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Geraghty type contraction mappings on Branciari b-metric spaces

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Abstract

In this paper fixed points of α-admissible contraction mappings of Geraghty type defined on Branciari b-metric spaces are studied. Existence and uniqueness theorems for these types of mappings of are proved. Some consequences of these theorems are given and specific examples are presented.

Keywords: fixed point, Branciari b-metric space, α-admissible contraction mappings, Geraghty contraction
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1. Introduction and preliminaries

Branciari metric spaces are among the recent generalizations of metric spaces and have been defined by Branciari [3]. The main feature of these spaces is the replacement of the triangular inequality by a rectangular inequality. The Branciari metric spaces are also referred to as rectangular or generalized metric spaces. Another recent generalization of the metric spaces called b-metric spaces has been introduced by Czerwik [4] and Bakhtin [2]. The difference between metric and b-metric shows itself in the triangle inequality which contains a constant $s \geq 1$. Combining these two concepts, George et.al. [5] defined Branciari b-metric spaces. This new metric space is also referred to as rectangular b-metric spaces. Several articles related with this new metric space have been published recently [5, 11, 7].

In this paper we discuss the problem of existence and uniqueness of fixed points for contraction mappings of Geraghty type defined on Branciari b-metric spaces.

We first introduce the basic notions used throughout the paper.

Branciari metric spaces are defined as follows [3].

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Definition 1.1. [3] Let X be a nonempty set and let $d : X \times X \to [0, +\infty)$ be a function such that for all $x, y \in X$ and all distinct $u, v \in X$ each of which is different from x and y, the following conditions are satisfied:

(1.) $d(x, y) = 0$ if and only if $x = y$;

(2.) $d(x, y) = d(y, x)$;

(3.) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$.

The map d is called a Branciari metric and the pair (X, d) is called a Branciari metric space.

Czerwik [4] and Bakhtin [2] defined the b-metric spaces as follows.

Definition 1.2. [2, 4] Let X be a nonempty set and let $d : X \times X \to [0, +\infty)$ be a mapping satisfying the following conditions for all $x, y, z \in X$:

(1.) $d(x, y) = 0$ if and only if $x = y$;

(2.) $d(x, y) = d(y, x)$;

(3.) $d(x, y) \leq s[d(x, z) + d(z, y)]$ for some real number $s \geq 1$.

Then the mapping d is called a b-metric and the pair (X, d) is called a b-metric space with a constant $s \geq 1$.

Combination of the Branciari and b-metric spaces results in the following definition of the Branciari b-metric spaces.

Definition 1.3. [5] Let X be a nonempty set and let $d : X \times X \to [0, +\infty)$ be a function such that for all $x, y \in X$ and all distinct $u, v \in X$ each of which is different from x and y, the following conditions are satisfied:

(1.) $d(x, y) = 0$ if and only if $x = y$;

(2.) $d(x, y) = d(y, x)$;

(3.) $d(x, y) \leq s[d(x, u) + d(u, v) + d(v, y)]$ for some real number $s \geq 1$.

The map d is called a Branciari b-metric and the pair (X, d) is called a Branciari b-metric space with a constant $s \geq 1$.

Convergent sequence, Cauchy sequence, completeness and continuity on Branciari b-metric space are defined as follows.

Definition 1.4. [5] Let (X, d) be a Branciari b-metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then

1. A sequence $\{x_n\} \subset X$ is said to converge to a point $x \in X$ if, for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n > n_0$. The convergence is also represented as
$$\lim_{n\to\infty} x_n = x \text{ or } x_n \to x \text{ as } n \to \infty.$$
2. A sequence $\{x_n\} \subset X$ is said to be a Cauchy sequence if, for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_{n+p}) < \varepsilon$ for all $n > n_0$, $p > 0$ or equivalently, if $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0$ for all $p > 0$.
3. (X, d) is said to be a complete Branciari b-metric space if every Cauchy sequence in X converges to some $x \in X$.
4. A mapping $T : X \to X$ on is said to be continuous with respect to the Branciari b-metric d if, for any sequence $\{x_n\} \subset X$ which converges to some $x \in X$, that is $\lim\limits_{n\to\infty} d(x_n, x) = 0$ we have $\lim\limits_{n\to\infty} d(Tx_n, Tx) = 0$.

One should be careful when working with the Branciari and Branciari b-metric spaces due to some of their properties listed below.

Remark 1.5. Let (X, d) be a Branciari or Branciari b-metric space.

1. If we denote an open ball of radius r centered at $x \in X$ as
$$B_r(x, r) = \{y \in X : |d(x, y) < r\},$$
such an open ball in (X, d) is not always an open set.
2. If $\mathcal{T}$ is the collection of all subsets $\mathcal{Y}$ of X such that for each $y \in \mathcal{Y}$ there exist $r > 0$ with $B_r(y) \subseteq \mathcal{Y}$, then $\mathcal{T}$ defines a topology for (X, d), which is not necessarily Hausdorff.
3. The limit of a convergent sequence $\{x_n\} \in X$ is not necessarily unique.
4. A convergent sequence in X is not necessarily a Cauchy sequence.
5. Branciari or Branciari b-metric is not necessarily continuous.

All these drawbacks are illustrated in the following example inspired by [5].

