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Circling-Point Curve in Minkowski Plane

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Abstract: The purpose of this paper is to study the circling-point curve and its degenerate cases at the initial position of motion in Minkowski plane. The first part of the paper is devoted to the determination Bottema's instantaneous invariants and trajectory of origin with respect to these invariants in Minkowski plane. The intersection points of the circling-point curve and inflection curve are called Ball points. Here the number and also the geometric location of Ball points in Minkowski plane have been determined. The fundamental geometric property of a trajectory of each point in a plane is its curvature function κ. Under consideration $\kappa = \kappa' = \kappa'' = 0$, the existence conditions of Ball points in Minkowski plane have been given.

Keywords: Circling-point curve, Ball point, Instantaneous Invariants, Burmester Theory.

1 Introduction

Oene Bottema (1901-1992), Dutch mathematician devised the method of instantaneous invariants in instantaneous kinematics. Various geometric and kinematic properties of Euclidean planar and spatial motions are introduced with respect to the instantaneous invariants. The concept of instantaneous invariants is characterizing the trajectory of any point on a moving rigid body with arbitrary degrees [1–3]. In the meantime, Veldkamp has called the aforementioned invariants as B-invariants [4] and has handled the application of B-invariants to Burmester theory [4–6]. Burmester theory deals with the formulation of special locus curves as inflection circle, circling point curve, twice circling curve, and their intersection points as Ball and Burmester point for planar or spatial motions. Although this analytical method is preferred in a great amount of study of the kinematics, there have been few investigations on non-Euclidean planar kinematics [7, 8].

In consideration of these studies, we investigate the circling-point curve and its degenerate cases of the motion of Minkowski planes and give the existence conditions of Ball points in Minkowski plane.

2 Preliminaries

The Minkowski plane L is the plane R^2 endowed with the Lorentzian scalar product given by $\langle u, w\rangle = u_1w_1 - u_2w_2$, where $u = (u_1, u_2)$ and $w = (w_1, w_2)$. The norm of a vector U is defined by $\|u\| = \sqrt{|\langle u, u\rangle|}$. Let L_m and L_f be two coincident Minkowski planes, L_m moving with respect to L_f. The motion can be represented by

$$\begin{aligned} X(\varphi) &= x\cosh\varphi + y\sinh\varphi + a(\varphi) \\ Y(\varphi) &= x\sinh\varphi + y\cosh\varphi + b(\varphi) \end{aligned}$$

such that Cartesian frames of reference xoy and XOY are located in L_m and L_f, respectively. The position corresponding to $\varphi = 0$ of L_m will be named zero-position. The value for zero-position of the nth $(n = 0, 1, 2, \ldots)$ derivative of a function f of φ with respect to φ will be denoted by f_n.

The derivatives $a_n,\ b_n\ (n = 0, 1, 2, \ldots)$ are known as Bottema's instantaneous invariants of the motion [2, 3]. It is well-known that the canonical relative system can be constructed by choose of

$$a = b = a_1 = b_1 = a_2 = 0 \quad \text{and} \quad b_2 = -1.$$

So, the instantaneous invariants $a_k\,(k = 3, 4, \ldots, n)$, $b_k\,(k = 2, 3, \ldots, n)$ completely characterize the infinitesimal properties of motion of Minkowski planes up to the $n-\text{th}$ order as

$$\begin{aligned} X &= x, & X_1 &= y, & X_2 &= x, & X_3 &= y + a_3, \\ Y &= y, & Y_1 &= x, & Y_2 &= y - 1, & Y_3 &= x + b_3, \end{aligned} \tag{1}$$

at the zero-position [7, 8].

