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On Ulam stability of the second order linear differential equation

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Abstract

We obtain a result on Ulam stability for a linear differential equation in Banach spaces. As application we give a result on the stability of Heun's differential equation.

Keywords: Ulam stability, Banach spaces, Heun equation
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1. Introduction

In chapter six of the book Problems in Modern Mathematics, S.M. Ulam formulated the following question: When is it true that by changing "a little" the hypothesis of a theorem one can still assert that the conclusion of the theorem remains true or approximately true? For very general functional equations one can ask the following question: when is it true that the solution of an equation differing slightly from a given one, must by necessity be close to the solution of the given equation? (see [13], pg. 63).

A precise formulation of such a question was made by S.M. Ulam in 1940 in the case of the equation of the homomorphism of groups, during a talk at Madison University, Wisconsin. S.M. Ulam called the equation of homomorphism *stable* if the answer to the previous question is affirmative. For more details and results on the stability of functional equations we refer the reader to [1, 2, 3]. The first answer to Ulam's question was given by D.H. Hyers who proved that Cauchy's functional equation is stable (see [3]). M. Obłoza investigated for the first time the Ulam stability of differential equations and the relation between Lyapunov and Ulam stability [7, 8].

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A characterization of Ulam stability of the first order linear differential equation was obtained by T. Miura, S. Miyajima and S.E. Takahasi [6]. Results on Ulam stability of some second order linear differential equations were obtained by S.M. Jung using the power series method [4, 5]. D. Popa, G. Pugna and I. Raşa studied the stability of the linear differential equation of order n and of Euler's linear differential equation [9, 10, 11].

Remark that, as far as we know, a characterization of Ulam stability for a second order linear differential equation of general form is not available, but there are many papers containing stability results for various particular forms of this equation.

In the present paper we consider a linear differential equation of second order of general form in Banach spaces and study its Ulam stability. The main result is obtained under the hypothesis that the corresponding scalar equation admits a solution different from zero.

Finally, we give an application to Ulam stability of Heun's differential equation.

2. Main result

In what follows let $I = (a, b)$, $a, b \in \mathbb{R} \cup \{\pm\infty\}$, $c \in I \cup (\{a, b\} \cap \mathbb{R})$, $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, $p, q \in C(I, \mathbb{K})$, $(X, \|\cdot\|)$ be a Banach space over the field $\mathbb{K}$. We give a result on generalized Ulam stability of the second order linear differential equation

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0, \qquad x \in I, \tag{2.1}$$

where $y \in C^2(I, X)$ is the unknown.

The equation (2.1) is called Ulam stable if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for every $y \in C^2(I, X)$ satisfying the relation

$$\left\| y''(x) + p(x)y'(x) + q(x)y(x) \right\| \le \varepsilon, \qquad x \in I, \tag{2.2}$$

there exists a solution $y_0 \in C^2(I, X)$ of the equation (2.1) with the property

$$\|y(x) - y_0(x)\| \le \delta, \qquad x \in I. \tag{2.3}$$

In other words the equation (2.1) is called Ulam stable if for every solution y of (2.2) there exists an exact solution y_0 of the equation (2.1) close to y. If in the previous definition ε and δ are replaced by some functions φ and ψ depending on $x \in I$, the equation (2.1) is called generalized Ulam stable. Recall first a result of D. Popa and I. Raşa on the stability of the first order linear differential equation which will be used in the sequel.

Theorem 2.1. *[9] Let $\lambda \in C(I, \mathbb{K})$, $f \in C(I, X)$, $\varepsilon \in C(I, \mathbb{R})$, with $\varepsilon > 0$ on I. Then for every $y \in C^1(I, X)$ satisfying the relation*

$$\left\| y'(x) - \lambda(x) \cdot y(x) - f(x) \right\| \le \varepsilon(x), \qquad x \in I, \tag{2.4}$$

there exists a unique $u \in C^1(I, X)$ such that $u'(x) - \lambda(x)u(x) - f(x) = 0$, $x \in I$, and

$$\|y(x) - u(x)\| \le \psi_c(x), \qquad x \in I, \tag{2.5}$$

where

$$\psi_c(x) := e^{\Re L(x)} \left| \int_c^x e^{-\Re L(t)} \varepsilon(t) dt \right|, \qquad x \in I, \tag{2.6}$$

and L is an antiderivative of λ, i.e., $L' = \lambda$ on I.

Throughout this paper we suppose that concerning the equation (2.1) the following hypothesis is satisfied:
(H) There exists a function $w \in C^2(I, \mathbb{K})$, $w(x) \neq 0$ for every $x \in I$, such that

$$w^{''}(x) + p(x)w^{'}(x) + q(x)w(x) = 0, \qquad x \in I. \tag{2.7}$$

In other words the hypothesis (H) says that the corresponding scalar equation associated to the equation (2.1) admits a solution which does not vanish on I.