Example 1.6. [5] Let $A = \left\{\frac{1}{n}, n \in \mathbb{N}\right\}$, $B = \{0, 3\}$ and $X = A \cup B$. Define the function $d(x, y) : X \times X \to [0, \infty)$ such that $d(x, y) = d(y, x)$ in the following way.

$$d(x, y) = \begin{cases} 0 & \text{if} \quad x = y, \\ 4 & \text{if} \quad x, y \in A, \\ \dfrac{1}{n} & \text{if} \quad x \in A, y \in B, \\ 2 & \text{if} \quad x, y \in B. \end{cases}$$

It is not difficult to see that the function $d(x, y)$ is not a metric, not a b-metric, not a Branciary metric but only a Branciari b-metric with $s = 2$. It is also clear that

$$\lim_{n\to\infty} d(\frac{1}{2n}, 0) = \lim_{n\to\infty} \frac{1}{2n} = 0,$$

and

$$\lim_{n\to\infty} d(\frac{1}{2n}, 3) = \lim_{n\to\infty} \frac{1}{2n} = 0,$$

that is, the sequence $\{\frac{1}{2n}\}$ has two different limits, the numbers 0 and 3.

In addition, the sequence $\{\frac{1}{2n}\}$ is convergent, but not a Cauchy sequence because

$$\lim_{p\to\infty} d(x_n, x_{n+p}) = \lim_{p\to\infty} d(\frac{1}{2n}, \frac{1}{2n+2p}) = \lim_{n\to\infty} 4 = 4.$$

Finally, note that the open set $B_1(\frac{1}{3})$ contains 0, that is $B_1(\frac{1}{3}) = \{0, 3, \frac{1}{3}\}$, but there is no positive r for which $B_r(0) \subset B_1(\frac{1}{3})$.

Therefore, when working on Branciari metric space, we need the following property stated in proved in [10].

Proposition 1.7. *[10] Let $\{x_n\}$ be a Cauchy sequence in a Branciari metric space (X, d) such that $\lim_{n\to\infty} d(x_n, x) = 0$, where $x \in X$. Then $\lim_{n\to\infty} d(x_n, y) = d(x, y)$, for all $y \in X$. In particular, the sequence $\{x_n\}$ does not converge to y if $y \neq x$.*

Remark 1.8. The Proposition 1.7 is valid if we replace Branciari metric space by a Branciari b-metric space.

Geraghty type contraction mappings have been introduced by Geraghty [6] who defined a class $\mathcal{F}$ of functions $\beta : [0, \infty) \to [0, 1)$ satisfying

$$\lim_{n\to\infty} \beta(t_n) = 1 \text{ implies } \lim_{n\to\infty} t_n = 0,$$

and with the help of these functions defined contraction mappings in the following manner.

Let (X, d) be a metric space and let $T : X \to X$ be a mapping satisfying

$$d(Tx, Ty) \le \beta(d(x, y))d(x, y), \tag{1.1}$$

for all $x, y \in X$ and some function $\beta \in \mathcal{F}$. He proved the existence and uniqueness of fixed points of such contractions on metric spaces.

In the context of b-metric spaces, Geraghty type contractions have been modified as follows [7]. Let $\mathcal{F}_s$ be the class of functions $\beta : [0, \infty) \to [0, \frac{1}{s})$ for which

$$\lim_{n\to\infty} \beta(t_n) = \frac{1}{s} \text{ implies } \lim_{n\to\infty} t_n = 0, \tag{1.2}$$

holds for some $s \ge 1$. On a b-metric space (X, d) with a constant $s \ge 1$ Geraghty type contraction is a self mapping $T : X \to X$ satisfying

$$d(Tx, Ty) \le \beta(d(x, y))d(x, y), \tag{1.3}$$

for all $x, y \in X$ and some function $\beta \in \mathcal{F}_s$.

As examples of functions from the class $\mathcal{F}_s$ we can give the following functions.

Example 1.9.

The function $\beta : [0, \infty) \to [0, \frac{1}{s})$ defined as $\beta(t) = \frac{\exp(-t)}{s}$ for some $s \ge 1$is in the class $\mathcal{F}_s$.

The function $\beta : [0, \infty) \to [0, \frac{1}{s})$ defined as $\beta(t) = \frac{1}{s(1+t^2)}$ is in the class $\mathcal{F}_s$.

Finally, we recall the concept of α-admissible mappings defined by Samet et al [12].

Definition 1.10. A mapping $T : X \to X$ is called α-admissible if for all $x, y \in X$ we have

$$\alpha(x, y) \ge 1 \Rightarrow \alpha(Tx, Ty) \ge 1, \tag{1.4}$$

where $\alpha : X \times X \to [0, \infty)$ is a given function.

2. Geraghty contractions on Branciari *b*-metric spaces

In many recent publications on fixed point on b-metric, quasi b-metric, Branciari b-metric, b-metric like spaces etc., the authors modify the contractive condition and the auxiliary functions involved in these conditions by taking into account the constant $s \ge 1$ of the space. In this sense, the Banach contractive condition on b-metric and related spaces becomes

$$d(Tx, Ty) \le kd(x, y), \text{ for all } x, y \in X$$

where $0 < k < \frac{1}{s}$.

In this paper, we deal with contractions of Geraghty type on Branciari b-metric spaces.

