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Finding the Fixed Points Inside Large Mapping Sets: Integral Equations

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Abstract

Let $xf(t,x) > 0$ for $x \neq 0$ and let $A(t-s)$ satisfy some classical properties yielding a nice resolvent. Using repeated application of a fixed point mapping and induction we develop an asymptotic formula showing that solutions of the Caputo equation

$$^cD^q x(t) = -f(t,x(t)), \quad 0 < q < 1, \quad x(0) \in \Re, \quad x(0) \neq 0,$$

and more generally of the integral equation

$$x(t) = x(0) - \int_0^t A(t-s)f(s,x(s))ds, x(0) \neq 0,$$

all satisfy $x(t) \to 0$ as $t \to \infty$.

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1. Introduction

In this paper $\Re$ denotes the set of real numbers and $(\mathcal{B}, \|\cdot\|)$ denotes the Banach space of bounded continuous functions $\phi : [0,\infty) \to \Re$ with the supremum norm.

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Integral equations of the form

$$x(t) = x(0) - \int_0^t A(t-s)f(s,x(s))ds$$

are found throughout applied mathematics and Caputo fractional differential equations under the "spring condition" that $xf(t,x) > 0$ for $x \neq 0$ and A satisfies properties parallel to those found in heat transfer problems. The study of such problems as found in the literature can be very challenging and it is certainly true that parts of the study are very deep. But the thesis here is that qualitative properties of solutions can be so similar to those of the elementary ordinary differential equation

$$x' = -f(t,x)$$

that simpler attacks may be very enlightening and fruitful.

Here is a sketch of our work. After a crucial transformation the natural mapping defined by the equation will map a closed ball in a Banach space $(\mathcal{B}, \|\cdot\|)$ into itself and there will be a fixed point. But that is very crude and we would like further properties and the location of that fixed point. If we try to impose additional conditions on the closed ball such as requiring all functions to tend to zero at infinity then the result fails because of problems with compactness. Here we discover a way out.

If P is the natural mapping defined by the integral equation and if M is the ball of radius $|x(0)|$ then $PM =: M_1 \subset M$ so $P : M_1 \to M$ and if P has a fixed point it will also reside in PM. In fact, P will have a fixed point and we seek its properties. It turns out that if we continue and repeat the mapping then using mathematical induction we can find an asymptotic formula of the fixed point and its limit as $t \to \infty$ is very simply calculated. In the next section we extend the introduction and add explicit details.

2. A sketch of the study

The vehicle for explaining the theory introduced here will be the scalar integral equation

$$x(t) = x(0) - \int_0^t A(t-s)f(s,x(s))ds \tag{2.1}$$

where $x(0) \in \Re$, $x(0) \neq 0$, $f\colon [0,\infty) \times \Re \to \Re$ is continuous, and A satisfies the following conditions found in Miller [9, p. 209]:

A1) The function $A \in C(0,\infty) \cap L^1(0,1)$.

A2) $A(t)$ is positive and nonincreasing for $t > 0$.

A3) For each $T > 0$ the function $A(t)/A(t+T)$ is nonincreasing in t for $0 < t < \infty$.

Under these conditions the resolvent equation

$$R(t) = A(t) - \int_0^t A(t-s)R(s)ds \tag{2.2}$$

has a continuous solution $R : (0,\infty) \to (0,\infty)$ satisfying

$$0 < R(t) \leq A(t) \tag{2.3}$$

for $t > 0$; the strict positivity is found in [8]. If $A \in L^1(0,\infty)$ and $\alpha = \int_0^\infty A(s)ds$ then

$$\int_0^\infty R(s)ds = \alpha(1+\alpha)^{-1} < 1 \tag{2.4}$$

while if $\int_0^\infty A(s)ds = \infty$ then

$$\int_0^\infty R(s)ds = 1. \tag{2.5}$$

Finally, there is a nonlinear variation of parameters formula [9, pp. 191-193] which we used in [2] to show that for every $J > 0$ (2.1) can be mapped into the equivalent equation

$$x(t) = x(0)\left[1 - \int_0^t R(s)ds\right] + \int_0^t R(t-s)\left[x(s) - \frac{f(s,x(s))}{J}\right]ds \tag{2.6}$$

and the mapping is reversible. There are more details given in [5]. R changes with $J > 0$, but (2.5) still holds and R is still positive.

Conditions A1)–A3) are not contrived, but rather appear widely in the literature. The Caputo fractional differential equation (see [6])

$$^{c}D^{q}x(t) = -f(t,x(t)), \quad x(0) \neq 0, \quad 0 < q < 1, \tag{2.7}$$

is known to invert [6, p. 86] as

$$x(t) = x(0) - \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s,x(s))ds \tag{2.8}$$

where Γ is the Euler gamma function. A selection of real-world problems modelled by (2.1) is found in [1], Section 5.

