u_0 + \frac{1}{\varphi(u_0)} \geq \int_\Omega \left[u_0 + \left(\frac{|f|}{\|f\|_{\mathcal{A}^\Phi}} - u_0\right)^+\right] dp \geq \int_\Omega \frac{|f|}{\|f\|_{\mathcal{A}^\Phi}} dp.$$

Whence, $\|f\|_{\mathcal{A}^1} \leq \left(u_0 + \frac{1}{\varphi(u_0)}\right) \|f\|_{\mathcal{A}^\Phi}$.

Part 3. Let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{A}^\Phi$ be any Cauchy sequence. Then we can extract from it a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that

$$\sum_{k=1}^{\infty} \left\|f_{n_{k+1}} - f_{n_k}\right\|_{\mathcal{A}^\Phi} < \infty$$

and hence by Part 2,

$$\sum_{k=1}^{\infty} \left\|f_{n_{k+1}} - f_{n_k}\right\|_{\mathcal{A}^1} < \infty.$$

Since $\mathcal{A}^1$ is a Banach space, the limit $\lim\limits_{k\to\infty} f_{n_k} = f$ exists almost everywhere. Clearly, for every $k \in \mathbb{N}$,

$$f_{n_k} = f_{n_1} + \sum_{j=1}^{k-1} \left(f_{n_{j+1}} - f_{n_j}\right),$$

Write

$$S_{n_k} = |f_{n_1}| + \sum_{j=1}^{k-1} \left|f_{n_{j+1}} - f_{n_j}\right|, \;\; k \in \mathbb{N}.$$

Obviously,

$$\|S_{n_k}\|_{\mathcal{A}^\Phi} \leq \|f_{n_1}\|_{\mathcal{A}^\Phi} + \sum_{j=1}^{k-1} \left\|f_{n_{j+1}} - f_{n_j}\right\|_{\mathcal{A}^\Phi}, \;\; k \in \mathbb{N}.$$

Since $(S_{n_k})_{k\in\mathbb{N}}$ is an increasing sequence it ensues that

$$\|f\|_{\mathcal{A}^\Phi} \leq \liminf_{k\to\infty} \|S_{n_k}\|_{\mathcal{A}^\Phi} \leq \|f_{n_1}\|_{\mathcal{A}^\Phi} + \sum_{j=1}^{\infty} \left\|f_{n_{j+1}} - f_{n_j}\right\|_{\mathcal{A}^\Phi} < \infty.$$

Hence $f \in \mathcal{A}^\Phi$. Note that

$$\|f - f_{n_k}\|_{\mathcal{A}^\Phi} \leq \sum_{j=k+1}^{\infty} \left\|f_{n_{j+1}} - f_{n_j}\right\|_{\mathcal{A}^\Phi}$$

which yields

$$\lim_{k\to\infty} \|f - f_{n_k}\|_{\mathcal{A}^\Phi} = 0.$$

By the triangle inequality we have

$$\|f - f_n\|_{\mathcal{A}^\Phi} \leq \|f - f_{n_k}\|_{\mathcal{A}^\Phi} + \|f_n - f_{n_k}\|_{\mathcal{A}^\Phi} \to 0,$$

as $k \to \infty$ and $n \to \infty$.

Part 4. Let $f \in \mathcal{A}^{\Phi}$ and $h \in \mathcal{A}^{\Psi}$ be arbitrary such that $\|f\|_{\mathcal{A}^{\Phi}} > 0$ and $\|h\|_{\mathcal{A}^{\Psi}} > 0$. Then by applying the fundamental inequality (6.1) to $u = \frac{|f|}{\|f\|_{\mathcal{A}^{\Phi}}}$ and $v = \frac{|h|}{\|h\|_{\mathcal{A}^{\Psi}}}$ yields

$$\bigvee_{\Omega} |fh|\, dp \leq \|f\|_{\mathcal{A}^{\Phi}} \cdot \|h\|_{\mathcal{A}^{\Psi}} \left(\bigvee_{\Omega} \Phi\left(\frac{|f|}{\|f\|_{\mathcal{A}^{\Phi}}}\right) dp \quad + \quad \bigvee_{\Omega} \Phi\left(\frac{|h|}{\|h\|_{\mathcal{A}^{\Psi}}}\right) dp \right) \leq$$
$$\leq 2\, \|f\|_{\mathcal{A}^{\Phi}} \cdot \|h\|_{\mathcal{A}^{\Psi}} .$$

Part 5. To show that $\|\cdot\|^{*}_{\mathcal{A}^{\Phi}}$ is a norm we shall only verify the biconditional $\|h\|^{*}_{\mathcal{A}^{\Phi}} = 0$ if and only if $h = 0$, p-a.e. because the two other norm axioms can be easily checked. To this end we need to prove first that $\|h\|^{*}_{\mathcal{A}^{\Phi}} = 0$ implies $h = 0$, p-a.e. In fact, suppose (by the contrapositive) that there is some $H \in \mathcal{H}(p)$ for which the inequality $p\,(H \cap (|h| > 0)) > 0$ holds. Write $A := H \cap (|h| > 0)$. Consider the measurable function $f_{\delta} = \delta \chi_A$ with $\delta > 0$ such that

$$\bigvee_{\Omega} \Psi\,(f_{\delta})\, dp = \Psi\,(\delta)\, p\,(A) \leq 1.$$

This can be done, because Ψ is a convex Young function. Then

$$\|h\|^{*}_{\mathcal{A}^{\Phi}} \geq \bigvee_{\Omega} |h|\, f_{\delta} dp > 0.$$