Definition 2.1. Let (X,d) be a Branciari b-metric space with a constant $s \geq 1$ and let $\alpha : X \times X \to [0,\infty)$ and $\beta \in \mathcal{F}_s$ be two given functions. A generalized Geraghty type α-admissible contractive mapping $T : X \to X$ is of type (I) if it is α-admissible and satisfies

$$\alpha(x,y)d(Tx,Ty) \leq \beta\left(M(x,y)\right)M(x,y), \text{ for all } x,y \in X, \tag{2.1}$$

where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}.$$

We will first prove an existence theorem for fixed point of the class of contractive mappings given in Definition 2.1.

Theorem 2.2. *Let (X,d) be a complete Branciari b-metric space with a constant $s \geq 1$ and $\alpha : X \times X \to [0,\infty)$ and $\beta \in \mathcal{F}_s$ be two given functions. Let $T : X \to X$ be a continuous α-admissible mapping satisfying*

$$\alpha(x,y)d(Tx,Ty) \leq \beta\left(M(x,y)\right)M(x,y), \textit{ for all } x,y \in X, \tag{2.2}$$

where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}.$$

Assume that there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$ and $\alpha(x_0,T^2x_0) \geq 1$. Then T has a fixed point.

Proof. Choosing $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$ and $\alpha(x_0,T^2x_0) \geq 1$ we define the sequence $\{x_n\}$ as

$$x_{n+1} = Tx_n \text{ for } n \in \mathbb{N}.$$

Suppose that $x_n \neq x_{n+1}$ for all $n \geq 0$. Otherwise, for some $k \in \mathbb{N}$ we would have $x_k = x_{k+1} = Tx_k$, that is, x_k would be a fixed point of T and the proof would be completed.

Since T is α-admissible, from $\alpha(x_0,Tx_0) \geq 1$ we have

$$\alpha(x_0,x_1) = \alpha(x_0,Tx_0) \geq 1 \Rightarrow \alpha(Tx_0,Tx_1) = \alpha(x_1,x_2) \geq 1,$$

and inductively,

$$\alpha(x_n,x_{n+1}) \geq 1, \text{ for all } n \in \mathbb{N}. \tag{2.3}$$

Also, from the condition $\alpha(x_0,T^2x_0) \geq 1$ we have

$$\alpha(x_0,x_2) = \alpha(x_0,T^2x_0) \geq 1 \Rightarrow \alpha(Tx_0,Tx_2) = \alpha(x_1,x_3) \geq 1,$$

and hence,

$$\alpha(x_n,x_{n+2}) \geq 1, \text{ for all } n \in \mathbb{N}. \tag{2.4}$$

We define the sequences $\{d_n\}$ and $\{e_n\}$ as

$$d_n = d(x_{n-1},x_n), \quad e_n = d(x_{n-1},x_{n+1}). \tag{2.5}$$

We will prove that both the sequence $\{d_n\}$ and $\{e_n\}$ converge to 0, that is,

$$\lim_{n\to\infty} d(x_{n-1},x_n) = \lim_{n\to\infty} d(x_{n-1},x_{n+1}) = 0 \tag{2.6}$$

Regarding (2.3) and the fact that $0 \leq \beta(t) < \dfrac{1}{s}$, the contractive condition (2.23) with $x = x_n$ and $y = x_{n+1}$ becomes

$$\begin{aligned} d(x_n,x_{n+1}) &= d(Tx_{n-1},Tx_n) \\ &\leq \alpha(x_{n-1},x_n)d(Tx_{n-1},Tx_n) \\ &\leq \beta(M(x_{n-1},x_n))M(x_{n-1},x_n) < \frac{1}{s}M(x_{n-1},x_n), \end{aligned} \tag{2.7}$$

for all $n \geq 1$, where

$$\begin{aligned} M(x_{n-1},x_n) &= \max\{d(x_{n-1},x_n), d(x_{n-1},Tx_{n-1}), d(x_n,Tx_n)\} \\ &= \max\{d(x_{n-1},x_n), d(x_{n-1},x_n), d(x_n,x_{n+1})\} \\ &= \max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\}. \end{aligned}$$

Suppose that $M(x_{n-1},x_n) = d(x_n,x_{n+1})$ for some $n \geq 1$. Then we have

$$d(x_n,x_{n+1}) \leq \beta(d(x_n,x_{n+1}))d(x_n,x_{n+1}) < \frac{1}{s}d(x_n,x_{n+1}),$$

which is not possible. Therefore, for all $n \geq 1$ $M(x_{n-1},x_n) = d(x_{n-1},x_n)$. In this case, the inequality (2.7) implies

$$\begin{aligned} d(x_n,x_{n+1}) &\leq \beta(d(x_{n-1},x_n))d(x_{n-1},x_n) < \tfrac{1}{s}d(x_{n-1},x_n) \\ &\leq d(x_{n-1},x_n), \text{ for all } n \geq 1. \end{aligned} \tag{2.8}$$

In other words, the sequence $\{d_n\} = \{d(x_{n-1},x_n)\}$ is positive and decreasing and hence, converges to some $d \geq 0$. If we take limit as $n \to \infty$ in (2.8) we obtain

$$d = \lim_{n\to\infty} d_{n+1} \leq \lim_{n\to\infty} \beta(d_n)d_n = d \lim_{n\to\infty} \beta(d_n) \leq \frac{1}{s}d. \tag{2.9}$$