A representation theorem for the solutions of the equation (2.1) is given in the next lemma.

Lemma 2.2. *Let $y \in C^2(I, X)$ be a solution of the equation (2.1) and $x_0 \in I \cup (\{a, b\} \cap \mathbb{R})$. Then there exist k_1, $k_2 \in X$ such that*

$$y(x) = w(x) \left(k_1 \int_{x_0}^{x} e^{-\Lambda(t) dt} + k_2 \right), \qquad x \in I, \tag{2.8}$$

where Λ is an antiderivative of the function

$$\lambda = \frac{2w^{'} + pw}{w}. \tag{2.9}$$

Proof. Let y be a solution of (2.1) and $u(x) = \frac{y(x)}{w(x)}$, $x \in I$. Then $y = wu$, $y^{'} = w^{'}u + wu^{'}$, $y^{''} = w^{''}u + 2w^{'}u^{'} + wu^{''}$ and

$$u^{''}(x) + \frac{2w^{'}(x) + p(x)w(x)}{w(x)} u^{'}(x) = 0, \qquad x \in I. \tag{2.10}$$

Then the function $z = u^{'}$ satisfies the relation

$$(z(x)e^{\Lambda(x)})^{'} = 0, \qquad x \in I,$$

so there exists $k_1 \in X$ such that $z(x) = k_1 e^{-\Lambda(x)}$, $x \in I$.
Finally, for a certain $k_2 \in X$,

$$u(x) = k_1 \int_{x_0}^{x} e^{-\Lambda(t)} dt + k_2, \qquad x \in I.$$

We conclude that (2.8) is true. □

The main result of this paper is contained in the next theorem.

Theorem 2.3. *Let $\varphi \in C(I, \mathbb{R})$, $\varphi > 0$. For every $y \in C^2(I, X)$ satisfying the relation*

$$\left\| y^{''}(x) + p(x)y^{'}(x) + q(x)y(x) \right\| \leq \varphi(x), \qquad x \in I, \tag{2.11}$$

and every $c_1 \in I \cup (\{a, b\} \cap \mathbb{R})$ there exists a solution $y_0 \in C^2(I, X)$ of the equation (2.1) such that

$$\|y(x) - y_0(x)\| \leq |w(x)| \cdot \left| \int_{c_1}^{x} \psi_c(t) dt \right|, \qquad x \in I, \tag{2.12}$$

where P is an antiderivative of p and

$$\psi_c(x) = \frac{1}{|w(x)|^2 e^{\Re P(x)}} \left| \int_{c}^{x} e^{\Re P(t)} \varphi(t) \cdot |w(t)| \, dt \right|, \qquad x \in I. \tag{2.13}$$

Moreover, if $c_1 = c$ then y_0 is uniquely determined.

Proof. Let $y \in C^2(I, X)$ satisfying (2.11) and $u(x) = \frac{y(x)}{w(x)}$, $x \in I$. Then (2.11) becomes

$$\left\| u''(x) + \frac{2w'(x) + p(x)w(x)}{w(x)} u'(x) \right\| \le \frac{\varphi(x)}{|w(x)|}, \qquad x \in I,$$

or

$$\left\| z'(x) + \lambda(x)z(x) \right\| \le \frac{\varphi(x)}{|w(x)|}, \qquad x \in I,$$

where

$$\lambda(x) = \frac{2w'(x) + p(x)w(x)}{w(x)}, \qquad z(x) = u'(x), \qquad x \in I.$$

Let $\varepsilon(x) := \frac{\varphi(x)}{|w(x)|}$, $x \in I$, and L an antiderivative of $-\lambda$, $L = -\Lambda$. Then

$$\Re L(x) = -2ln\,|w(x)| - \Re P(x), \qquad x \in I.$$

According to Theorem 2.1, it follows that there exists a unique $z_0 \in C^1(I, X)$ such that

$$z_0'(x) + \lambda(x)z_0(x) = 0, \qquad x \in I,$$

and

$$\|z(x) - z_0(x)\| \le \psi_c(x), \qquad x \in I.$$

Now let $c_1 \in I \cup (\{a, b\} \cap \mathbb{R})$, $u_0 \in C^1(I, X)$ such that $u_0'(x) = z_0(x)$, $x \in I$, and $u_0(c_1) - u(c_1) = 0$. Hence

$$\left\| u'(x) - u_0'(x) \right\| \le \psi_c(x), \qquad x \in I,$$

and

$$\left\| \int_{c_1}^{x} (u'(t) - u_0'(t))dt \right\| \le \left| \int_{c_1}^{x} \left\| u'(t) - u_0'(t) \right\| dt \right| \le \left| \int_{c_1}^{x} \psi_c(t)dt \right|,$$

therefore

$$\|u(x) - u_0(x)\| \le \left| \int_{c_1}^{x} \psi_c(t)dt \right|, \qquad x \in I.$$