It is assumed that f satisfies what may be called the spring condition

$$x \neq 0 \implies xf(t,x) > 0. \tag{2.9}$$

In view of (2.7) we take our pattern from the ordinary differential equation

$$x'(t) = -f(t,x(t)), \quad x(0) \neq 0. \tag{2.10}$$

We can define a Liapunov function by

$$V(x) = x^2$$

so that if (2.10) has a solution, $x(t)$, then we can take the derivative of V along the solution using the chain rule and have

$$\frac{dV(x(t))}{dt} = 2x(t)[-f(t,x(t))] < 0 \tag{2.11}$$

when $x \neq 0$. Clearly the solution then satisfies

$$x^2(t) \leq x^2(0) \tag{2.12}$$

and if f is bounded away from zero for $x \neq 0$ then an integration of (2.11) drives $x(t)$ to zero.

Goal

Use the transformation and a fixed point map to show parallel properties for (2.1) transformed to (2.6) with a strengthened form of (2.9) holding.

3. The mappings

From now on our focus is on the transformed equation (2.6) which we now designate by

$$x(t) = x(0)\left[1 - \int_0^t R(s)ds\right] + \int_0^t R(t-s)\left[x(s) - \frac{f(s,x(s))}{J}\right]ds \tag{3.1}$$

with all the past continuity conditions on f and the conditions on R. But we will always assume that the integral of A is infinite so that we have

$$\int_0^\infty R(s)ds = 1. \tag{3.2}$$

Everything will be based on showing that the natural mapping defined by (3.1) will map the closed ball in $\mathcal{B}$

$$M = \{\phi : [0,\infty) \to \Re\colon |\phi(t)| \le |x(0)|\} \tag{3.3}$$

into itself and then successive mappings by P will send all fixed points to zero as $t \to \infty$. The proof rests on a repeated fixed point map, induction, and a very simple lemma which now follows.

Lemma 3.1. *Let R be the resolvent with $\int_0^\infty R(s)ds = 1$ and let $\phi : [0,\infty) \to [0,\infty)$ with $0 \le \phi(t) < 1$. Suppose that for each $\epsilon > 0$ there exists $T > 0$ such that $t \ge T \implies 1 - \epsilon \le \phi(t) < 1$. Then*

$$\int_0^t R(t-s)\phi(s)ds \to 1$$

as $t \to \infty$. In particular,

$$\int_0^t R(t-s)\int_0^s R(s-u)duds \to 1$$

as $t \to \infty$.

Proof. Clearly

$$\int_0^t R(t-s)\phi(s)ds < 1.$$

Now for an $\epsilon > 0$ and the corresponding T then $0 < T < t$ implies that

$$\int_0^t R(t-s)\phi(s)ds \ge \int_T^t R(t-s)(1-\epsilon)ds = (1-\epsilon)\int_0^{t-T} R(t-T-s)ds$$
$$= (1-\epsilon)\int_0^{t-T} R(u)du \to 1-\epsilon$$

as $t \to \infty$. As $\epsilon \to 0$ we find that

$$\int_0^t R(t-s)\phi(s)ds \to 1$$

as $t \to \infty$.

For the final conclusion take $\phi(t) = \int_0^t R(t-u)du$ and conclude that

$$\int_0^t R(t-s)\int_0^s R(s-u)duds \to 1$$

as $t \to \infty$. □

For our work below, we set

$$\begin{aligned} R_1(t) \quad &: \quad = \int_0^t R(s)\,ds = \int_0^t R(t-s)\,ds, \quad t \ge 0, \\ R_i(t) \quad &= \quad \int_0^t R(t-s)\,R_{i-1}(s)\,ds, \quad t \ge 0, \quad i = 2,3,.... \end{aligned}$$

Clearly

$$\lim_{t\to\infty} R_2(t) := \lim_{t\to\infty}\int_0^t R(t-s)\,R_1(s)\,ds = 1.$$

By repeated use of Lemma 3.1 we may see that for any integer $n > 0$ then

$$\lim_{t\to\infty} R_n(t) = \lim_{t\to\infty}\int_0^t R(t-s)\,R_{n-1}(s)\,ds = 1.$$

Note that by the definition of R_n and the fact that $R_1(t) < 1, t \geq 0$, for any such n we have

$$0 < ... \leq R_{n+1}(t) \leq R_n(t) \leq ... \leq R_1(t) < 1, \quad t \geq 0.$$

Refer now to (3.1) and refine the conditions on f as

$$0 \leq 1 - \frac{f(t,x)}{Jx} \leq 1 - \frac{k}{J}, \tag{3.4}$$

for $0 < k < J$ and for $|x| \leq |x(0)|$.