Hence, $\|h\|^{*}_{\mathcal{A}^{\Phi}} = 0$ implies $h = 0$, p-a.e. Note that the converse conditional is straightforward.
By applying the Optimal Hölder Inequality, we observe from (6.5) that

$$\|h\|^{*}_{\mathcal{A}^{\Phi}} = \sup_{\left\{f \in \mathfrak{M} \colon \|f\|_{\mathcal{A}^{\Phi}} \leq 1\right\}} \bigvee_{\Omega} |fh|\, dp \leq 2\, \|h\|_{\mathcal{A}^{\Psi}} .$$

Next, we shall show the inequality $\lambda \|h\|_{\mathcal{A}^{\Psi}} \leq \|h\|^{*}_{\mathcal{A}^{\Phi}}$ for some constant $\lambda \in (0,\, 2]$ and all $h \in \mathcal{A}^{\Psi}$. In fact, assume the contrary, i.e. for every constant $\lambda \in (0,\, 2]$ we can find an $h \in \mathcal{A}^{\Psi}$ for which $\lambda \|h\|_{\mathcal{A}^{\Psi}} > \|h\|^{*}_{\mathcal{A}^{\Phi}}$. Now, choose $f_0 = \frac{\|h\|_{\mathcal{A}^{\Psi}}}{\rho p\,(H)} \chi_H$, where $H \in \mathcal{H}(p)\,,\ \rho > 0$ and $p\,(H \cap (|h| = \rho)) = p\,(H)$. Then $f_0 \in \mathcal{A}^{\Phi}$, via Remark 6.15. Consequently,

$$\lambda \|h\|_{\mathcal{A}^{\Psi}} > \|h\|^{*}_{\mathcal{A}^{\Phi}} \ = \sup_{\left\{f \in \mathfrak{M} \colon \|f\|_{\mathcal{A}^{\Phi}} \leq 1\right\}} \bigvee_{\Omega} |fh|\, dp \geq \bigvee_{\Omega} |f_0|\, |h|\, dp = \|h\|_{\mathcal{A}^{\Psi}}$$

so that $\lambda > 1$ for all $\lambda \in (0,\, 2]$. Letting $\lambda \to 0$ would entail $0 > 1$ which is absurd, indeed. Therefore, the inequality $\lambda \|h\|_{\mathcal{A}^{\Psi}} \leq \|h\|^{*}_{\mathcal{A}^{\Phi}}$ fulfils for some constant $\lambda \in (0,\, 2]$ and all $h \in \mathcal{A}^{\Psi}$.

Part 6. Let $F \circ |\cdot| \in \widetilde{\mathcal{A}^{\Phi}}$. Define the function $q : \mathcal{F} \to [0,\, \infty)$ by $q\,(A) = F\,(\chi_A)$. Via the assumption for every $A \in \mathcal{F}$,

$$q\,(A) \leq C\, \|\chi_A\|_{\mathcal{A}^{\Phi}} .$$

Consider the continuous function

$$\eta\,(t) = \begin{cases} \frac{1}{\Phi^{-1}\left(\frac{1}{t}\right)} & \text{whenever } t > 0 \\ 0 & \text{if } \ t = 0. \end{cases}$$

A simple calculus shows that

$$\int_{\Omega} \Phi\left(\frac{\chi_A}{\eta\left(p\left(A\right)\right)}\right) dp = \Phi\left(\frac{1}{\eta\left(p\left(A\right)\right)}\right) p\left(A\right) = 1.$$

Hence $q\left(A\right) \leq C\eta\left(p\left(A\right)\right)$, whenever $A \in \mathcal{F}$. Consequently, $q \ll p$, i.e. q is absolutely continuous with respect to p. Then by Theorem 2.4 of [2],

$$h = \max\left\{\frac{q\left(H\right)}{p\left(H\right)} \cdot \chi_H : H \in \mathcal{H}\left(p\right), \, q\left(H\right) > 0\right\}$$

is the unique measurable function such that $dq = h \cdot dp$ almost everywhere. Consequently, for every measurable simple function $s = \sum\limits_{i=1}^{n} b_i \chi_{B_i} = \bigvee\limits_{i=1}^{n} b_i \chi_{B_i}$ we have

$$\bigvee_{i=1}^{n} \left|b_i\right| F\left(\chi_{B_i}\right) = \bigvee_{i=1}^{n} F\left(\left|b_i\right| \chi_{B_i}\right) = F\left(\bigvee_{i=1}^{n} \left|b_i\right| \chi_{B_i}\right) = \int_{\Omega} h \bigvee_{i=1}^{n} \left|b_i\right| \chi_{B_i} dp =$$

$$= \int_{\Omega} h\left|s\right| dp = F\left(\left|s\right|\right).$$

Next, we show that $\left\|h\right\|_{\mathcal{A}^{\Psi}} \leq 2C$. To this end, let $\left(s_n\right)$ be a sequence of non-negative measurable simple functions tending increasingly to h. Then by the Young equality (6.2) one can observe that

$$\Psi\left(\frac{s_n}{2C}\right) + \Phi\left(\psi\left(\frac{s_n}{2C}\right)\right) = \frac{s_n}{2C}\psi\left(\frac{s_n}{2C}\right).$$