The non-null trajectory of the points satisfying $\kappa = 0$ is the inflection circle where $X' \neq \pm Y'$ in the Minkowski plane. Then the equation of the inflection circle can be obtained from $X'' : Y'' = X' : Y'$ since the curvature function is

$$\kappa = \frac{X'Y'' - X''Y'}{\left|(X')^2 - (Y')^2\right|^{\frac{3}{2}}}. \tag{2}$$

If we substitute the equalities of (1) into (2) at zero position we get the equation of the inflection circle during planar motion of L_m with respect to L_f as follows

$$x^2 - y^2 + y = 0. \tag{3}$$

where $(x, y) \neq (0, 0)$, $x \neq \mp y$ or $y \neq 0$ [7, 8].

3 The Trajectory of Origin of Minkowski Plane

The trajectory of the point $(0, 0)$ of the Minkowski plane L_m, which is coincident with the pole, can be given by

$$X = \sum_{n=3}^{\infty} \frac{a_n}{n!}\varphi^n, \quad Y = \frac{-1}{2}\varphi^2 + \sum_{n=3}^{\infty} \frac{b_n}{n!}\varphi^n \tag{4}$$

for sufficiently small values of $|\varphi|$ at the zero-position with respect to canonical relative systems.

Case 1. Let $a_3 \neq 0$. If ε is a sufficiently small positive number, then the trajectory described through the time interval $[-\varepsilon, \varepsilon]$ has a cusp at the pole of zero-position since $\lim_{\varphi \to 0} |\kappa| = \infty$ and the tangent of the trajectory is pole normal.

Case 2. Let $a_3 = 0,\ a_4 \neq 0$. In this case $a_2 b_3 - a_3 b_2 = 0$ and $a_2 b_4 - a_4 b_2 \neq 0$. So two branches of the trajectory stay at the same side of the tangent. If ε is a sufficiently small positive number, then the trajectory described through the time interval $[-\varepsilon, \varepsilon]$ has a ramphoid cusp at the pole of the zero-position. In this case the curvature is obtained as

$$\begin{gathered}\kappa = \tfrac{a_4}{3} + \left(\tfrac{5a_4 b_3}{12} + \tfrac{a_5}{8}\right)\varphi + \left(\tfrac{-a_4\, {b_3}^2}{8} + \tfrac{a_4\, b_4}{6} + \tfrac{7a_5\, b_3}{48} + \tfrac{a_6}{30}\right)\varphi^2 \\ + \left(\tfrac{7a_4\, b_5}{144} - \tfrac{a_4\, {b_3}^2}{12} - \tfrac{a_4\, b_3\, b_4}{24} + \tfrac{a_5\, b_4}{18} - \tfrac{a_5\, {b_3}^2}{16} + \tfrac{3a_6\, b_3}{80} + \tfrac{a_7}{144}\right)\varphi^3 + \ldots\end{gathered}$$

The successive curvatures of the trajectory at the pole are

$$\kappa_0 = \tfrac{a_4}{3}, \tag{5}$$

$$\kappa_1 = \frac{5a_4 b_3}{12} + \frac{a_5}{8}, \tag{6}$$

$$\kappa_2 = \frac{-a_4 {b_3}^2}{4} + \frac{a_4 b_4}{3} + \frac{7a_5 b_3}{24} + \frac{a_6}{15}, \tag{7}$$

$$\kappa_3 = \frac{7a_4 b_5}{24} - \frac{a_4 {b_3}^2}{2} - \frac{a_4 b_3 b_4}{4} + \frac{a_5 b_4}{3} - \frac{3a_5 {b_3}^2}{8} + \frac{9a_6 b_3}{40} + \frac{a_7}{24}.$$

