Advances in the Theory of Nonlinear Analysis and its Applications **2** (2018) No. 2, 81-84.
https://doi.org/10.31197/atnaa.417783
Available online at www.atnaa.org
Research Article

Advances in the Theory of Nonlinear Analysis and its Applications

ISSN: 2587-2648 Peer-Reviewed Scientific Journal

A Self-Similar Dendrite with One-Point Intersection and Infinite Post-Critical Set

Prabhjot Singh[a], Andrei Tetenov[b]

[a] *Central University of Rajasthan, India*
[b] *Novosibirsk State University, Russia*

Abstract

We build an example of a system $\mathcal{S}$ of similarities in $\mathbb{R}^2$ whose attractor is a plane dendrite $K \supset [0,1]$ which satisfies one point intersection property, while the post-critical set of the system $\mathcal{S}$ is a countable set whose natural projection to K is dense in the middle-third Cantor set.

Keywords: self-similar set, post-critical finiteness, self-similar dendrite, open set condition
2010 MSC: 28A80

---

1. Introduction

Let $\mathcal{S} = \{S_1 \dots S_m\}$ be a system of contraction maps of a complete metric space X. A non-empty compact set $K \in X$ satisfying $K = S_1(K) \cup \dots \cup S_m(K)$ is called invariant set, or attractor defined on the system $\mathcal{S}$. The uniqueness and existence of the attractor K is provided by Hutchinson's Theorem [3].

Let $I = \{1, 2, \dots, m\}$, $I^* = \bigcup_{n=1}^{\infty} I^n$ be the set of all finite I-tuples and $I^\infty = \{\alpha = \alpha_1\alpha_2 \dots, \quad \alpha_i \in I\}$ be the index space and $\pi : I^\infty \to K$ be the address map.

A system $\mathcal{S}$ satisfies open set condition (OSC) if there is non-empty open O such that $S_i(O) \subset O$ and $S_i(O) \cap S_j(O) = \varnothing$ for any $i, j \in I, i \neq j$ [3, 5]. We say that the system $\mathcal{S}$ satisfies one point intersection property [1] if for any $i, j \in I, i \neq j$, $\#(S_i(K) \cap S_j(K)) \leq 1$.

Let $\mathcal{C}$ be the union of all $S_i(K) \cap S_j(K)$, $i, j \in I, i \neq j$. The post-critical set $\mathcal{P}$ of the system $\mathcal{S}$ is the set of all $\alpha \in I^\infty$ such that for some $\mathbf{j} \in I^*$, $S_{\mathbf{j}}(\alpha) \in \mathcal{C}$. In other words, $\mathcal{P} = \{\sigma^k(\alpha) | \alpha \in \mathcal{C}, k \in \mathbb{N}\}$, where the map

---

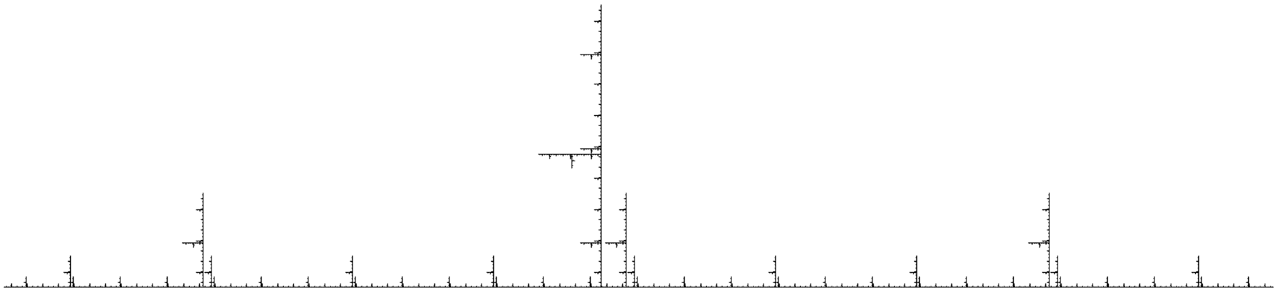
Email addresses: prabhjot198449@gmail.com (Prabhjot Singh), atet@mail.ru (Andrei Tetenov)
Supported by Russian Foundation of Basic Research projects 16-01-00414, 18-501-51021

Received February 12, 2018, Accepted: April 22, 2018 Online: April 25, 2018.