On the one hand,

$$\int_{\Omega} \left[\Psi\left(\frac{s_n}{2C}\right) + \Phi\left(\psi\left(\frac{s_n}{2C}\right)\right)\right] dp \geq \int_{\Omega} \max\left\{\Psi\left(\frac{s_n}{2C}\right) ; \Phi\left(\psi\left(\frac{s_n}{2C}\right)\right)\right\} dp =$$

$$= \max\left\{\int_{\Omega} \Psi\left(\frac{s_n}{2C}\right) dp \quad ; \quad \int_{\Omega} \Phi\left(\psi\left(\frac{s_n}{2C}\right)\right) dp\right\}.$$

On the other hand we observe via Remark 6.13 that

$$\int_{\Omega} \frac{s_n}{2C}\psi\left(\frac{s_n}{2C}\right) dp \leq \frac{1}{2C}\int_{\Omega} h\psi\left(\frac{s_n}{2C}\right) dp = \frac{1}{2C} F\left(\psi\left(\frac{s_n}{2C}\right)\right) \leq$$

$$\leq \frac{1}{2}\left\|\psi\left(\frac{s_n}{2C}\right)\right\|_{\mathcal{A}^{\Phi}} \leq \frac{1}{2}\max\left\{1 ; \int_{\Omega} \Phi\left(\psi\left(\frac{s_n}{2C}\right)\right) dp\right\}.$$

Consequently,

$$\int_{\Omega} \Psi\left(\frac{s_n}{2C}\right) dp + \int_{\Omega} \Phi\left(\psi\left(\frac{s_n}{2C}\right)\right) dp \leq$$

$$\leq 2\max\left\{\int_{\Omega} \Psi\left(\frac{s_n}{2C}\right) dp ; \int_{\Omega} \Phi\left(\psi\left(\frac{s_n}{2C}\right)\right) dp\right\} \leq$$

$$\leq \max\left\{1 ; \int_{\Omega} \Phi\left(\psi\left(\frac{s_n}{2C}\right)\right) dp\right\}$$

$$\leq 1 + \int_{\Omega} \Phi\left(\psi\left(\frac{s_n}{2C}\right)\right) dp.$$

This implies that

$$\int_{\Omega} \Psi \left(\frac{s_n}{2C} \right) dp \leq 1, \ n \in \mathbb{N}, \tag{6.7}$$

since $\int_{\Omega} \Phi \left(\psi \left(\frac{s_n}{2C} \right) \right) dp < \infty$. Finally, letting $n \to \infty$ in (6.7), the Optimal Monotone Convergence Theorem (cf. [1], Theorem 3.1/i) implies that $\int_{\Omega} \Psi \left(\frac{h}{2C} \right) dp \leq 1$. Therefore, $h \in \mathcal{A}^{\Psi}$.

□

7. Cauchy-type functional equation in lattice environments

The most famous functional equation by Cauchy and known as *linear functional equation* reads:

$$f(x+y) = f(x) + f(y), \tag{7.1}$$

where f is a real function.
We should point out that equation (7.1) has been investigated for many spaces and in various perspectives such as its stability which has been intensively considered in the literature. The stability problem was first posed by M. Ulam (see [36]) in the terms: "Give conditions in order for a linear mapping near an approximately linear mapping to exist." More precisely the problem can be formulated as follows:
Given two Banach algebras E and $E^{'}$, a transformation $f : E \to E^{'}$ is called δ-linear if

$$\| f(x+y) - f(x) - f(y) \| < \delta, \tag{7.2}$$

for all $x, y \in E$.

The stability problem of equation (7.1) can be stated as follows. Does there exist for each $\varepsilon \in (0, 1)$ some $\delta > 0$ such that to each δ-linear transformation $f : E \to E'$ there corresponds a linear transformation $l : E \to E'$ satisfying the inequality $\| f(x) - l(x) \| < \varepsilon$ for all $x \in E$? This question was answered in the affirmative by Hyers [23] and then generalized by Aoki [12]. Ever since various problems of stability on various spaces have come to light. We shall list just few of them: [22, 31, 27, 29, 35].

7.1. Functional equation with both lattice operations

In the sequel $(\mathcal{X}, \wedge_{\mathcal{X}}, \vee_{\mathcal{X}})$ will denote a vector lattice and $(\mathcal{Y}, \wedge_{\mathcal{Y}}, \vee_{\mathcal{Y}})$ a Banach lattice with $\mathcal{X}^+$ and $\mathcal{Y}^+$ their respective positive cones.

We recall that a functional $H : \mathcal{X} \to \mathcal{Y}$ is *cone-related* if $H(\mathcal{X}^+) = \{H(|x|) : x \in \mathcal{X}\} \subset \mathcal{Y}^+$ (see more about this notion in [6]).

In the image of the Cauchy functional equation we consider the following operator equation

$$T(|x| \Delta_{\mathcal{X}}^* |y|) \Delta_{\mathcal{Y}}^* T(|x| \Delta_{\mathcal{X}}^{**} |y|) = T(|x|) \Delta_{\mathcal{Y}}^{**} T(|y|) \tag{7.3}$$

to hold true for all $x, y \in \mathcal{X}$, where $\Delta_{\mathcal{X}}^*, \Delta_{\mathcal{X}}^{**} \in \{\wedge_{\mathcal{X}}, \vee_{\mathcal{X}}\}$ and $\Delta_{\mathcal{Y}}^*, \Delta_{\mathcal{Y}}^{**} \in \{\wedge_{\mathcal{Y}}, \vee_{\mathcal{Y}}\}$ are fixed lattice operations.