Case 3. Let $a_3 = a_4 = 0$. For sufficiently small values of ε, the trajectory described through the time interval $[-\varepsilon, \varepsilon]$ has cusp or ramphoid cusp, provided that the smallest value of n, where $a_n \neq 0$, is odd or even, respectively. In this case the curvature is given by

$$\kappa = 0 + \tfrac{a_5}{8}\varphi + \left(\tfrac{7a_5 b_3}{48} + \tfrac{a_6}{30}\right)\varphi^2 + \left(\tfrac{a_5 b_4}{18} - \tfrac{a_5 {b_3}^2}{16} + \tfrac{3a_6 b_3}{80} + \tfrac{a_7}{144}\right)\varphi^3 + \ldots$$

that is, the successive curvatures at pole are

$$\kappa_0 = 0,$$

$$\kappa_1 = \frac{a_5}{8},$$

$$\kappa_2 = \frac{7a_5 b_3}{24} + \frac{a_6}{15},$$

$$\kappa_3 = \frac{a_5 b_4}{3} - \frac{3a_5 {b_3}^2}{8} + \frac{9a_6 b_3}{40} + \frac{a_7}{24}.$$

4 Circling-Point Curve of Motions in Minkowski Plane

Definition 1. *The locus of the points with constant non-null trajectory curvature at the zero-position of the Minkowski plane L_m is called circling-point curve or cubic of stationary curvatures and denoted by cp.*

This means that the locus of the points satisfying $\kappa' = 0$ where $\left(X'\right)^2 - \left(Y'\right)^2 \neq 0$ is the circling-point curve in Minkowski plane. The differentiation of the equation (2) is

$$\kappa' = \frac{\left(X'Y''' - X'''Y'\right)\left(\left(X'\right)^2 - \left(Y'\right)^2\right) - 3\left(X'Y'' - X''Y'\right)\left(X'X'' - Y'Y''\right)}{\left|(X')^2 - (Y')^2\right|^{\frac{3}{2}}}.$$

In this regard, if we consider the equations of (1) and the last equation together, one can prove the following theorem.

Theorem 1. *In Minkowski plane the equation of the circling-point curve cp of the original motion L_m/L_f is*

$$\left(x^2 - y^2\right)(a_3 x - b_3 y) + 3x\left(x^2 - y^2 + y\right) = 0 \tag{8}$$

where $(x, y) \neq (0, 0)$ or $x \neq \mp y$.

If we recall the equation (6) for the case of $a_3 = 0$ and $a_4 \neq 0$, we can prove the following theorem.

Theorem 2. *The trajectory of the points different from the origin is the circling-point curve if and only if is*

$$10a_4 b_3 + 3a_5 = 0.$$

in case of $a_3 = 0$ and $a_4 \neq 0$.

The graphics of the circling point curves for special cases in the Minkowski plane are drawn hereinafter and further detailed analysis of the graphics enables us to compare them with each other.

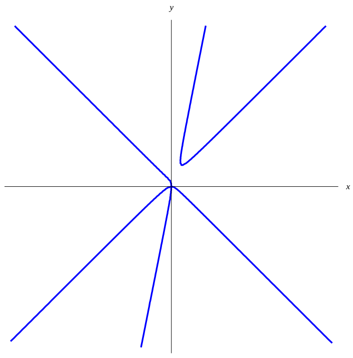


Fig. 1: The circling point curve cp for $a_3 = 2$ and $b_3 = 1$.

The circling point curve cp has node point at the pole. At the same time, tangents of the circling point curve cp are pole tangent and pole normal. Consequently, the cubic curve cp is a strophoid in Minkowski plane. Now let us investigate the degenerate cases of the circling point curve cp.

i. If $a_3 \neq -3$ and $b_3 = 0$ the equation of the circling-point curve cp in Minkowski plane is

$$x\left((a_3 + 3)\left(x^2 - y^2\right) + 3y\right) = 0. \tag{9}$$

This geometrically means that cp consists of the pole normal and the circle, which is donated by Γ, with the imaginary radius $\frac{3i}{2(a_3+3)}$. The center of Γ is $\left(0, \frac{3}{2(a_3+3)}\right)$ at the pole normal, see Figure 2a.

In addition, if $a_3 = 0$ when $b_3 = 0$, then the equation (9) becomes $x^2 - y^2 + y = 0$, that is, the circling point curve cp coincides with the inflection circle in the case of $a_3 = 0$ and $b_3 = 0$.