$\sigma^k : I^\infty \to I^\infty$ is defined by $\sigma^k(\alpha_1\alpha_2\ldots) = \alpha_{k+1}\alpha_{k+2}\ldots$.

A system $\mathcal{S}$ is called *post-critically finite*(pcf) [4] if its post-critical set is finite. This obviously implies finite intersection property.

Our aim is to show that the converse need not be true even in the case of plane dendrites. We construct an example of non-pcf system $\mathcal{S}$, whose attractor is a dendrite $K\subset\mathbb{R}^2$, satisfying one point intersection property.



So we prove the following

Theorem 1.1. *There is a system $\mathcal{S} = \{S_1, S_2, S_3, S_h\}$ in $\mathbb{R}^2$, whose attractor K is a dendrite, which satisfies OSC and 1-point intersection property and has infinite post-critical set whose projection to K is dense in the middle-third Cantor set.*

2. Construction

Take a system $\mathcal{S} = \{S_0, S_1, S_2, S_h\}$ of contraction similarities of $\mathbb{R}^2$, defined by

$$S_j(x, y) = ((x + j)/3, y/3), \quad j = 0, 1, 2 \text{ and } S_h(x, y) = (-hy + c, hx) \tag{2.1}$$

and let K be the attractor of $\mathcal{S}$. Here c is infinite base-3 fraction beginning with 0.11 and containing all finite tuples, consisting of 0 and 2: $c = 0.110200220020222\ldots$

We will show that if h is sufficiently small, then all the images $S_{i_1i_2\ldots i_nh}(K)$, $i_k = 0, 1, 2$, are disjoint.

Put $I = \{0, 1, 2\}$ and denote by $I^* = \bigcup_{n=1}^{\infty} I^n$ the set of all tuples formed by $\{0, 1, 2\}$. Consider the images of c under the maps $S_{\mathbf{j}}, \mathbf{j} \in I^*$. Using base-3 fractions, we can write them as $S_{\mathbf{j}}(c) = 3^{-n}c+0.j_1\ldots j_n$, $\mathbf{j} \in I^*$, so $(c_{\mathbf{j}})_k = j_k$ for $k \le n$ and $(c_{\mathbf{j}})_k = c_{k-n}$ for $k > n$.

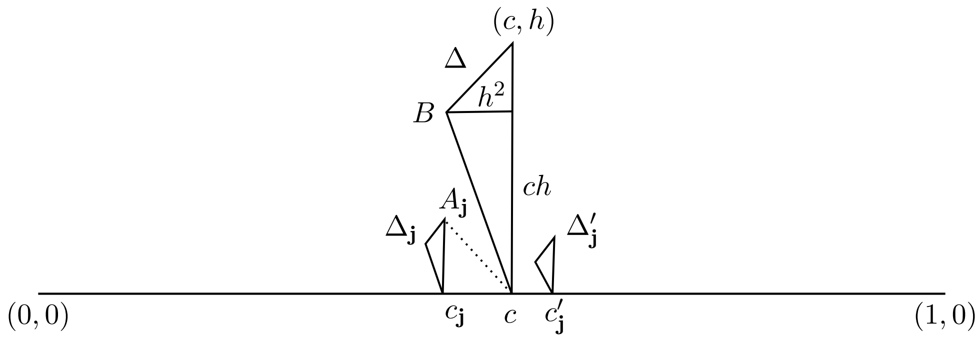


Fig. 1.

Let D be a triangle with vertices $\{(0, 0), (1, 0), (c, h)\}$ and $\Delta = S_h(D)$. Since c is not rational, the point $c_{\mathbf{j}} = S_{\mathbf{j}}(c)$ is not equal to c for any $\mathbf{j} \in I^*$. So either $c_{\mathbf{j}} < c$ or $c_{\mathbf{j}} > c$.

For $c_{\mathbf{j}} < c$, $\Delta_{\mathbf{j}} \cap \Delta \ne \varnothing$ if $A_{\mathbf{j}}(c_j, h3^{-n})$ lies in Δ. To avoid this, the slope of the line Bc has to be steeper than that of $A_{\mathbf{j}}c$ (See Fig 1.):

$$\frac{3^{-n}h}{(c-c_{\mathbf{j}})} < \frac{ch}{h^2} \quad \text{or} \quad h^2 < 3^n(c-c_j)c \tag{1}$$

similarly, for $c < c'_{\mathbf{j}}$, we have to require that $S_{\mathbf{j}}(B)$ does not lie in Δ

$$\frac{h^2}{3^n} < (c'_{\mathbf{j}} - c) \quad \text{or} \quad h^2 < 3^n(c'_{\mathbf{j}} - c) \tag{2}$$

So we need to estimate $3^n(c - c_{\mathbf{j}})$ and $3^n(c'_{\mathbf{j}} - c)$.