Note that if in the special case the above four lattice operations are at the same time the supremum (join) or the infimum (meet), then the functional equation (7.3) is just a join-homomorphism or a meet-homomorphism. Moreover, if operations $\Delta_{\mathcal{X}}^*$ and $\Delta_{\mathcal{X}}^{**}$ are the same, then the left hand side of (7.3) is the maps of the meets or the joins, which are just in the image of (7.1).
Problem: Given lattice operations $\Delta_{\mathcal{X}}^*, \Delta_{\mathcal{X}}^{**} \in \{\wedge_{\mathcal{X}}, \vee_{\mathcal{X}}\}$ and $\Delta_{\mathcal{Y}}^*, \Delta_{\mathcal{Y}}^{**} \in \{\wedge_{\mathcal{Y}}, \vee_{\mathcal{Y}}\}$, a vector lattice G_1, a vector lattice G_2 endowed with a metric $d(\cdot, \cdot)$ and a positive number ε, does there exist some $\delta > 0$ such that, if a mapping $F : G_1 \to G_2$ satisfies

$$d\left(F(|x| \Delta_{\mathcal{X}}^* |y|) \Delta_{\mathcal{Y}}^* F(|x| \Delta_{\mathcal{X}}^{**} |y|), \ F(|x|) \Delta_{\mathcal{Y}}^{**} F(|y|)\right) \leq \delta$$

for all x, $y \in G_1$, then an operation-preserving functional $T : G_1 \to G_2$ exists with the property that

$$d\left(T(x), F(x)\right) \leq \varepsilon$$

for all $x \in G_1$?

One can view this problem as a lattice version of the Ulam's stability problem formulated in [36]. We shall present here only one type of clauses leading to a unique solution.

Theorem 7.1 ([8], Theorem 2.1)**.** *Consider a cone-related functional $F : \mathcal{X} \to \mathcal{Y}$ for which there are numbers $\vartheta > 0$ and $\alpha \in (-\infty, 1)$ such that*

$$\left\| \frac{F\left(|x| \, \Delta^{*}_{\mathcal{X}} \, |y|\right) \Delta^{*}_{\mathcal{Y}} \, F\left(|x| \, \Delta^{**}_{\mathcal{X}} \, |y|\right)}{\tau} - F\left(\frac{|x|}{\tau}\right) \Delta^{**}_{\mathcal{Y}} F\left(\frac{|y|}{\tau}\right) \right\| \leq \frac{\vartheta}{4}\left(\|x\|^{\alpha} + \|y\|^{\alpha}\right) \tag{7.4}$$

for all x, $y \in \mathcal{X}$ and $\tau \in (0, \infty)$, where $\Delta^{*}_{\mathcal{X}}$, $\Delta^{**}_{\mathcal{X}} \in \{\wedge_{\mathcal{X}}, \vee_{\mathcal{X}}\}$ and $\Delta^{*}_{\mathcal{Y}}$, $\Delta^{**}_{\mathcal{Y}} \in \{\wedge_{\mathcal{Y}}, \vee_{\mathcal{Y}}\}$ are fixed lattice operations. Then the sequence $\left(2^{-n} F\left(2^{n} |x|\right)\right)_{n \in \mathbb{N}}$ is a Cauchy sequence for every $x \in \mathcal{X}$. Moreover, let the functional $T : \mathcal{X} \to \mathcal{Y}$ be defined by

$$T\left(|x|\right) = \lim_{n \to \infty} 2^{-n} F\left(2^{n} |x|\right). \tag{7.5}$$

Then

(a.) *T is semi-homogeneous, i.e. $T\left(\gamma \, |x|\right) = \gamma T\left(|x|\right)$, for all $x \in \mathcal{X}$ and all $\gamma \in [0, \infty)$;*

(b.) *T is the unique cone-related functional satisfying both identity* (7.3) *and inequality*

$$\left\| T\left(|x|\right) - F\left(|x|\right) \right\| \leq \frac{2^{\alpha} \vartheta}{2 - 2^{\alpha}} \|x\|^{\alpha} \tag{7.6}$$

for every $x \in \mathcal{X}$.

Before we start the proof the following obvious remarks are worth being mentioned, as they will be used multiple times.

Remark 7.2 ([8], Remark 2.1). If the conditions of Theorem 7.1 holds true, then $F(0) = 0$.

Remark 7.3 ([8], Remark 2.2). Let Z be a set closed under the scalar multiplication, i.e. $bz \in Z$ whenever $b \in \mathbb{R}$ and $z \in Z$. Given a number $c \in \mathbb{R}$ let the function $\gamma : Z \to Z$ be defined by $\gamma(z) = cz$. Then $\gamma^{j} : Z \to Z$ the j-th iteration of γ is given by $\gamma^{j}(z) = c^{j} z$ for every counting number $j \geq 2$.