Let

$$y_0(x) = w(x)u_0(x), \qquad x \in I.$$

Then y_0 satisfies the equation (2.1) and

$$\|y(x) - y_0(x)\| \le |w(x)| \cdot \left| \int_{c_1}^{x} \psi_c(t)dt \right|, \qquad x \in I.$$

The existence is proved.
Uniqueness. Suppose that $c_1 = c$ and for an $y \in C^2(I, X)$ satisfying (2.11) there exist two solutions y_1, $y_2 \in C^2(I, X)$ of the equation (2.1) such that

$$\|y(x) - y_k(x)\| \le |w(x)| \cdot \left| \int_{c}^{x} \psi_c(t)dt \right|, \qquad x \in I.$$

Then, according to Lemma 2.1

$$\begin{aligned} y_1(x) &= w(x)\left(k_1 \int_c^x e^{-\Lambda(t)}dt + k_2\right), \\ y_2(x) &= w(x)\left(k_3 \int_c^x e^{-\Lambda(t)}dt + k_4\right), \end{aligned} \qquad x \in I, \tag{2.14}$$

where $k_1, k_2, k_3, k_4 \in X$. We get

$$\begin{aligned}\|y_1(x) - y_2(x)\| &\leq \|y_1(x) - y(x)\| + \|y(x) - y_2(x)\| \\ &\leq 2\,|w(x)| \cdot \left| \int_c^x \psi_c(t)dt \right|, \qquad x \in I,\end{aligned}$$

or taking account of (2.14) it follows

$$\left\| (k_1 - k_3) \int_c^x e^{-\Lambda(t)} dt + k_2 - k_4 \right\| \leq 2 \left| \int_c^x \psi_c(t)dt \right|, \qquad x \in I. \tag{2.15}$$

Letting $x \to c$ in (2.15) we get $k_2 = k_4$. Thus (2.15) becomes

$$\left\| (k_1 - k_3) \int_c^x e^{-\Lambda(t)} dt \right\| \leq 2 \left| \int_c^x \psi_c(t)dt \right|, \qquad x \in I.$$

Since $e^{-\Lambda(c)} \neq 0$, it follows that there exists a neighborhood V of c such that

$$\|k_1 - k_3\| \leq 2 \left| \frac{\int_c^x \psi_c(t)dt}{\int_c^x e^{-\Lambda(t)}dt} \right|, \quad x \in I \cap V, \quad x \neq c.$$

Taking account of

$$\lim_{x \to c} \frac{\int_c^x \psi_c(t)dt}{\int_c^x e^{-\Lambda(t)}dt} = \lim_{x \to c} \frac{\left(\int_c^x \psi_c(t)dt\right)'}{\left(\int_c^x e^{-\Lambda(t)}dt\right)'} = \frac{\psi_c(c)}{e^{-\Lambda(c)}} = \frac{0}{e^{-\Lambda(c)}} = 0,$$

it follows $k_1 = k_3$, therefore $y_1 = y_2$.

Uniqueness is proved. □

3. Application

We give an application concerning the Ulam stability of Heun's differential equation.

Heun differential equation is usually written in the form

$$y''(x) + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\varepsilon}{x-a} \right) y'(x) + \frac{\alpha\beta x - q}{x(x-1)(x-a)} y(x) = 0,$$

where $a, \alpha, \beta, \gamma, \delta, \varepsilon, q$ are real (or complex) constants, $a \neq 0$, $a \neq 1$ and $\alpha + \beta + 1 = \gamma + \delta + \varepsilon$. Heun equation has many applications in physical science. It is recently encountered in problems in general relativity and astrophysics. We deal with Ulam stability of a particular case of Heun's equation, namely

$$y''(x) + \left(\frac{1}{x} + \frac{1}{x-1} + \frac{-2n}{x-\frac{1}{2}} \right) y'(x) + \frac{-2nx + n}{x(x-1)(x-\frac{1}{2})} y(x) = 0, \tag{3.1}$$

where n is a positive integer and $y \in C^2(I, \mathbb{R})$.

