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Compactness of Matrix Operators on the Banach Space $\ell_p(T)$

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Abstract: In this study, by using the Hausdorff measure of non-compactness, we obtain the necessary and sufficient conditions for certain matrix operators on the spaces $\ell_p(T)$ and $\ell_\infty(T)$ to be compact, where $1 \leq p < \infty$.

Keywords: Compact operators, Hausdorff measure of non-compactness, Sequence spaces.

1 Introduction

By ω, we denote the space of all real sequences. Any subset of ω is called a sequence space. Let Ψ, ℓ_∞, c and c_0 denote the sets of all finite, bounded, convergent and null sequences, respectively and $\ell_p = \{u = (u_n) \in \omega : \sum_n |u_n|^p < \infty\}$ for $1 \leq p < \infty$. Throughout the study, we assume that $p, q \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

A B-space is a complete normed space. A topological sequence space in which all coordinate functionals π_k, $\pi_k(u) = u_k$, are continuous is called a K-space. A BK-space is defined as a K-space which is also a B-space, that is, a BK-space is a Banach space with continuous coordinates. A BK-space $\lambda \supset \psi$ is said to have AK if every sequence $u = (u_k) \in \lambda$ has a unique representation $u = \sum_k u_k e^{(k)}$, where $e^{(k)}$ is the sequence whose only non-zero term is 1 in the nth place for each $k \in \mathbb{N}$. For example, the space ℓ_p $(1 \leq p < \infty)$ is a BK-space with the norm $\|u\|_p = (\sum_k |u_k|^p)^{1/p}$ and c_0 and ℓ_∞ is a BK-space with the norm $\|u\|_\infty = \sup_k |u_k|$. Also, the BK-spaces c_0 and ℓ_p have AK but c and ℓ_∞ do not have AK.

The β-dual of a sequence space λ is defined by

$$\lambda^\beta = \{z = (z_k) \in \omega : zu = (z_k u_k) \in cs \text{ for all } u = (u_k) \in \lambda\}.$$

Let $\mathcal{A}$ be the sequence of n^{th} row of an infinite matrix $\mathcal{A} = (\mathfrak{a}_{nk})$ with real numbers $\mathfrak{a}_{nk}$ for each $n \in \mathbb{N}$. For a sequence $u = (u_k) \in \omega$, the $\mathcal{A}$-transform of u is the sequence $\mathcal{A}u = (\mathcal{A}_n(u))$, where

$$\mathcal{A}_n(u) = \sum_{n=0}^{\infty} \mathfrak{a}_{nk} u_k$$

provided that the series is convergent for each $n \in \mathbb{N}$.

(λ, μ) stands for the class of all infinite matrices from a sequence space λ into another sequence space μ. Hence, $\mathcal{A} \in (\lambda, \mu)$ if and only if $\mathcal{A}_n \in \lambda^\beta$ for all $n \in \mathbb{N}$.

Let λ be a normed space and S_λ be the unit sphere in λ. For a BK-space $\lambda \supset \psi$ and $z = (z_k) \in \omega$, we use the notation

$$\|z\|_\lambda^* = \sup_{u \in S_\lambda} \left| \sum_k z_k u_k \right|$$

under the assumption that the supremum is finite. In this case observe that $z \in \lambda^\beta$.

Lemma 1. *[1, Theorem 1.29] $\ell_1^\beta = \ell_\infty$, $\ell_p^\beta = \ell_q$ and $\ell_\infty^\beta = \ell_1$, where $1 < p < \infty$. If $\lambda \in \{\ell_1, \ell_p, \ell_\infty\}$, then $\|z\|_\lambda^* = \|z\|_{\lambda^\beta}$ holds for all $z \in \lambda^\beta$, where $\|.\|_{\lambda^\beta}$ is the natural norm on λ^β.*

By $\mathcal{B}(\lambda, \mu)$, we denote the set of all bounded (continuous) linear operators from λ to μ.