Case 1. If $c_{\mathbf{j}} < c$, there are the following possibilities:

(a) $j_1 \dots j_n = i_1 \dots i_n$. Then $(n+1)$-th entry $(c_{\mathbf{j}})_{n+1} = i_1 = 1$. Since $i_{n+1} > 1$, then $i_{n+1} = 2$. So $c_{\mathbf{j}} < 0.i_1 \dots i_n 12$, $c > 0.i_1 \dots i_n 20$, then $c - c_{\mathbf{j}} > 3^{-n-2}$.

(b) $j_1 \dots j_k = i_1 \dots i_k$ for some $k < n$ and $j_{k+1} < i_{k+1}$. Since the only entries allowed here are 0 and 2, so $j_{k+1} = 0$ and $i_{k+1} = 2$. So $c > 0.i_1 \dots i_k 2$ and $c_{\mathbf{j}} < 0.i_1 \dots i_k 1$, therefore $c - c_{\mathbf{j}} < 3^{-k-1}$.

Case 2. $c < c'_{\mathbf{j}}$, then
(a) $j_1 \dots j_n = i_1 \dots i_n$. Since $(c_{\mathbf{j}})_{n+1} = i_1 = 1$, $c < c'_{\mathbf{j}}$ implies $i_{n+1} = 0$, so $c'_{\mathbf{j}} > 0.i_1 \dots i_n 1102$, $c < 0.i_1 \dots i_n 0(2)$, then $c - c'_{\mathbf{j}} > 3^{-n-2}$.

(b) $j_1 \dots j_k = i_1 \dots i_k$ for some $k < n$ and $j_{k+1} > i_{k+1}$. Then $j_{k+1} = 2$, $i_{k+1} = 0$, so $c'_{\mathbf{j}} > 0.i_1 \dots i_k 2$, $c < 0.i_1 \dots i_k 1$, so $c'_j - c > 3^{-k-1}$.

(c) if $n = 1$ and $j_1 = 1$ i.e. $c_{\mathbf{j}} = 0.11102002\dots$, then $c_{\mathbf{j}} - c > 0.0001121$.
Therefore $(c_{\mathbf{j}} - c)3^n > 4/81$.

(d) if $n = 2$ and $j_1 j_2 = 11$, i.e. $c_{\mathbf{j}} = 0.11110200$, we similarly get $c_{\mathbf{j}} - c > 0.000201$. Thus, $(c_{\mathbf{j}} - c)3^n > 2/9$.

So if $h \le 2/9$ the inequalities (1) and (2) are satisfied. Further we show that if $h \le 2/9$, the system $\mathcal{S}$ satisfies open set condition(OSC) and one point intersection property and the attractor K is a dendrite.

3. Proof of the Theorem.

Lemma 3.1. *If $h \le 2/9$, for any $\mathbf{i}, \mathbf{j} \in I^*$, $\Delta_{\mathbf{i}} \bigcap \Delta_{\mathbf{j}} = \varnothing$.*

Proof. Let $\mathbf{i} = i_1 \dots i_n$, $\mathbf{j} = j_1 \dots j_m$. If $j_1 \dots j_k = i_1 \dots i_k$ for some $k < n$ and $j_{k+1} \ne i_{k+1}$, then $\Delta_{\mathbf{i}} \bigcap \Delta_{\mathbf{j}} = S_{i_1 \dots i_k}(\Delta_{i_{k+1} \dots i_n} \bigcap \Delta_{j_{k+1} \dots j_m})$. It follows from $\Delta_{i_{k+1} \dots i_n} \subset S_{i_{k+1}}(D)$ and $\Delta_{j_{k+1} \dots j_m} \subset S_{j_{k+1}}(D)$ that $\Delta_{i_{k+1} \dots i_n}$ and $\Delta_{j_{k+1} \dots j_m}$ are disjoint. Thus $\Delta_{\mathbf{i}} \cap \Delta_{\mathbf{j}} = \varnothing$.
If $m > n$ and $i_s = j_s$ for $s = 1, \dots, m$, then $\Delta_{\mathbf{i}} \bigcap \Delta_{\mathbf{j}} = S_{\mathbf{i}}(\Delta \bigcap \Delta_{j_{n+1} \dots j_m})$. By the construction, if $h \le 2/9$, $\Delta \bigcap \Delta_{j_{n+1} \dots j_m} = \varnothing$. ■

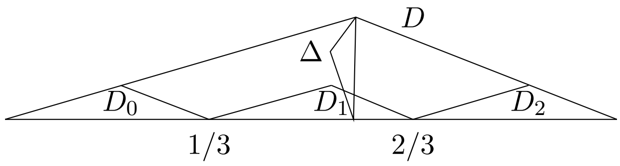


Fig. 2.