This implies $\lim_{n\to\infty} \beta(d_n) = \dfrac{1}{s}$ and hence, by (1.2),

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} d(x_{n-1},x_n) = 0. \tag{2.10}$$

On the other hand, we observe that repeated application of (2.8) leads to

$$d_{n+1} < \frac{1}{s}d_n < \frac{1}{s^2}d_{n-1} < \cdots < \frac{1}{s^{n+1}}d_0. \tag{2.11}$$

Now, taking into account (2.4), we substitute $x = x_{n-1}$ and $x = x_{n+1}$ in (2.23). This yields

$$\begin{aligned} d(x_n,x_{n+2}) &= d(Tx_{n-1},Tx_{n+1}) \\ &\leq \alpha(x_{n-1},x_{n+1})d(Tx_{n-1},Tx_{n+1}) \\ &\leq \beta(M(x_{n-1},x_{n+1}))M(x_{n-1},x_{n+1}) < \frac{1}{s}M(x_{n-1},x_{n+1}), \end{aligned} \tag{2.12}$$

for all $n \geq 1$, where

$$\begin{aligned} M(x_{n-1},x_{n+1}) &= \max\{d(x_{n-1},x_{n+1}), d(x_{n-1},Tx_{n-1}), d(x_{n+1},Tx_{n+1})\} \\ &= \max\{d(x_{n-1},x_{n+1}), d(x_{n-1},x_n), d(x_{n+1},x_{n+2})\}, \end{aligned} \tag{2.13}$$

Regarding (2.8), the maximum $M(x_{n-1},x_{n+1})$ is either $d(x_{n-1},x_{n+1})$ or $d(x_{n-1},x_n)$, that is, either e_n or d_n. From the inequality (2.12) we have

$$e_{n+1} = d(x_n,x_{n+2}) < \frac{1}{s}M(e_n) = \frac{1}{s}\max\{e_n,d_n\} \tag{2.14}$$

for all $n \in \mathbb{N}$. In addition, from (2.8) we have

$$d_{n+1} < d_n \leq \max\{e_n,d_n\},$$

from which we deduce

$$\max\{e_{n+1},d_{n+1}\} \leq \max\{e_n,d_n\} \text{ for all } n \geq 1,$$

that is, the sequence $\{\max\{e_n, d_n\}\}$ is non increasing and hence, it converges to some $l \geq 0$. Assume that $l > 0$. Taking into account (2.10) we obtain

$$\begin{aligned} l &= \lim_{n\to\infty} \max\{e_n, d_n\} = \max\{\lim_{n\to\infty} e_n, \lim_{n\to\infty} d_n\} \\ &= \max\{\lim_{n\to\infty} e_n, 0\} = \lim_{n\to\infty} e_n. \end{aligned}$$

On the other hand, letting $n \to \infty$ in (2.14) we conclude

$$l = \lim_{n\to\infty} e_{n+1} < \lim_{n\to\infty} \max\{e_n, d_n\} = l,$$

which contradicts the assumption $l > 0$. Hence, $l = 0$, and then we have

$$\lim_{n\to\infty} e_n = \lim_{n\to\infty} d(x_{n-1}, x_{n+1}) = 0. \tag{2.15}$$

Next, we will prove that $x_n \neq x_m$ for all $n \neq m$. Assume that $x_n = x_m$ for some $m, n \in \mathbb{N}$ with $n \neq m$. By the initial assumption, we have $d(x_n, x_{n+1}) > 0$ for each $n \in \mathbb{N}$. Without loss of generality we may take $m > n + 1$. The assumption $x_n = x_m$ implies

$$d(x_n, Tx_n) = d(x_m, Tx_m).$$

Recalling the inequality (2.7) we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(x_n, Tx_n) = d(x_m, Tx_m) \\ &= d(Tx_{m-1}, Tx_m) \leq \alpha(x_{m-1}, x_m) d(Tx_{m-1}, Tx_m) \\ &\leq \beta(M(x_{m-1}, x_m)) M(x_{m-1}, x_m) < \frac{1}{s} M(x_{m-1}, x_m), \end{aligned} \tag{2.16}$$

where

$$\begin{aligned} M(x_{m-1}, x_m) &= \max\{d(x_{m-1}, x_m), d(x_{m-1}, Tx_{m-1}), d(x_m, Tx_m)\} \\ &= \max\{d(x_{m-1}, x_m), d(x_{m-1}, x_m), d(x_m, x_{m+1})\} \\ &= \max\{d(x_{m-1}, x_m), d(x_m, x_{m+1})\} = d(x_{m-1}, x_m), \end{aligned} \tag{2.17}$$

because of (2.8). Then we have,

$$d(x_m, x_{m+1}) < \frac{1}{s} d(x_{m-1}, x_m) \leq d(x_{m-1}, x_m),$$

for all $m > n + 1$. Continuing the process we conclude,

$$d(x_m, x_{m+1}) < d(x_{m-1}, x_m) < d(x_{m-1}, x_m) < \ldots < d(x_n, x_{n+1}), \tag{2.18}$$

which contradicts the assumption $x_n = x_m$ for some $m \neq n$. Therefore, our initial assumption is incorrect and we should have $x_n \neq x_m$ for all $m \neq n$.