Proof of Theorem 7.1. First, if we choose $\tau = 2$, $y = x$ and replace x by $2x$ in inequality (7.4) then we obviously have

$$\left\| \frac{F\left(2 \, |x|\right)}{2} - F\left(|x|\right) \right\| \leq \vartheta 2^{\alpha - 1} \|x\|^{\alpha}. \tag{7.7}$$

Next, let us define the following functions:

1.) $G : \mathcal{X} \to \mathcal{X}, \quad G\left(|x|\right) = 2\,|x|$.
2.) $\delta : \mathcal{X} \to [0, \infty), \quad \delta\left(|x|\right) = \vartheta 2^{\alpha - 1} \|x\|^{\alpha}$.
3.) $\varphi : [0, \infty) \to [0, \infty), \quad \varphi(t) = 2^{-1} t$.
4.) $H : \mathcal{Y} \to \mathcal{Y}, \quad H\left(|y|\right) = 2^{-1} |y|$.
5.) $d(\cdot, \cdot) : \mathcal{Y} \times \mathcal{Y} \to [0, \infty), \quad d\left(y_1, y_2\right) = \|y_1 - y_2\|$.

We shall verify the fulfilment all the three conditons of the first Forti's theorem (cf. [18, Theorem 1]) as follows.

(I.) From inequality (7.7) we obviously have

$$d\left(H\left(F\left(G\left(|x|\right)\right)\right), F\left(|x|\right)\right) = \left\| \frac{F\left(2\,|x|\right)}{2} - F\left(|x|\right) \right\| \leq \vartheta 2^{\alpha - 1} \|x\|^{\alpha} = \delta\left(|x|\right).$$

(II.) $d(H(|y_1|)\ , H(|y_2|)) = 2^{-1}\|y_1 - y_2\| = \phi(d(y_1\ , y_2))$ for all $y_1, y_2 \in \mathcal{Y}$.

(III.) Clearly, on the one hand φ is a non-decreasing subadditive function on the positive half line, and on other hand by applying Remark 7.3 on both the iterations G^j and φ^j of G and φ respectively, one can observe that

$$\sum_{j=0}^{\infty}\varphi^j\left(\delta\left(G^j(|x|)\right)\right) = \vartheta 2^{\alpha-1}\|x\|^\alpha \sum_{j=0}^{\infty} 2^{(\alpha-1)j} = \vartheta\|x\|^\alpha \frac{2^\alpha}{2-2^\alpha} < \infty.$$

Then in virtue of Forti's first theorem in [18] sequence $(H^n(F(G^n|x|)))$ is a Cauchy sequence for every $x \in \mathcal{X}$ and thus so is sequence $(2^{-n}F(2^n|x|))$ and furthermore, the mapping (7.5) is the unique functional which satisfies inequatility (7.6).

Next, we prove the validity of inequality (7.3). In fact, in (7.4) substitute x with $2^n x$ and y with $2^n y$, and also let $\tau = 1$. Then

$$\left\|F(2^n(|x|\,\Delta^*_{\mathcal{X}}|y|))\,\Delta^{**}_{\mathcal{Y}}F(2^n(|x|\,\Delta^{**}_{\mathcal{X}}|y|)) - F(2^n|x|)\,\Delta^{**}_{\mathcal{Y}}F(2^n|y|)\right\| \le \frac{\vartheta}{4}2^{n\alpha}(\|x\|^\alpha + \|y\|^\alpha).$$

Dividing both sides of this last inequality by 2^n yields

$$\begin{aligned}\left\|\frac{F(2^n(|x|\,\Delta^*_{\mathcal{X}}|y|))\,\Delta^{**}_{\mathcal{Y}}F(2^n(|x|\,\Delta^{**}_{\mathcal{X}}|y|))}{2^n} - \frac{F(2^n|x|)\,\Delta^{**}_{\mathcal{Y}}F(2^n|y|)}{2^n}\right\| &\le \\ \le \frac{\vartheta}{4}(\|x\|^\alpha + \|y\|^\alpha)\,2^{(\alpha-1)n}&.\end{aligned} \tag{7.8}$$

Taking the limit in (7.8) we have via (7.5) that

$$\left\|T(|x|\,\Delta^*_{\mathcal{X}}|y|)\,\Delta^*_{\mathcal{Y}}T(|x|\,\Delta^{**}_{\mathcal{X}}|y|) - T(|x|)\,\Delta^{**}_{\mathcal{Y}}T(|y|)\right\| = 0$$

which is equivalent to

$$T(|x|\,\Delta^*_{\mathcal{X}}|y|)\,\Delta^*_{\mathcal{Y}}T(|x|\,\Delta^{**}_{\mathcal{X}}|y|) = T(|x|)\,\Delta^{**}_{\mathcal{Y}}T(|y|).$$

Because of Remark 7.2 identity $\gamma F(|x|) = F(\gamma|x|)$ is trivial on the one hand for $\gamma = 0$ and all $x \in \mathcal{X}$, on the other hand for $x = 0$ and all $\gamma \in [0, \infty)$. Without loss of generality let us thus fix arbitrarily a number $\gamma \neq 0$ and an $x \in \mathcal{X}\setminus\{0\}$. In (7.4) choose $y = x$, $\tau = \gamma^{-1}$ and change x to $2^n x$. Then

$$\|\gamma F(2^n|x|) - F(\gamma 2^n|x|)\| \le \frac{\vartheta}{2}\|x\|^\alpha 2^{n\alpha}.$$

Divide both sides of this last inequality by 2^n to get

$$\left\|\gamma 2^{-n}F(2^n|x|) - 2^{-n}F(\gamma 2^n|x|)\right\| \le \frac{\vartheta}{2}\|x\|^\alpha 2^{(\alpha-1)n}. \tag{7.9}$$

By taking the limit in (7.9) we have via (7.5) that

$$\|\gamma T(|x|) - T(\gamma|x|)\| = 0$$

or equivalently,

$$T(\gamma|x|) = \gamma T(|x|)$$

for all $x \in \mathcal{X}$. We have thus shown the semi-homogeneity of operator T. We can conclude on the validity of the argument. □

Next, we shall provide an example showing that if in (7.4) the parameter τ is omitted and the power p of the norms equals the unity, then stability cannot always be guaranteed. We remind that in the addition environments Gajda in [21] and Găvruţa in [**?**] gave some interesting examples to show how stability fails when the power of the norms is equal to 1.