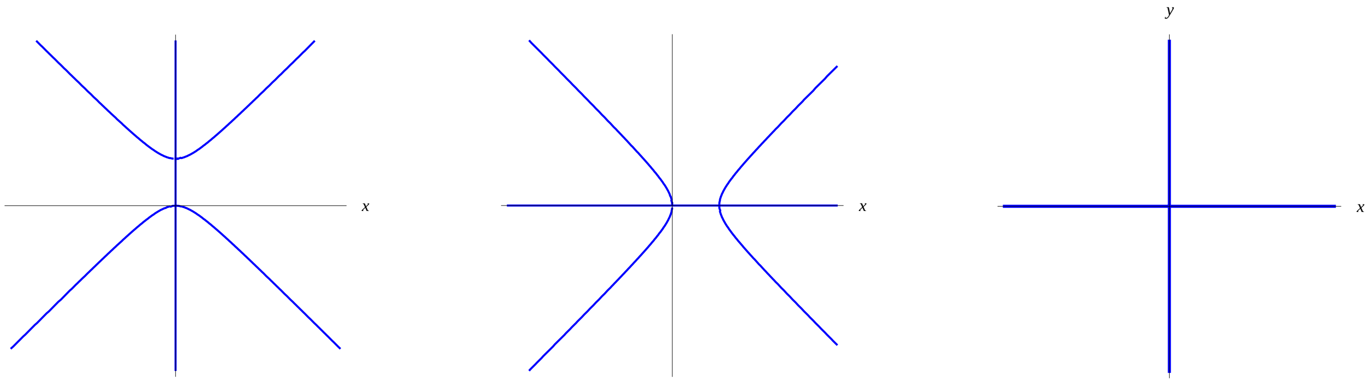
ii. If $a_3 = -3$ and $b_3 \neq 0$ the equation of the circling-point curve cp in Minkowski plane is

$$y\left(b_3\left(x^2 - y^2\right) - 3x\right) = 0.$$

Thus, the circling-point curve cp consists of pole tangent and the circle, which is donated by Γ_0, with the real radius $\frac{3}{2b_3}$. The center of Γ_0 is $\left(\frac{3}{2b_3}, 0\right)$ at the pole tangent, see Figure 2b.

iii. If $a_3 = -3$ and $b_3 = 0$, the equation of the circling-point curve cp is $xy = 0$. The curve consists of pole tangent and pole normal, see Figure 2c.

The circles Γ and Γ_0 are the circles of curvature of the circling-point curve cp at its node. From here the geometrical interpretation of the invariants a_3 and b_3 can be given as in the following theorem.



(a) $a_3 = 2,\ b_3 = 0$ (b) $a_3 = -3,\ b_3 = 1$ (c) $a_3 = -3,\ b_3 = 0$

Fig. 2: The circling-point curves in Minkowski plane

Theorem 3. *a_3 equals $3/2$ times the curvature of that branch of cp that touches the pole tangent and similarly b_3 equals $3/2$ times the curvature of that branch of cp that touches the pole normal.*

The equation of the real asymptote of the circling-point curve cp is obtained as

$$\left((a_3+3)^2 - {b_3}^2\right)(b_3 y - (a_3+3)\,x) + 3\,(a_3+3)\,b_3 = 0. \tag{10}$$

The real asymptotes of the circling-point curve cp drawn in the Figure 1 can be seen in the undermentioned figure.

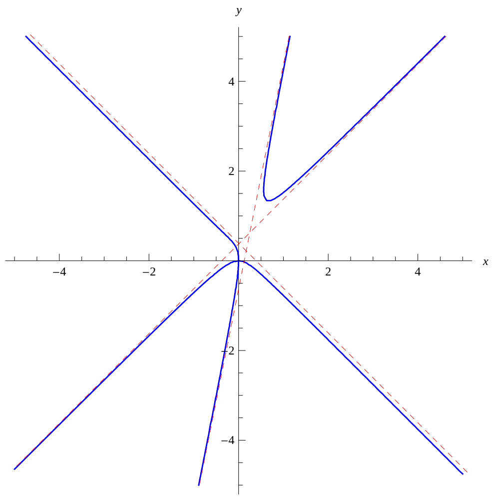


Fig. 3: The real asymptotes of the circling point curve cp for $a_3 = 2$ and $b_3 = 1$.