Theorem 3.2. *Let M be defined by (3.3) and let (3.4) hold. Let $P : M \to \mathcal{B}$ be defined by $\phi \in M$ implies that*

$$(P\phi)(t) = x(0)\big[1 - \int_0^t R(s)ds\Big] + \int_0^t R(t-s)\phi(s)\Big[1 - \frac{f(s,\phi(s))}{J\phi(s)}\Big]ds. \tag{3.5}$$

Then $P : M \to M$ and if $M_1 = PM$ then $\phi \in M_1$implies

$$\begin{aligned}|(P\phi)(t)| \leq& |x(0)|\Big[1 - \int_0^t R(s)ds\Big] + |x(0)|\Big[1 - \frac{k}{J}\Big]\int_0^t R(s)ds\\ &= |x(0)|\Big[1 - \frac{k}{J}\int_0^t R(s)ds\Big].\end{aligned}$$

Finally, P has at least one fixed point ξ in M which, of course, also resides in M_1. Thus $P\xi = \xi$, $\xi \in M_1$, and ξ satisfies (3.1) on $[0, \infty)$.

Proof. Using the natural mapping defined by (3.1), if $\phi \in M$ then

$$\begin{aligned}|(P\phi)(t)| &\leq |x(0)|\Big[1 - \int_0^t R(s)ds\Big] + \int_0^t R(t-s)|x(0)|\Big[1 - \frac{k}{J}\Big]ds\\ &= |x(0)| - |x(0)|\int_0^t R(s)ds + |x(0)|\int_0^t R(s)ds - |x(0)|\frac{k}{J}\int_0^t R(s)ds\\ &= |x(0)|\Big[1 - \frac{k}{J}\int_0^t R(s)ds\Big].\end{aligned}$$

There is a long list of papers dealing with existence of solutions of this equation. First, [3, p. 95,Theorem 4.1] as corrected in [4, p. 234] yields a fixed point $\xi \in M$ when $A(t-s) = (t-s)^{q-1}, 0 < q < 1$. The general case under A1)–A3) is proved in exactly the same way. The main points are that P maps M into an equicontinuous set, P is continuous, and M is a ball. The last two points are the same for the general A as for the $(t-s)^{q-1}$ case. More detail on equicontinuity and continuity is found in Dwiggins [7]. □

Theorem 3.3. *Under the same conditions if $P : M \to \mathcal{B}$ and $\phi \in M$, for any positive integer n we have*

$$\left|P^{(n)}(\phi)(t)\right| \leq |x(0)|\left[1 - \frac{k}{J}\sum_{i=1}^{i=n}\left(1 - \frac{k}{J}\right)^{i-1} R_i(t)\right], \quad t \geq 0, \tag{3.6}$$

where

$$\begin{aligned}R_1(t) \ &: = \int_0^t R(s)ds = \int_0^t R(t-s)ds,\\ R_i(t) \ &: = \int_0^t R(t-s)R_{i-1}(s)ds, \quad i = 2, 3, ...\end{aligned}$$

In particular, if ξ is a fixed point of P then

$$|\xi(t)| \leq |x(0)|\left[1 - \frac{k}{J}\sum_{i=1}^{\infty}\left(1 - \frac{k}{J}\right)^{i-1} R_i(t)\right], \quad t \geq 0 \tag{3.7}$$

and

$$\lim_{t\to\infty} \xi(t) = 0. \tag{3.8}$$

Proof. Inequality (3.6) for $n = 1$ becomes

$$|P(\phi(t))| \le |x(0)| \left[1 - \frac{k}{J} R_1(t)\right], \quad t \ge 0,$$

which holds true by Theorem 3.2. To prove (3.6) by induction, we assume that (3.6) holds for some positive integer m, i.e., that

$$\left|P^{(m)}(\phi)(t)\right| \le |x(0)| \left[1 - \sum_{i=1}^{i=m} \left(1 - \frac{k}{J}\right)^{i-1} \frac{k}{J} R_i(t)\right], \quad t \ge 0, \tag{3.9}$$

and we want to prove that (3.6) holds for $n = m + 1$.