Example 7.4 ([8], Example 1)**.** Consider the Lipschitz-continuous function

$$F : [0,\, \infty) \to [0,\, \infty)\,,\; F(x) = \sqrt{x^2 + 1}.$$

Fix arbitrarily two numbers $x,\, y \in [0,\, \infty)$. Since F is an increasing function the very first equality in the chain of relations here below is valid, implying the subsequent relations in the chain:

$$\begin{aligned}
|F(x \vee y) - (F(x) \wedge F(y))| &= |F(x \vee y) - F(x \wedge y)| \\
&= \left| \sqrt{(x \vee y)^2 + 1} - \sqrt{(x \wedge y)^2 + 1} \right| \\
&= \frac{(x \vee y)^2 - (x \wedge y)^2}{\sqrt{(x \vee y)^2 + 1} + \sqrt{(x \wedge y)^2 + 1}} = \\
|x - y| \cdot \frac{(x \vee y) + (x \wedge y)}{\sqrt{(x \vee y)^2 + 1} + \sqrt{(x \wedge y)^2 + 1}} &\le |x - y| \le x + y
\end{aligned}$$

for all $x,\, y \in [0,\, \infty)$. Now, let $T : [0,\, \infty) \to [0,\, \infty)$ be a function such that $T(x) = xT(1)$ for all $x \in [0,\, \infty)$. Then a simple argument shows

$$\sup_{x \in (0,\, \infty)} \frac{|F(x) - T(x)|}{x} = \sup_{x \in (0,\, \infty)} \left| \sqrt{1 + x^{-2}} - T(1) \right| = \infty.$$

7.2. Schwaiger's type functional equation

Schwaiger's theorem reads [34]:

Theorem 7.5 (Schwaiger's Stability Theorem)**.** *Given a real vector space E_1 and a real Banach space E_2, let $f : E_1 \to E_2$ be a mapping for which inequality*

$$\|f(x + \alpha y) - f(x) - \alpha f(y)\| \le b(\alpha) \tag{7.10}$$

is satisfied for all $\alpha \in \mathbb{R}$. Then there exists a unique linear function $g : E_1 \to E_2$ such that $f - g$ is bounded.

In the sequel $(\mathcal{X},\, \wedge_{\mathcal{X}},\, \vee_{\mathcal{X}})$ will denote a vector lattice and $(\mathcal{Y},\, \wedge_{\mathcal{Y}},\, \vee_{\mathcal{Y}})$ a Banach lattice with $\mathcal{X}^+$ and $\mathcal{Y}^+$ their respective positive cones.
Given two positive real numbers p and q consider the functional equation

$$T((\tau^q |x|) \vee |y|) = (\tau^p T(|x|)) \vee T(|y|) \tag{7.11}$$

for all $x,\, y \in \mathcal{X}$ and $\tau \in [0,\, \infty)$, where T maps $\mathcal{X}$ into $\mathcal{Y}$.

The following simple examples show that the functional equation (7.11) has at least one solution. This can easily checked from the monotonicity of the functions.

Example 7.6 ([9], Example 1)**.** The function $T_1 : [0,\, \infty) \to [0,\, \infty)$ defined by $T_1(x) = x$ is a solution of (7.11), for all $\tau,\, q,\, x,\, y \in [0,\, \infty)$ with the choice $p = q$.

Example 7.7 ([9], Example 2)**.** The function $T_2 : [0,\, \infty) \to [0,\, \infty)$ defined by $T_2(x) = \sqrt{x}$ is a solution of (7.11), for all $\tau,\, q,\, x,\, y \in [0,\, \infty)$ with the choice $p = \frac{q}{2} < q$.

Example 7.8 ([9], Example 3)**.** The function $T_3 : [0,\, \infty) \to [0,\, \infty)$ defined by $T_3(x) = x^2$ is a solution of (7.11), for all $\tau,\, q,\, x,\, y \in [0,\, \infty)$ with the choice $p = 2q > q$.

Example 7.9 ([9], Example 4)**.** Let $\mathcal{X} = B(M,\, \mathbb{R})$ be the space of all bounded real-valued functions defined on M. Then the functional $T : \mathcal{X} \to \mathcal{X}$, such that $T(|f|) = |f|^{\alpha}$, solves (7.11) for arbitrary positive numbers q and α with $p = q\alpha$.

Our essential goal in this part is to prove the stability of the functional equation (7.11) to be viewed as a counterpart of the Schwaiger type stability theorem (cf. [34]).

We recall that a functional $H : \mathcal{X} \to \mathcal{Y}$ is *cone-related* if $H(\mathcal{X}^+) = \{H(|x|) : x \in \mathcal{X}\} \subset \mathcal{Y}^+$ (see more about this notion in [6]).