Lemma 2. *[1, Theorem 1.23 (a)] Let λ and μ be BK-spaces. Then, for every $\mathcal{A} \in (\lambda, \mu)$, there exists a linear operator $L_\mathcal{A} \in \mathcal{B}(\lambda, \mu)$ such that $L_\mathcal{A}(u) = \mathcal{A}u$ for all $u \in \lambda$.*

Lemma 3. *[1] Let $\lambda \supset \psi$ be a BK-space and $\mu \in \{c_0, c, \ell_\infty\}$. If $\mathcal{A} \in (\lambda, \mu)$, then*

$$\|L_{\mathcal{A}}\| = \|\mathcal{A}\|_{(\lambda,\mu)} = \sup_n \|\mathcal{A}_n\|^*_\lambda < \infty.$$

The Hausdorff measure of noncompactness of a bounded set Q in a metric space λ is defined by

$$\chi(Q) = \inf\{\varepsilon > 0 : Q \subset \cup_{i=1}^n B(x_i, r_i), x_i \in \lambda, r_i < \varepsilon, n \in \mathbb{N}\},$$

where $B(x_i, r_i)$ is the open ball centered at x_i and radius ε for each $i = 1, 2, ..., n$.

The following theorem is useful to compute the Hausdorff measure of non-compactness in ℓ_p for $1 \leq p < \infty$.

Theorem 1. *[2] Let Q be a bounded subset in ℓ_p for $1 \leq p < \infty$ and $P_r : \ell_p \to \ell_p$ be the operator defined by $P_r(u) = (u_0, u_1, u_2, ..., u_r, 0, 0, ...)$ for all $u = (u_k) \in \ell_p$ and each $r \in \mathbb{N}$. Then, we have*

$$\chi(Q) = \lim_r \left(\sup_{u \in Q} \|(I - P_r)(u)\|_{\ell_p} \right),$$

where I is the identity operator on ℓ_p.

Let λ and μ be Banach spaces. Then, a linear operator $L : \lambda \to \mu$ is is said to be compact if the domain of L is all of λ and $L(Q)$ is a totally bounded subset of μ for every bounded subset Q in λ. Equivalently, we say that L is compact if its domain is all of λ and for every bounded sequence $u = (u_n)$ in λ, the sequence $(L(u_n))$ has a convergent subsequence in μ.

The idea of compact operators between Banach spaces is closely related to the Hausdorff measure of non-compactness. For $L \in \mathcal{B}(\lambda, \mu)$, the Hausdorff measure of non-compactness of L denoted by $\|L\|_\chi$ is given by

$$\|L\|_\chi = \chi(L(S_\lambda))$$

and we have

$$L \text{ is compact if and only if } \|L\|_\chi = 0.$$

Several authors have studied compact operators on the sequence spaces and given very important results related to the Hausdorff measure of non-compactness of a linear operator. For example [3]-[9].

The main purpose of this study is to obtain necessary and sufficient conditions for some matrix operators to be compact. For this purpose, we use the Banach spaces $\ell_p(T)$ and $\ell_\infty(T)$ introduced in [10] as

$$\ell_p(T) = \left\{ u = (u_n) \in \omega : \sum_n \left| t_n u_n - \frac{1}{t_n} u_{n-1} \right|^p < \infty \right\} \quad (1 \leq p < \infty)$$

and

$$\ell_\infty(T) = \left\{ u = (u_n) \in \omega : \sup_n \left| t_n u_n - \frac{1}{t_n} u_{n-1} \right| < \infty \right\}.$$

Here, the difference matrix matrix $T = (t_{nk})$ is defined by

$$t_{nk} = \begin{cases} t_n & , \quad k = n \\ -\frac{1}{t_n} & , \quad k = n-1 \\ 0 & , \quad k > n \text{ or } 0 \leq k < n-1, \end{cases}$$

where $t_n > 0$ for all $n \in \mathbb{N}$ and $t = (t_n) \in c \backslash c_0$.

Note that we use the sequence $v = (v_n)$ for the T-transform of a sequence $u = (u_n)$, that is,

$$v_n = T_n(u) = \begin{cases} t_0 u_0 & , \quad n = 0 \\ t_n u_n - \frac{1}{t_n} u_{n-1} & , \quad n \geq 1 \end{cases} \quad (n \in \mathbb{N}).$$

2 Compact Operators on the Spaces $\ell_p(T)$ and $\ell_\infty(T)$

For a sequence $a = (a_k) \in \omega$, we define a sequence $\tilde{a} = (\tilde{a}_k)$ as $\tilde{a}_k = \sum_{j=k}^\infty t_k \prod_{i=k}^j \frac{1}{t_i^2} a_j$ for all $k \in \mathbb{N}$.