Employing (3.9) in the definition of P we have

$$\begin{aligned}
& \left|P^{(m+1)}(\phi)(t)\right| = \left|P\left(P^{(m)}(\phi)\right)(t)\right| \\
\le \;& |x(0)| \left[1 - \int_0^t R(s)\, ds\right] + \int_0^t R(t-s) \left|P^{(m)}(\phi)(s)\right| \left(1 - \frac{k}{J}\right) ds \\
\le \;& |x(0)| \left[1 - R_1(t)\right] + \\
& + \int_0^t R(t-s) |x(0)| \left[1 - \sum_{i=1}^{i=m} \left(1 - \frac{k}{J}\right)^{i-1} \frac{k}{J} R_i(s)\right] \left(1 - \frac{k}{J}\right) ds \\
= \;& |x(0)| \left\{1 - R_1(t) + \left(1 - \frac{k}{J}\right) \times \right. \\
& \left. \times \left[\int_0^t R(t-s)\, ds - \int_0^t R(t-s) \sum_{i=1}^{i=m} \left(1 - \frac{k}{J}\right)^{i-1} \frac{k}{J} R_i(s)\, ds\right]\right\} \\
= \;& |x(0)| \left\{1 - R_1(t) + \left(1 - \frac{k}{J}\right) R_1(t) \right. \\
& \left. - \sum_{i=1}^{i=m} \left(1 - \frac{k}{J}\right)^{i} \frac{k}{J} \int_0^t R(t-s) R_i(s)\, ds\right\} \\
= \;& |x(0)| \left\{1 - \frac{k}{J} R_1(t) - \sum_{i=1}^{i=m} \left(1 - \frac{k}{J}\right)^{i} \frac{k}{J} R_{i+1}(t)\right\} \\
= \;& |x(0)| \left\{1 - \frac{k}{J} \left(1 - \frac{k}{J}\right)^{1-1} R_1(t) - \sum_{i=2}^{i=m+1} \left(1 - \frac{k}{J}\right)^{i-1} \frac{k}{J} R_i(t)\right\},
\end{aligned}$$

i.e.,

$$\left|P^{(m+1)}(\phi)(t)\right| \le |x(0)| \left\{1 - \sum_{i=1}^{i=m+1} \left(1 - \frac{k}{J}\right)^{i-1} \frac{k}{J} R_i(t)\right\}, \quad t \ge 0,$$

which is (3.6) for $n = m + 1$, thus induction is completed and inequality (3.6) is proved.

Next, note that if ξ is a fixed point of P then $P(\xi) = \xi$, thus for any positive integer n we have $P^{(n)}(\xi) = \xi$, and so

$$|\xi(t)| \le |x(0)| \left[1 - \frac{k}{J} \sum_{i=1}^{n} \left(1 - \frac{k}{J}\right)^{i-1} R_i(t)\right], \quad t \ge 0. \tag{3.10}$$

Since

$$0 \leq \sum_{i=1}^{n} \left(1 - \frac{k}{J}\right)^{i-1} R_i\left(t\right) \leq \sum_{i=1}^{n} \left(1 - \frac{k}{J}\right)^{i-1} < \infty$$

we see that the series of nonnegative terms at the right hand side of (3.10) converges uniformly so taking $n \to \infty$ in (3.10) leads to (3.7).

Before we prove (3.8) we recall that

$$\lim_{t\to\infty} R_1\left(t\right) := \lim_{t\to\infty} \int_0^t R\left(s\right) ds = 1,$$

so by use of Lemma 3.1 we take

$$\lim_{t\to\infty} R_2\left(t\right) := \lim_{t\to\infty} \int_0^t R\left(t-s\right) R_1\left(s\right) ds = 1.$$

By a simple induction we see that for any positive integer n we have

$$\lim_{t\to\infty} R_n\left(t\right) = \lim_{t\to\infty} \int_0^t R\left(t-s\right) R_{n-1}\left(s\right) ds = 1.$$

It follows that the limit of the right hand side of (3.6) exists and

$$\begin{aligned} \lim_{t\to\infty} \left[1 - \frac{k}{J} \sum_{i=1}^{i=n} \left(1 - \frac{k}{J}\right)^{i-1} R_i\left(t\right)\right] &= 1 - \frac{k}{J} \sum_{i=1}^{i=n} \left(1 - \frac{k}{J}\right)^{i-1} \\ &= 1 - \frac{k}{J} \frac{1 - \left(1 - \frac{k}{J}\right)^n}{\frac{k}{J}}, \end{aligned}$$

i.e.,

$$\lim_{t\to\infty} \left[1 - \sum_{i=1}^{i=n} \left(1 - \frac{k}{J}\right)^{i-1} \frac{k}{J} R_i\left(t\right)\right] = \left(1 - \frac{k}{J}\right)^n.$$

Then (3.8) follows by observing that the series at the left hand side of (3.7) converges uniformly and that

$$\lim_{n\to\infty} \left(1 - \frac{k}{J}\right)^n = 0.$$

□

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