Now we will prove that $\{x_n\}$ is a Cauchy sequence, that is,

$$\lim_{n\to\infty} d(x_n, x_{n+k}) = 0, \text{ for all } k \in \mathbb{N}. \tag{2.19}$$

Notice that (2.19) holds for $k = 1$ and $k = 2$ due to (2.10) and (2.15). Therefore, we assume that $k \geq 3$. We consider separately the cases with odd and even $k \in \mathbb{N}$.

Case 1. Let $k = 2m + 1$ where $m \geq 1$. We have $x_l \neq x_s$ for all $l \neq s$ and $x_l \neq x_{l+1}$ for all $l \geq 0$, so that we can apply repeatedly the condition 3. in Definition 1.3 which implies

$$\begin{aligned} d(x_n, x_{n+k}) &= d(x_n, x_{n+2m+1}) \leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2m+1})] \\ &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] \\ &+ s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+2m+1})] \\ &\vdots \\ &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4})] \\ &+ s^3[d(x_{n+4}, x_{n+5}) + d(x_{n+5}, x_{n+6})] + \ldots + s^{m+1}[d(x_{n+2m}, x_{n+2m+1})] \\ &\leq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^3 d(x_{n+2}, x_{n+3}) \\ &+ \ldots + s^{n+2m-1} d(x_{n+2m}, x_{n+2m+1}). \end{aligned}$$

Then, by the inequality (2.11) we conclude

$$\begin{aligned} d(x_n, x_{n+k}) &\leq \frac{1}{s^{n-1}} d(x_0, x_1) + \frac{1}{s^n} d(x_0, x_1) + \ldots + \frac{1}{s^{n+2m}} d(x_0, x_1) \\ &= d(x_0, x_1) \left[\sum_{k=0}^{n+2m} \frac{1}{s^k} - \sum_{k=0}^{n-2} \frac{1}{s^k} \right] \\ &= d(x_0, x_1) \left[\frac{s^{n+2m+1} - 1}{s^{n+2m}(s-1)} - \frac{s^{n-1} - 1}{s^{n-2}(s-1)} \right]. \end{aligned}$$

Letting $n \to \infty$ in the last inequality we obtain

$$0 \leq \lim_{n\to\infty} d(x_n, x_{n+k}) \leq \lim_{n\to\infty} d(x_0, x_1) \left[\frac{s^{n+2m+1} - 1}{s^{n+2m}(s-1)} - \frac{s^{n-1} - 1}{s^{n-2}(s-1)} \right] = 0. \tag{2.20}$$

Case 2. Let $k = 2m$ where $m \geq 2$. Again, repeated application of the inequality 3. in Definition 1.3 yields

$$\begin{aligned} d(x_n, x_{n+k}) &= d(x_n, x_{n+2m}) \leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2m})] \\ &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] \\ &+ s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+2m})] \\ &\vdots \\ &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4})] \\ &+ \ldots + s^{m-1}\left[d(x_{n+2m-4}, x_{n+2m-3}) + d(x_{n+2m-3}, x_{n+2m-2})\right. \\ &+ \left. d(x_{n+2m-2}, x_{n+2m})\right] \\ &\leq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^3 d(x_{n+2}, x_{n+3}) \\ &+ \ldots + s^{n+2m-3} d(x_{n+2m-3}, x_{n+2m-2}) + s^{m-1} d(x_{n+2m-2}, x_{n+2m}). \end{aligned}$$

By the inequality in (2.11), we have

$$\begin{aligned} d(x_n, x_{n+k}) &\leq \frac{1}{s^{n-1}} d(x_0, x_1) + \frac{1}{s^n} d(x_0, x_1) + \ldots + \frac{1}{s^{n+2m-2}} d(x_0, x_1) \\ &+ s^{m-1} d(x_{n+2m-2}, x_{n+2m}) \\ &= d(x_0, x_1) \left[\sum_{k=0}^{n+2m-2} \frac{1}{s^k} - \sum_{k=0}^{n-2} \frac{1}{s^k} \right] + s^{m-1} d(x_{n+2m-2}, x_{n+2m}) \\ &= d(x_0, x_1) \left[\frac{s^{n+2m-1} - 1}{s^{n+2m}(s-1)} - \frac{s^{n-1} - 1}{s^{n-2}(s-1)} \right] \\ &+ s^{m-1} d(x_{n+2m-2}, x_{n+2m}). \end{aligned} \tag{2.21}$$

From (2.15) we have $\lim_{n\to\infty} s^{m-1} d(x_{n+2m-2}, x_{n+2m}) = 0$ and hence, letting $n \to \infty$ in (2.21) we deduce

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} d(x_n, x_{n+k}) \\ &= \lim_{n\to\infty} \left\{ d(x_0, x_1) \left[\frac{s^{n+2m-1} - 1}{s^{n+2m}(s-1)} - \frac{s^{n-1} - 1}{s^{n-2}(s-1)} \right] + s^{m-1} d(x_{n+2m-2}, x_{n+2m}) \right\} \\ &= 0. \end{aligned}$$

As a result, for any $k \in \mathbb{N}$, we have

$$\lim_{n\to\infty} d(x_n, x_{n+k}) = 0,$$

that is, the sequence $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete Branciari b-metric space, there exists $u \in X$ such that

$$\lim_{n\to\infty} d(x_n, u) = 0. \tag{2.22}$$

Since T is a continuous mapping, then, from (2.22) we have

$$\lim_{n\to\infty} d(Tx_n, Tu) = \lim_{n\to\infty} d(x_{n+1}, Tu) = 0,$$

that is, the sequence $\{x_n\}$ converges to Tu. Then the Proposition 1.7 implies that $Tu = u$, that is, u is a fixed point of T. □

Adding an additional condition to the statement of the Theorem 2.2, we can prove the uniqueness of the fixed point.