A particular solution of the equation (3.1), related to Bernstein polynomials, is

$$F_n(x) = \sum_{k=0}^{n} \left(\binom{n}{k} (x^k (1-x)^{n-k}) \right)^2, \qquad x \in [0,1].$$

See for more details [12]. We will prove that equation (3.1) is Ulam stable in classical sense on the interval $I = (0, \frac{1}{2})$.

Theorem 3.1. *Let $\varepsilon > 0$. For every $y \in C^2(I, \mathbb{R})$ such that*

$$\left| y''(x) + \left(\frac{1}{x} + \frac{1}{x-1} + \frac{-2n}{x-\frac{1}{2}} \right) y'(x) + \frac{-2nx+n}{x(x-1)(x-\frac{1}{2})} y(x) \right| \leq \varepsilon, \qquad x \in I, \tag{3.2}$$

there exists a unique solution y_0 of the equation (3.1) with the property

$$|y(x) - y_0(x)| \leq \frac{M\varepsilon}{2}, \qquad x \in I,$$

where $M = \max_{x \in [0, \frac{1}{2}]} \psi(x)$, and

$$\psi(x) = \frac{(\frac{1}{2}-x)^{2n}}{x(1-x)F_n^2(x)} \int_0^x \frac{t(1-t)}{(\frac{1}{2}-t)^{2n}} F_n(t)dt, \qquad x \in \left(0, \frac{1}{2}\right)$$

$$\psi(0) = \psi\left(\frac{1}{2}\right) = 0.$$

Proof. Let $y \in C^2(I, \mathbb{R})$ be a function satisfying (3.2) and

$$p(x) = \frac{1}{x} + \frac{1}{x-1} + \frac{-2n}{x-\frac{1}{2}}, \quad q(x) = \frac{-2nx+n}{x(x-1)(x-\frac{1}{2})}, \qquad x \in I.$$

Then $P(x) = \ln \frac{x(1-x)}{(\frac{1}{2}-x)^{2n}}$, and for $c = 0$, $\varphi(x) = \varepsilon$, $x \in I$, in (2.13) we have

$$\psi_0(x) = \frac{\varepsilon(\frac{1}{2}-x)^{2n}}{x(1-x)F_n^2(x)} \int_0^x \frac{t(1-t)}{(\frac{1}{2}-t)^{2n}} F_n(t)dt, \qquad x \in I.$$

The function $\frac{\psi_0}{\varepsilon}$ can be extended to a continuous function on the interval $[0, \frac{1}{2}]$. Indeed

$$\begin{aligned} \lim_{x\to 0} \psi_0(x) &= \frac{\varepsilon}{2^{2n}} \lim_{x\to 0} \frac{1}{x} \int_0^x \frac{t(1-t)}{(\frac{1}{2}-t)^{2n}} F_n(t)dt \\ &= \frac{\varepsilon}{2^{2n}} \lim_{x\to 0} \frac{x(1-x)}{(\frac{1}{2}-x)^{2n}} F_n(x) = 0, \end{aligned}$$

according to l'Hospital's theorem. Analogously

$$\begin{aligned} \lim_{x\to \frac{1}{2}} \psi_0(x) &= \frac{4\varepsilon}{F_n^2(\frac{1}{2})} \lim_{x\to \frac{1}{2}} \left(\frac{1}{2} - x\right)^{2n} \int_0^x \frac{t(1-t)}{(\frac{1}{2}-t)^{2n}} F_n(t)dt \\ &= \frac{4\varepsilon}{F_n^2(\frac{1}{2})} \lim_{x\to \frac{1}{2}} \frac{\int_0^x \frac{t(1-t)}{(\frac{1}{2}-t)^{2n}} F_n(t)dt}{(\frac{1}{2}-x)^{-2n}} \\ &= \frac{4\varepsilon}{F_n^2(\frac{1}{2})} \lim_{x\to \frac{1}{2}} \left(\frac{1}{2} - x\right) \cdot \frac{x(1-x)F_n(x)}{2n} = 0. \end{aligned}$$

Therefore we can extend $\frac{\psi_0}{\varepsilon}$ to a continuous function ψ on $[0, \frac{1}{2}]$ with

$$\psi(0) = \psi\left(\frac{1}{2}\right) = 0.$$

Taking $w(x) = F_n(x)$, $x \in I$, $c = c_1 = 0$ in Theorem 2.2 it follows that there exists a unique solution y_0 of the equation (3.1) such that

$$|y(x) - y_0(x)| \leq \varepsilon F_n(x) \int_0^x \psi(t)dt, \qquad x \in I.$$

On the other hand $sup_{x\in I}F_n(x)=1$, hence

$$F_n(x)\int_0^x \psi(t)dt \le \int_0^{\frac{1}{2}} M dt = \frac{M}{2}, \qquad x\in I.$$

The theorem is proved. □

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