Remark 7.10 ([9], Remark 1.1). Given two positive real numbers p and q, if a cone-related operator $T : \mathcal{X} \to \mathcal{Y}$ satisfies the functional equation (7.11), then

1.) $T(|x| \vee |y|) = T(|x|) \vee T(|y|)$ for all $x, y \in \mathcal{X}$ and $\tau = 1$;

2.)
$$T(\tau^q |x|) = \tau^p T(|x|) \tag{7.12}$$
for all $x \in \mathcal{X}$ and all $\tau \in [0, \infty) \setminus \{1\}$.

Proof. Note that by letting $\tau = 1$ in (7.11) shows that T is trivially a join-homomorphism. To show the second part we first prove that $T(0) = 0$. In fact, take $x = y = 0$ in (7.11). Then $T(0) = (\tau^p T(0)) \vee T(0)$. But since τ runs over the non-negative real line, by choosing $\tau = 2$ yields $T(0) = (2T(0)) \vee T(0)$, which is possible only if $T(0) = 0$. Consequently, (7.12) follows if we select $y = 0$ in (7.11). $\square$

Theorem 7.11 ([9], Theorem 2.1)**.** *Given a pair of positive real numbers (p, q), consider a cone-related functional $F : \mathcal{X} \to \mathcal{Y}$ for which there are numbers $\vartheta > 0$ and α with $q\alpha \in (0, p)$ such that*
$$\|F((\tau^q |x|) \vee |y|) - (\tau^p F(|x|)) \vee F(|y|)\| \leq 2^{-p} \vartheta (\|x\|^\alpha + \|y\|^\alpha) \tag{7.13}$$
for all $x, y \in \mathcal{X}$ and all $\tau \in [0, \infty)$. Then the sequence $(2^{-np} F(2^{nq} |x|))_{n \in \mathbb{N}}$ is a Cauchy sequence for every $x \in \mathcal{X}$. Let the functional $T : \mathcal{X} \to \mathcal{Y}$ be defined by
$$T(|x|) = \lim_{n \to \infty} 2^{-np} F(2^{nq} |x|). \tag{7.14}$$
Then

a.) *T is a solution of the functional equation (7.11);*

b.) *T is the unique cone-related functional which satisfies inequality*
$$\|T(|x|) - F(|x|)\| \leq \frac{2^{q\alpha} \vartheta}{2^p - 2^{q\alpha}} \|x\|^\alpha \tag{7.15}$$
for every $x \in \mathcal{X}$.

Moreover, assume that $\mathcal{X}$ is a Banach lattice and F is continuous from below on the positive cone $\mathcal{X}^+$. Then in order that the limit operator T be continuous from below on $\mathcal{X}^+$, it is necessary and sufficient that
$$\lim_{n \to \infty} \lim_{k \to \infty} \frac{F(2^{nq} x_k)}{2^{np}} \leq \lim_{k \to \infty} \lim_{n \to \infty} \frac{F(2^{nq} x_k)}{2^{np}}, \tag{7.16}$$
for any increasing sequence $(x_k) \subset \mathcal{X}^+$., provided that the limits exist.

Before we start the proof the following obvious remarks are worth being mentioned, as they will be used multiple times. The first will be checked and the second one can be found in [8] without proof.

Remark 7.12 ([9], Remark 2.1). If the condition of Theorem 7.11 hold true, then $F(0) = 0$.

Proof. In (7.13) choose $x = y = 0$ and observe that $\|F(0) - (\tau^p F(0)) \vee F(0)\| = 0$ so that $F(0) = (\tau^p F(0)) \vee F(0)$. But since τ runs over the non-negative real line, by choosing $\tau = 2$ yields $F(0) = (2F(0)) \vee F(0)$, which is possible only if $F(0) = 0$. $\square$

Remark 7.13 ([9], Remark 2.2). Let Z be a set closed under the scalar multiplication, i.e. $bz \in Z$ whenever $b \in (0, \infty)$ and $z \in Z$. Given a number $c \in (0, \infty)$ let the function $\gamma : Z \to Z$ be defined by $\gamma(z) = cz$. Then $\gamma^j : Z \to Z$ the j-th iteration of γ is given by $\gamma^j(z) = c^j z$ for every counting number $j \geq 2$.

Proof of Theorem 7.11. First, we choose $\tau = 2^{-1}$, $y = 0$ and replacing x by $2^q x$ in (7.13) we obviously have

$$\left\| \frac{F(2^q |x|)}{2^p} - F(|x|) \right\| \leq \vartheta 2^{q\alpha - p} \|x\|^\alpha . \tag{7.17}$$

Next, let us define the following functions:

1.) $G : \mathcal{X} \to \mathcal{X}, \quad G(|x|) = 2^q |x|.$
2.) $\delta : \mathcal{X} \to [0, \infty), \quad \delta(|x|) = \vartheta 2^{q\alpha - p} \|x\|^\alpha.$
3.) $\varphi : [0, \infty) \to [0, \infty), \quad \varphi(t) = 2^{-p} t.$
4.) $H : \mathcal{Y} \to \mathcal{Y}, \quad H(|y|) = 2^{-p} |y|.$
5.) $d(\cdot, \cdot) : \mathcal{Y} \times \mathcal{Y} \to [0, \infty), \quad d(y_1, y_2) = \|y_1 - y_2\|.$

We shall verify the fulfilment of all the three conditions of the first Forti's theorem (cf. [18, Theorem 1]) as follows.