Furthermore, we can obtain a parametric representation of and irreducible curve cp by putting $y = ux$. This parametric equation is

$$x = \frac{3u}{(u^2-1)\,(-b_3 u + a_3 + 3)}, \quad y = \frac{3u^2}{(u^2-1)\,(-b_3 u + a_3 + 3)}. \tag{11}$$

If we substitute the equation (11) into the equation (10) we find parameter-value $u = \frac{(a_3+3)}{b_3}$. This parameter-value corresponds to the point of intersection of cp with its asymptote.

In the case of $a_3 \neq -3$ and $b_3 = 0$, the equation (11) takes the form

$$x = \frac{3u}{(u^2-1)\,(a_3+3)}, \quad y = \frac{3u^2}{(u^2-1)\,(a_3+3)},$$

which is the parametric representation of the circle Γ.

In a similar vein, if $a_3 = -3$ and $b_3 \neq 0$, under consideration the equation (11)the parametric representation of Γ_0 is given by

$$x = \frac{3}{b_3\,(1-u^2)}, \quad y = \frac{3u}{b_3\,(1-u^2)}.$$

5 Ball Points in Minkowski Plane

Definition 2. *The intersection points of the circling-point curve and inflection curve are called Ball points and denoted by Bl points.*

From this definition and the equations (3) and (8) the coordinates of a Bl point in Minkowski Plane is found as

$$\left(\frac{a_3b_3}{a_3^2-b_3^2},\quad \frac{a_3^2}{a_3^2-b_3^2}\right). \tag{12}$$

The pole is not a Bl point if $a_3 \neq 0$. Therefore we may draw the conclusion that in the case of $a_3 \neq 0$ and $a_3 \neq \pm b_3$ there is only one point in the zero position given by (12).

From the equation (12), if $a_3 = 0,\ b_3 \neq 0$ we cannot directly say that the origin is Bl point. Therefore in the case of $a_3 = 0,\ b_3 \neq 0$, if $a_4 = a_5 = 0$ we know that $\kappa_0 = \kappa_1 = 0$ is satisfied from the equations (5) and (6). From here if $a_3 = a_4 = a_5 = 0$ the origin is Bl point. Providing that $a_3 = 0,\ b_3 \neq 0$, there is no Bl point if and only if $a_4 \neq 0$ or $a_5 \neq 0$ (because of $\kappa_0 \neq 0$ or $\kappa_1 \neq 0$). Finally, we can say that there is no Bl point if $a_3 = 0,\ b_3 \neq 0,\ a_4^2 + a_5^2 \neq 0$. On the other hand if $a_3 = b_3 = 0$ the circling point curve splits up into the inflection circle and the pole normal. In the case of $a_4^2 + a_5^2 \neq 0$ any point on the inflection circle with the possible exception of the origin is a Bl point of the zero position, the origin being a Bl point too, if $a_4 = a_5 = 0$ at the same time.

The aforementioned analysis of Bl points in Minkowski plane is outlined in the following table.

Conditions	Bl point(s)
$a_3 \neq 0, a_3 \neq \pm b_3$	$\left(\frac{a_3b_3}{a_3^2-b_3^2},\quad \frac{a_3^2}{a_3^2-b_3^2}\right)$
$a_3 = a_4 = a_5 = 0, b_3 \neq 0$	the origin
$a_3 = 0,\ b_3 \neq 0,\ a_4^2 + a_5^2 \neq 0$	none
$a_3 = b_3 = 0, a_4^2 + a_5^2 \neq 0$	the points on the inflection circle with the exception of the origin
$a_3 = a_4 = a_5 = b_3 = 0$	all points of the inflection circle

As a consequence, if $a_3 \neq 0$ and $a_3 \neq \pm b_3$ the Bl point of the zero position is in the parametric representation (11) of *cp* indicated by the parameter value $u = a_3/b_3$.