Theorem 2.3. *Let all the conditions of Theorem 2.2 hold. Assume that for every pair x and y of fixed points of T, $\alpha(x,y) \geq 1$. Then the fixed point of the mapping T is unique.*

Proof. Since the existence of a fixed point is already proved in Theorem 2.2, we need to prove only the uniqueness. Assume that the map T has two distinct fixed points, say $x, y \in X$, such that $x \neq y$, or $d(x,y) > 0$. We put these two points in the contractive condition (2.23) and use the fact that $\alpha(x,y) \geq 1$ which gives

$$d(x,y) = \alpha(x,y)d(Tx,Ty) \leq \beta(M(x,y))M(x,y) < \frac{1}{s}M(x,y),$$

where,

$$M(x,y) = \max\{d(x,y), d(Tx,x), d(Ty,y)\} = d(x,y).$$

This implies

$$d(x,y) < \frac{1}{s}d(x,y),$$

which is a contradiction and hence, $d(x,y) = 0$, or, $x = y$. This completes the proof of the uniqueness. □

In the next theorem we replace the continuity of the mapping T by the so-called α-regularity of the Branciari b-metric space.

Theorem 2.4. *Let (X,d) be a complete Branciari b-metric space with a constant $s \geq 1$ and $\alpha : X \times X \to [0,\infty)$ and $\beta \in \mathcal{F}_s$ be two given functions. Let $T : X \to X$ be an α-admissible mapping satisfying*

$$\alpha(x,y)d(Tx,Ty) \leq \beta\left(M(x,y)\right)M(x,y), \text{ for all } x,y \in X, \tag{2.23}$$

where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}.$$

Suppose also that

(*i*) *There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$.*
(*ii*) *For any sequence $\{x_n\} \subset X$ such that $\lim_{n\to\infty} d(x_n,x) = 0$ and satisfying $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, we have $\alpha(x_n,x) \geq 1$ for all $n \in \mathbb{N}$.*
(*iii*) *For every pair x and y of fixed points of T, $\alpha(x,y) \geq 1$.*

Then T has a unique fixed point.

Proof. Taking $x_0 \in X$ as the element satisfying the condition (i), we construct the sequence $\{x_n\}$ as usual, that is, $x_n = Tx_{n-1}$, for $n \in \mathbb{N}$.

The convergence of this sequence can be shown exactly as in the proof of Theorem 2.2.

Let u be the limit of $\{x_n\}$, that is,

$$\lim_{n\to\infty} d(x_n,u) = 0.$$

We will show that u is a fixed point of T. For the sequence $\{x_n\}$ which converges to u we have from (2.3) that $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N}_0$. Then, the condition (ii) in the statement of the theorem implies that

$$\alpha(x_n,u) \geq 1, \text{ for all } n \in \mathbb{N}_0.$$

We write the contractive inequality (2.23) for x_n and u, that is,

$$\begin{aligned} d(x_{n+1}, Tu) &= d(Tx_n, Tu) \leq \alpha(x_n, u) d(Tx_n, Tu) \\ &\leq \beta(M(x_n, u)) M(x_n, u) < \frac{1}{s} M(x_n, u), \end{aligned} \tag{2.24}$$

where

$$M(x_n, u) = \max\{d(x_n, u), d(x_n, x_{n+1}), d(u, Tu)\}.$$

Since the sequence $\{x_n\}$ is Cauchy and $\lim_{n\to\infty} d(x_n, u) = 0$, by the Proposition 1.7 we have,

$$\lim_{n\to\infty} d(x_{n+1}, Tu) = d(u, Tu). \tag{2.25}$$

On the other hand,

$$\lim_{n\to\infty} M(x_n, u) = \lim_{n\to\infty} \max\{d(x_n, u), d(x_n, x_{n+1}), d(u, Tu)\} = d(u, Tu). \tag{2.26}$$

Therefore, by letting $n \to \infty$ in (2.24) and regarding (2.25) and (2.26) we obtain

$$d(u, Tu) = \lim_{n\to\infty} d(x_{n+1}, Tu) < \frac{1}{s} \lim_{n\to\infty} M(x_n, u) = \frac{1}{s} d(u, Tu). \tag{2.27}$$

This yields $d(u, Tu) = 0$, hence, u is a fixed point of T. We skip the uniqueness proof since it is identical to the proof of Theorem 2.3. □

We next define another class of Geraghty type mappings on Branciari b-metric spaces.