(I.) From inequality (7.17) we obviously have

$$d(H(F(G(|x|))), F(|x|)) = \left\| \frac{F(2^q |x|)}{2^p} - F(|x|) \right\| \leq \vartheta 2^{q\alpha - p} \|x\|^\alpha = \delta(|x|).$$

(II.) $d(H(|y_1|), H(|y_2|)) = 2^{-p} \|y_1 - y_2\| = \varphi(d(y_1, y_2))$ for all $y_1, y_2 \in \mathcal{Y}$.

(III.) Clearly, on the one hand φ is a non-decreasing subadditive function on the positive half line, and on other hand by applying Remark 7.13 on both the iterations G^j and φ^j of G and φ respectively, one can observe that

$$\sum_{j=0}^{\infty} \varphi^j \left(\delta \left(G^j (|x|) \right) \right) = \vartheta 2^{(q\alpha - p)} \|x\|^\alpha \sum_{j=0}^{\infty} 2^{(q\alpha - p)j} = \vartheta \|x\|^\alpha \frac{2^{q\alpha}}{2^p - 2^{q\alpha}} < \infty.$$

Then in virtue of Forti's first theorem in [18], sequence $(H^n(F(G^n |x|)))_{n \in \mathbb{N}}$ is a Cauchy sequence for every $x \in \mathcal{X}$ and thus so is sequence $(2^{-np} F(2^{nq} |x|))_{n \in \mathbb{N}}$ and furthermore, the mapping (7.14) is the unique functional which satisfies inequatility (7.15). Next, we prove that the mapping T, defined in (7.14), satisfies the functional equation (7.11). In fact, in (7.13) substitute x with $2^{nq} x$ also y with $2^{nq} y$, and fix arbitarily $\tau \in [0, \infty)$. Then

$$\|F(2^{nq}((\tau^q |x|) \vee |y|)) - (\tau^p F(2^{nq} |x|)) \vee F(2^{nq} |y|)\| \leq \vartheta 2^{-p} 2^{q\alpha n} (\|x\|^\alpha + \|y\|^\alpha).$$

Dividing both sides of this last inequality by 2^{np} yields

$$\left\| \frac{F(2^{nq}((\tau^q |x|) \vee |y|))}{2^{np}} - \frac{(\tau^p F(2^{nq} |x|)) \vee F(2^{nq} |y|)}{2^{np}} \right\| \leq \vartheta 2^{-p} 2^{(q\alpha - p)n} (\|x\|^\alpha + \|y\|^\alpha). \tag{7.18}$$

Taking the limit in (7.18) we have via (7.14) that for all $\tau \in [0, \infty)$ and all $x, y \in \mathcal{X}$

$$\|T((\tau^q |x|) \vee |y|) - (\tau^p T(|x|)) \vee T(|y|)\| = 0$$

which is equivalent to (7.11).

The moreover part can be proved the same way the moreover parts of the theorems in [6] were, after we will have shown that the limits on both sides of (7.16) exist. In fact, on the one hand, the existence of the limit on the left hand side follows from the combination of the monotonicity of F and (7.14). On the other hand, because of (7.14) the inner limit on the right hand side equals $T(x_k)$ for every $k \in \mathbb{N}$. But since the limit operator T is a join-homomorphism, it is also isotonic or increasing. Consequently, $(T(x_k))_{k \in \mathbb{N}}$ is a convergent sequence. We have thus proved that the limits on both sides of (7.16) exist.

Therefore, we can conclude on the validity of the argument. □

The example hereafter is to show that stability fails in some cases. if the range of the parameters p and q is retricted and the power α of the norms equals the ratio of p and q To end the section we shall provide some example showing that if in (7.13) parameter τ does not range over the whole non-negative half-line and the power α of the norms equals the ratio of p and q, then stability cannot always be guaranteed. A similar example can be found in [8].

Example 7.14 ([9], Example 5)**.** Fix arbitrarily three numbers p, q, $c \in (0, \infty)$ and consider the function

$$F : \mathbb{R} \to \mathbb{R},\ F(|x|) = c.$$

Then whenever $\tau \in (0, 1]$ we have:

$$|F((\tau^q |x|) \vee |y|) - (\tau^p F(|x|)) \vee F(|y|)| = |c - (\tau^p c) \vee c| = 0 \le |x|^\alpha + |y|^\alpha, \text{ where } \quad \alpha = \frac{p}{q}.$$

Since $|x| = \left(|x|^{\frac{1}{q}}\right)^q$, for any function $T : \mathbb{R} \to \mathbb{R}$ which solves (7.11) the following consecutive relations are true:

$$\sup_{|x|\in(0,\infty)} \frac{|F(|x|) - T(|x|)|}{|x|^\alpha} = \sup_{|x|\in(0,\infty)} \frac{\left|c - T\left(\left(|x|^{\frac{1}{q}}\right)^q\right)\right|}{|x|^\alpha} = \sup_{|x|\in(0,\infty)} \frac{|c - |x|^\alpha T(1)|}{|x|^\alpha} =$$

$$= \sup_{|x|\in(0,\infty)} \left| \frac{c}{|x|^\alpha} - T(1) \right| = \infty.$$

8. Concluding Remarks

We would like to pinpoint that Riesz spaces can offer a very fertile soil for proving addition dependent results in addition-free environments. We believe that this is yet to come to an end. So broad can be the spectrum of questions to ask and to answer that we judge not to cite any of them here.

9. Competing Interests

The author declares that he has no competing interests.

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