We need the following results in the sequel.

Lemma 4. *Let $a = (a_k) \in (\ell_p(T))^\beta$, where $1 \leq p \leq \infty$. Then $\tilde{a} = (\tilde{a}_k) \in \ell_q$ and*

$$\sum_k a_k u_k = \sum_k \tilde{a}_k v_k \tag{1}$$

for all $u = (u_k) \in \ell_p(T)$.

Lemma 5. *The following statements hold.*

(a) $\|a\|^*_{\ell_1(T)} = \sup_k |\tilde{a}_k| < \infty$ *for all* $a = (a_k) \in (\ell_1(T))^\beta$.
(b) $\|a\|^*_{\ell_p(T)} = \left(\sum_k |\tilde{a}_k|^q\right)^{1/q} < \infty$ *for all* $a = (a_k) \in (\ell_p(T))^\beta$, *where* $1 \le p \le \infty$.
(c) $\|a\|^*_{\ell_\infty(T)} = \sum_k |\tilde{a}_k| < \infty$ *for all* $a = (a_k) \in (\ell_\infty(T))^\beta$.

Proof: We only prove part (a) and the others can be proved analogously. Choose $a = (a_k) \in (\ell_1(T))^\beta$. Then, by Lemma 4, we have $\tilde{a} = (\tilde{a}_k) \in \ell_\infty$ and (1) holds. Since $\|u\|_{\ell_1(T)} = \|v\|_{\ell_1}$ holds, we obtain that $u \in S_{\ell_1(T)}$ if and only if $v \in S_{\ell_1}$. Hence, we deduce that $\|a\|^*_{\ell_1(T)} = \sup_{u \in S_{\ell_1(T)}} |\sum_k a_k u_k| = \sup_{v \in S_{\ell_1}} |\sum_k \tilde{a}_k v_k| = \|\tilde{a}\|^*_{\ell_1}$. From Lemma 1, it follows that $\|a\|^*_{\ell_1(T)} = \|\tilde{a}\|^*_{\ell_1} = \|\tilde{a}\|_{\ell_\infty} = \sup_k |\tilde{a}_k|$. $\square$

Throughout this section, we use the matrix $\tilde{\mathcal{A}} = (\tilde{\mathfrak{a}}_{nk})$ defined by an infinite matrix $\mathcal{A} = (\mathfrak{a}_{nk})$ via

$$\tilde{\mathfrak{a}}_{nk} = \sum_{j=k}^{\infty} t_k \prod_{i=k}^{j} \frac{1}{t_i^2} \mathfrak{a}_{nj}$$

for all $n, k \in \mathbb{N}$ under the assumption that the series is convergent.

Lemma 6. *Let* λ *be a sequence space. If* $\mathcal{A} \in (\ell_p(T), \lambda)$, *then* $\tilde{\mathcal{A}} \in (\ell_p, \lambda)$ *and* $\mathcal{A}u = \tilde{\mathcal{A}}v$ *for all* $u \in \ell_p(T)$, *where* $1 \le p \le \infty$.

Lemma 7. *If* $\mathcal{A} \in (\ell_1(T), \ell_p)$, *then we have*

$$\|L_{\mathcal{A}}\| = \|\mathcal{A}\|_{(\ell_1(T), \ell_p)} = \sup_k \left(\sum_n |\tilde{\mathfrak{a}}_{nk}|^p\right)^{1/p} < \infty,$$

where $1 \le p \le \infty$.