Lemma 3.2. *The system $\mathcal{S}$ satisfies one point intersection property and open set condition(OSC).*

Proof. Let $\dot{D}, \dot{\Delta}$ be the interiors of D and Δ. Define $O = \dot{\Delta} \cup \bigcup_{\mathbf{i} \in I^*} S_{\mathbf{i}}(\dot{\Delta})$. Obviously, for $i \in I$, $O_i = S_i(O) \subset O$. Moreover, $O_h = S_h(O) \subset \dot{\Delta} \subset O$.
Observe that with the only exception $S_1(\dot{D}) \cap S_h(\dot{D}) \ne \varnothing$, the sets $S_0(\dot{D}), S_1(\dot{D})$, $S_2(\dot{D})$ and $S_h(\dot{D})$ are disjoint. Since $O \subset \dot{D}$, the same is true for the sets O_0, O_1, O_2 and O_h. But $O_h \subset \dot{\Delta}$, so $O_h \bigcap O_1 = \varnothing$ too, therefore O_0, O_1, O_2, O_h are disjoint and (OSC) is fulfilled.
It follows from Lemma 2 that $\Delta \bigcap S_1(\overline{O}) = \{c\}$ and therefore $S_1(K) \bigcap S_h(K) = \{c\}$, which implies one point intersection property.■

Lemma 3.3. *The system $\mathcal{S}$ is post critically infinite and its post critical set is dense in the middle-third Cantor set $\mathcal{C}$.*

Proof. Let $y = .y_1 y_2 ..., y_i \in \{0, 2\}$ be base 3 representation for some point from the middle-third Cantor set $\mathcal{C}$. Since the representation of c contains all possible tuples of symbols 0 and 2, then for any $n \in \mathbb{N}$ there is $k = k(n)$ such that $c_{k+i} = y_i$ for $i = 1, \dots, n$. Therefore $|\sigma_k(c) - y| < 3^{-n}$. So, the sequence $\sigma_{k(n)}(c)$, converges to the point $y \in \mathcal{C}$. ■

To finish the proof of the Theorem 1, we need only to check that the set K is a dendrite. Let $\Delta_0 = \bigcup\limits_{\mathbf{i} \in I^*} S_{\mathbf{j}}(\Delta) \cup \Delta \cup [0, 1]$. This set is compact and it is simply-connected, because the sets $S_{\mathbf{j}}(\Delta)$ are disjoint. It is a strong deformation retract of the set D. Define $\Delta_{k+1} = \bigcup\limits_{\mathbf{i} \in I^*} S_{\mathbf{j}} * S_h(\Delta_k) \cup S_h(\Delta_k) \cup [0, 1]$. The sets Δ_k form a nested sequence of compact simply-connected sets, each being a strong deformation retract of the previous one. Then the intersection $\bigcap\limits_{k=1}^{\infty} \Delta_k = K$ is a strong deformation retract of the set D. By Kigami's theorem [4] it is locally connected and arcwise connected. Since the interior of K is empty, it contains no simple closed curve, therefore it is a dendrite [2, Theorem 1.1]. ■

References

[1] C. Bandt, N.V. Hung and H. Rao, On the open set condition for self-similar fractals, Proc. Amer. Math. Soc. 134, 1369–1374 (2005) 1

[2] J.Charatonik, W.Charatonik, Dendrites, Aportaciones Mat. Comun. 22, 227–253(1998) 3

[3] J.E.Hutchinson, Fractals and self-similarity, Indiana Univ. Math. J. 30, 713–747 (1981) 1

[4] J.Kigami, Analysis on Fractals. Cambridge University Press, 2001 1, 3

[5] P.A.P. Moran, Additive functions of intervals and Hausdorff measure, Proc. Cambridge Philos. Soc. 42, 15-23 (1946) 1