Definition 2.5. Let (X, d) be a Branciari b-metric space with a constant $s \geq 1$ and let $\alpha : X \times X \to [0, \infty)$ and $\beta \in \mathcal{F}_s$ be two given functions. A generalized Geraghty type α-admissible contractive mapping $T : X \to X$ is of type (II) if it is α-admissible and satisfies

$$\alpha(x, y) d(Tx, Ty) \leq \beta(N(x, y)) N(x, y), \text{ for all } x, y \in X, \tag{2.28}$$

where

$$N(x, y) = \max\{d(x, y), \frac{1}{2s}[d(x, Tx) + d(y, Ty)]\}.$$

Remark 2.6. For all $x, y \in X$ the relation $d(x, y) \leq N(x, y) \leq M(x, y)$ holds.

An existence-uniqueness theorem for the class of contraction mappings introduced in Definition 2.5 is stated below. We observe that the proof of this theorem is trivial once we take into account the Remark 2.6.

Theorem 2.7. *Let (X, d) be a complete Branciari b-metric space with a constant $s \geq 1$ and let $\alpha : X \times X \to [0, \infty)$ and $\beta \in \mathcal{F}_s$ be two given functions. Let $T : X \to X$ be an α-admissible mapping satisfying*

$$\alpha(x, y) d(Tx, Ty) \leq \beta(N(x, y)) N(x, y), \text{ for all } x, y \in X,$$

where

$$N(x, y) = \max\{d(x, y), \frac{1}{2s}[d(x, Tx) + d(y, Ty)]\}.$$

Suppose also that

- *(i) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2 x_0) \geq 1$.*
- *(ii) Either T is continuous or, for any sequence $\{x_n\} \subset X$ with $\lim_{n\to\infty} d(x_n, x) = 0$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, we have $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*
- *(iii) For every pair x and y of fixed points of T, $\alpha(x, y) \geq 1$.*

Then T has a unique fixed point.

By the Remark 2.6 we also easily conclude the following existence-uniqueness result.

Theorem 2.8. *Let (X, d) be a complete Branciari b-metric space with a constant $s \geq 1$ and let $\alpha : X \times X \to [0, \infty)$ and $\beta \in \mathcal{F}_s$ be two given functions. Let $T : X \to X$ be an α-admissible mapping satisfying*

$$\alpha(x,y)d(Tx,Ty) \leq \beta\left(d(x,y)\right)d(x,y), \text{ for all } x,y \in X.$$

Suppose also that

(i) *There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$.*
(ii) *Either T is continuous or for any sequence $\{x_n\} \subset X$ with $\lim_{n\to\infty} d(x_n, x) = 0$ and satisfying $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, we have $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*
(iii) *For every pair x and y of fixed points of T, $\alpha(x, y) \geq 1$.*

Then T has a unique fixed point.

3. Consequences

In this section, we give some consequences of the Theorem2.2. First, we notice that a Branciari b-metric spaces with $s = 1$ is simply a Branciari metric space.

Corollary 3.1. *Let (X, d) be a complete Branciari metric space and let $\alpha : X \times X \to [0, \infty)$ and $\beta \in \mathcal{F}$ be two given functions. Let $T : X \to X$ be an α-admissible mapping satisfying*

$$\alpha(x,y)d(Tx,Ty) \leq \beta\left(M(x,y)\right)M(x,y), \text{ for all } x,y \in X,$$

where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}.$$

Suppose also that

(i) *There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$.*
(ii) *Either T is continuous or for any sequence $\{x_n\} \subset X$ with $\lim_{n\to\infty} d(x_n, x) = 0$ and satisfying $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, we have $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*
(iii) *For every pair x and y of fixed points of T, $\alpha(x, y) \geq 1$.*

Then T has a unique fixed point.

Corollary 3.2. *Let (X, d) be a complete Branciari metric space and let $\alpha : X \times X \to [0, \infty)$ and $\beta \in \mathcal{F}$ be two given functions. Let $T : X \to X$ be an α-admissible mapping satisfying*

$$\alpha(x,y)d(Tx,Ty) \leq \beta\left(N(x,y)\right)N(x,y), \text{ for all } x,y \in X,$$

where

$$N(x,y) = \max\{d(x,y), \frac{1}{2}[d(x,Tx), d(y,Ty)]\}.$$

Suppose also that

(i) *There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$.*
(ii) *Either T is continuous or for any sequence $\{x_n\} \subset X$ with $\lim_{n\to\infty} d(x_n, x) = 0$ and satisfying $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, we have $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*
(iii) *For every pair x and y of fixed points of T, $\alpha(x, y) \geq 1$.*

Then T has a unique fixed point.

Corollary 3.3. *Let (X,d) be a complete Branciari metric space and let $\alpha : X \times X \to [0,\infty)$ and $\beta \in \mathcal{F}$ be two given functions. Let $T : X \to X$ be an α-admissible mapping satisfying*

$$\alpha(x,y)d(Tx,Ty) \leq \beta\left(d(x,y)\right)d(x,y), \text{ for all } x,y \in X.$$

Suppose also that

(i) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$.
(ii) Either T is continuous or for any sequence $\{x_n\} \subset X$ with $\lim_{n\to\infty} d(x_n,x) = 0$ and satisfying $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, we have $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.
(iii) For every pair x and y of fixed points of T, $\alpha(x,y) \geq 1$.

Then T has a unique fixed point.

Also the choice $\alpha(x,y) = 1$ gives fixed point results for self mappings on Branciari b-metric spaces. We list some of these consequences below.