6 Ball Points with Excess in Minkowski Plane

Definition 3. *If we have for a Ball point of a given position*

$$\kappa = \kappa' = ... = \kappa^{(r+1)} = 0,\ \kappa^{(r+2)} \neq 0$$

this point is called a Ball point with excess r and denoted by Bl_r point.

In the case of $a_3 \neq 0$, the zero position has a Bl point. Under this consideration the following theorem can be given.

Theorem 4. *In the case $a_3 \neq 0$, the Bl point is a Bl_1 point if and only if*

$$a_4b_3 - a_3b_4 = a_3.$$

Proof: From the equation (2), $\kappa = \kappa' = \kappa'' = 0$ if and only if $X_1Y_4 - X_4Y_1 = 0$. If we substitute the equation (1) into $X_1Y_4 - X_4Y_1 = 0$ we get

$$x^2 - y^2 + a_4x - b_4y = 0.$$

If the Bl_1 point has the coordinates (x_0, y_0) this last equation takes form of

$$x_0^2 - y_0^2 + a_4x_0 - b_4y_0 = 0. \tag{13}$$

In virtue of Bl_1 point is also on the inflection circle, the common solution of $x_0^2 - y_0^2 + y_0 = 0$ and the equation (4) gives us

$$a_4x_0 + (-b_4 - 1)\, y_0 = 0. \tag{14}$$

Substituting the equation (12) into the equation (14) completes the proof. □

This relation represents a necessary and sufficient condition for the Bl point of the zero position to be a Bl_1 point for the case of $a_3 \neq 0$. In the zero position if $a_3 = a_4 = a_5 = 0$, $b_3 \neq 0$ the origin is the only Bl point. From the equation (7) this point is a Bl_1 point if and only if $a_6 = 0$. In the case of $a_3 = b_3 = 0$, $a_4^2 + a_5^2 \neq 0$ any point of the inflection circle with the exception of the origin is a Bl point of the zero position. From the equation (13) and the equation (14) it follows that all these points are Bl_1 points if and only if $a_4 = 0$, $b_4 = -1$ whereas in the case $a_4 \neq 0$ the only Bl_1 point of the zero position is given by:

$$\left(\frac{(b_4+1)\, a_4}{{a_4}^2 - (b_4+1)^2},\quad \frac{{a_4}^2}{{a_4}^2 - (b_4+1)^2}\right).$$

In the case $a_3 = a_4 = a_5 = b_3 = 0$ any point of the inflection circle is a Bl point of the zero point.

If $b_4 = -1$ at the same time, all these points with exception of the origin are Bl_1 points, the origin being in this case is a Bl_1 point if moreover $a_6 = 0$. If, however, $b_4 \neq -1$ there is no Bl_1 point unless $a_6 = 0$ in which case the origin is the only Bl_1 point of the zero position. From here, we give conditions of being a Bl_1 point in Minkowski plane in the following table.

Condition(s)	Bl_1 point(s)
$a_3 = a_4 b_3 - a_3 b_4 \neq 0$, $a_3 \neq \pm b_3$	$\left(\frac{a_3 b_3}{a_3^2 - b_3^2}, \frac{a_3^2}{a_3^2 - b_3^2}\right)$
$a_3 = b_3 = 0,\ a_4 \neq 0$	$\left(\frac{a_4(b_4+1)}{a_4^2-(b_4+1)^2}, \frac{a_4^2}{a_4^2-(b_4+1)^2}\right)$
$a_3 = a_4 = a_5 = a_6 = 0$, $a_4^2 - (b_4+1)^2 \neq 0$	Origin

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