Lemma 8. *[11, Theorem 3.7] Let* $\lambda \supset \psi$ *be a BK-space. Then, the following statements hold.*
(a) $\mathcal{A} \in (\lambda, \ell_\infty)$, *then* $0 \le \|L_{\mathcal{A}}\|_\chi \le \limsup_n \|\mathcal{A}_n\|^*_\lambda$.
(b) $\mathcal{A} \in (\lambda, c_0)$, *then* $\|\mathcal{A}_S\|_\chi \le \limsup_n \|\mathcal{A}_n\|^*_\lambda$.
(c) If λ *has AK or* $\lambda = \ell_\infty$ *and* $\mathcal{A} \in (\lambda, c)$, *then*

$$\frac{1}{2} \limsup_n \|\mathcal{A}_n - \alpha\|^*_\lambda \le \|L_{\mathcal{A}}\|_\chi \le \limsup_n \|\mathcal{A}_n - \alpha\|^*_\lambda,$$

where $\alpha = (\alpha_k)$ *and* $\alpha_k = \lim_n \mathfrak{a}_{nk}$ *for all* $k \in \mathbb{N}$.

Theorem 2.

1. *For* $\mathcal{A} \in (\ell_1(T), \ell_\infty)$,

$$0 \le \|L_{\mathcal{A}}\|_\chi \le \limsup_n \left(\sup_k |\tilde{\mathfrak{a}}_{nk}|\right)$$

holds.
2. *For* $\mathcal{A} \in (\ell_1(T), c)$,

$$\frac{1}{2} \limsup_n \left(\sup_k |\tilde{\mathfrak{a}}_{nk} - \tilde{\alpha}_k|\right) \le \|L_{\mathcal{A}}\|_\chi \le \limsup_n \left(\sup_k |\tilde{\mathfrak{a}}_{nk} - \tilde{\alpha}_k|\right)$$

holds.
3. *For* $\mathcal{A} \in (\ell_1(T), c_0)$,

$$\|L_{\mathcal{A}}\|_\chi = \limsup_n \left(\sup_k |\tilde{\mathfrak{a}}_{nk}|\right)$$

holds.
4. *For* $\mathcal{A} \in (\ell_1(T), \ell_1)$,

$$\|L_{\mathcal{A}}\|_\chi = \lim_m \left(\sup_k \sum_{n=m}^{\infty} |\tilde{\mathfrak{a}}_{nk}|\right)$$

holds.

Corollary 1.

1. $L_{\mathcal{A}}$ *is compact for* $\mathcal{A} \in (\ell_1(T), \ell_\infty)$ *if*

$$\lim_n \left(\sup_k |\tilde{\mathfrak{a}}_{nk}|\right) = 0.$$

2. $L_{\mathcal{A}}$ *is compact for* $\mathcal{A} \in (\ell_1(T), c)$, *if and only if*

$$\lim_n \left(\sup_k |\tilde{\mathfrak{a}}_{nk} - \tilde{\alpha}_k|\right) = 0.$$

3. $L_{\mathcal{A}}$ *is compact for* $\mathcal{A} \in (\ell_1(T), c_0)$ *if and only if*

$$\lim_n \left(\sup_k |\tilde{\mathfrak{a}}_{nk}| \right) = 0.$$

4. $L_{\mathcal{A}}$ *is compact for* $\mathcal{A} \in (\ell_1(T), \ell_1)$ *if and only if*

$$\lim_m \left(\sup_k \sum_{n=m}^{\infty} |\tilde{\mathfrak{a}}_{nk}| \right) = 0.$$

Lemma 9. *Let* $\lambda \supset \psi$ *be a BK-space. If* $\mathcal{A} \in (\lambda, \ell_1)$*, then*

$$\lim_r \left(\sup_{N \in \mathcal{K}_r} \left\| \sum_{n \in N} \mathcal{A}_n \right\|_{\lambda}^{*} \right) \le \|L_{\mathcal{A}}\|_{\chi} \le 4 \lim_r \left(\sup_{N \in \mathcal{K}_r} \left\| \sum_{n \in N} \mathcal{A}_n \right\|_{\lambda}^{*} \right)$$

and $L_{\mathcal{A}}$ *is compact if and only if* $\lim_r \left(\sup_{N \in \mathcal{K}_r} \| \sum_{n \in N} \mathcal{A}_n \|_{\lambda}^{*} \right) = 0$*, where* $\mathcal{K}_r$ *is the subcollection of* $\mathcal{K}$ *consisting of subsets of* $\mathbb{N}$ *with elements that are greater than* r*.*

Theorem 3. *Let* $1 < p < \infty$*.*

1. *For* $\mathcal{A} \in (\ell_p(T), \ell_\infty)$*,*

$$0 \le \|L_{\mathcal{A}}\|_{\chi} \le \limsup_n \left(\sum_k |\tilde{\mathfrak{a}}_{nk}|^q \right)^{1/q}$$

holds.