Corollary 3.4. *Let (X,d) be a complete Branciari b-metric space with a constant $s \geq 1$ and let $\beta \in \mathcal{F}_s$ be a given function. Let $T : X \to X$ be a continuous self mapping satisfying*

$$d(Tx,Ty) \leq \beta\left(M(x,y)\right)M(x,y), \text{ for all } x,y \in X,$$

where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}.$$

Then T has a unique fixed point.

Corollary 3.5. *Let (X,d) be a complete Branciari b-metric space with a constant $s \geq 1$ and let $\beta \in \mathcal{F}_s$ be a given function. Let $T : X \to X$ be a continuous self mapping satisfying*

$$d(Tx,Ty) \leq \beta\left(N(x,y)\right)N(x,y), \text{ for all } x,y \in X,$$

where

$$N(x,y) = \max\{d(x,y), \frac{1}{2s}[d(x,Tx), d(y,Ty)]\}.$$

Then T has a unique fixed point.

Corollary 3.6. *Let (X,d) be a complete Branciari b-metric space with a constant $s \geq 1$ and let $\beta \in \mathcal{F}_s$ be a given function. Let $T : X \to X$ be a continuous self mapping satisfying*

$$d(Tx,Ty) \leq \beta\left(d(x,y)\right)d(x,y), \text{ for all } x,y \in X,$$

Then T has a unique fixed point.

Finally, we give the following example which illustrates the theoretical results discussed above.

Example 3.7. Let $X = A \cup B$ where $A = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}\right\}$ and $B = [1,2]$. Define the function $d : X \times X \to [0,\infty)$ such that $d(x,y) = d(y,x)$ as follows.

For $x,y \in B$, or $x \in A$ and $y \in B$, $d(x,y) = |x-y|$ and

$$d(\tfrac{1}{2}, \tfrac{1}{4}) = d(\tfrac{1}{6}, \tfrac{1}{8}) = 0.2$$

$$d(\tfrac{1}{2}, \tfrac{1}{6}) = d(\tfrac{1}{4}, \tfrac{1}{6}) = d(\tfrac{1}{4}, \tfrac{1}{8}) = 0.1$$

$$d(\tfrac{1}{2}, \tfrac{1}{8}) = 1$$

Clearly, d is a Branciari b-metric with $s = \dfrac{10}{3}$.

Let $T : X \to X$ be defined as

$$Tx = \begin{cases} \frac{x}{8} & \text{if} \quad x \in B \\ \\ \frac{1}{6} & \text{if} \quad x \in A \end{cases}$$

We see that

$$d(Tx, Ty = \begin{cases} 0 & \text{if} \quad x, y \in A \\ 0.2 & \text{if} \quad x \in A, y = 1 \\ 0.1 & \text{if} \quad x \in A, y = 2 \\ 0.1 & \text{if} \quad x, y \in B \end{cases}.$$

Then, for all $x, y \in X$ the mapping T satisfies the condition

$$d(Tx, Ty) \leq \frac{3}{20} d(x, y) = \frac{1/2}{10/3} d(x, y),$$

Hence, the conditions of the Corollary 3.6 hold with $\beta(t) = \dfrac{1}{2s} = \dfrac{3}{20}$ and T has a unique fixed point which is $x = \frac{1}{6}$.

4. Concluding Remarks

The general structure of the mappings discussed in this paper makes it possible to deduce many particular existence and uniqueness results.

As it was already mentioned, by taking $s = 1$ and/or $\alpha(x, y) = 1$ in all the theorems and corollaries, various existing results on Branciari b-metric and Branciari metric spaces can be obtained.

On the other hand, it should be noticed that by choosing the function α in the definition of α-admissible mappings in a particular way, it is possible to obtain existence and uniqueness results for maps defined on partially ordered Branciari or Branciari b-metric space.

Indeed, if we define a partial ordering $\preceq$ on a Branciari b-metric space (X, d) and take $T : X \to X$ as an increasing mapping, we can easily proof the following fixed point theorem.

Theorem 4.1. *Let $(X, d, \preceq)$ be a complete Branciari b-metric space with a constant $s \geq 1$ on which a partial ordering $\preceq$ is defined. Suppose that $T : X \to X$ is an increasing mapping satisfying the following:*

(i)

$$d(Tx, Ty) \leq \beta(M(x, y))M(x, y),$$

for all x, y in X with $x \preceq y$ and some function $\beta \in \mathcal{F}_s$ where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}.$$

(ii) There exists $x_0 \in X$ such that $x_0 \preceq Tx_0$ and $x_0 \preceq T^2 x_0$.

(iii) Either T is continuous or, for any increasing sequence $\{x_n\} \in X$ which converges to x we have $x_n \preceq x$ for all $n \in \mathbb{N}$.

Then T has a fixed point. If, in addition any two fixed points of T are comparable, that is, $x \preceq y$ or $y \preceq x$, then the fixed point of T is unique.

Proof. Observe that all the conditions of Theorems 2.2, 2.3 and 2.4 hold if we choose the function α as

$$\alpha(x, y) = \begin{cases} 1 & \text{if} \quad x \preceq y \text{ or } y \preceq x \\ 0 & \text{if} \quad \text{otherwise} \end{cases}.$$

Then, the mapping T has a unique fixed point. □

Finally, we note that all the consequent results of Theorems 2.2, 2.3 and 2.4 can be written on Branciari b-metric spaces with a partial ordering and proved in a similar way.

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