2. *For* $\mathcal{A} \in (\ell_p(T), c)$*,*

$$\frac{1}{2} \limsup_n \left(\sum_k |\tilde{\mathfrak{a}}_{nk} - \tilde{\alpha}_k|^q \right)^{1/q} \le \|L_{\mathcal{A}}\|_{\chi} \le \limsup_n \left(\sum_k |\tilde{\mathfrak{a}}_{nk} - \tilde{\alpha}_k|^q \right)^{1/q}$$

holds.

3. *For* $\mathcal{A} \in (\ell_p(T), c_0)$*,*

$$\|L_{\mathcal{A}}\|_{\chi} = \limsup_n \left(\sum_k |\tilde{\mathfrak{a}}_{nk}|^q \right)^{1/q}$$

holds.

4. *For* $\mathcal{A} \in (\ell_p(T), \ell_1)$*,*

$$\lim_m \|\mathcal{A}\|_{(\ell_p(T),\ell_1)}^{(m)} \le \|L_{\mathcal{A}}\|_{\chi} \le 4 \lim_m \|\mathcal{A}\|_{(\ell_p(T),\ell_1)}^{(m)}$$

holds, where $\|\mathcal{A}\|_{(\ell_p(T),\ell_1)}^{(m)} = \sup_{N \in \mathcal{K}_m} \left(\sum_k | \sum_{n \in N} \tilde{\mathfrak{a}}_{nk}|^q \right)^{1/q}$.

Corollary 2. *Let* $1 < p < \infty$*.*

1. $L_{\mathcal{A}}$ *is compact for* $\mathcal{A} \in (\ell_p(T), \ell_\infty)$ *if*

$$\lim_n \left(\sum_k |\tilde{\mathfrak{a}}_{nk}|^q \right)^{1/q} = 0.$$

2. $L_{\mathcal{A}}$ *is compact for* $\mathcal{A} \in (\ell_p(T), c)$ *if and only if*

$$\lim_n \left(\sum_k |\tilde{\mathfrak{a}}_{nk} - \tilde{\alpha}_k|^q \right)^{1/q} = 0.$$

3. $L_{\mathcal{A}}$ *is compact for* $\mathcal{A} \in (\ell_p(T), c_0)$ *if and only if*

$$\lim_n \left(\sum_k |\tilde{\mathfrak{a}}_{nk}|^q \right)^{1/q} = 0.$$

4. $L_{\mathcal{A}}$ *is compact for* $\mathcal{A} \in (\ell_p(T), \ell_1)$ *if and only if*

$$\lim_m \|\mathcal{A}\|_{(\ell_p(T),\ell_1)}^{(m)} = 0,$$

where $\|\mathcal{A}\|_{(\ell_p(T),\ell_1)}^{(m)} = \sup_{N \in \mathcal{K}_m} \left(\sum_k | \sum_{n \in N} \tilde{\mathfrak{a}}_{nk}|^q \right)^{1/q}$.

Theorem 4.

1. *For* $\mathcal{A} \in (\ell_\infty(T), \ell_\infty)$,

$$0 \leq \|L_{\mathcal{A}}\|_\chi \leq \limsup_n \sum_k |\tilde{\mathfrak{a}}_{nk}|$$

holds.

2. *For* $\mathcal{A} \in (\ell_\infty(T), c)$,

$$\frac{1}{2} \limsup_n \sum_k |\tilde{\mathfrak{a}}_{nk} - \tilde{\alpha}_k| \leq \|L_{\mathcal{A}}\|_\chi \leq \limsup_n \sum_k |\tilde{\mathfrak{a}}_{nk} - \tilde{\alpha}_k|$$

holds.

3. *For* $\mathcal{A} \in (\ell_\infty(T), c_0)$,

$$\|L_{\mathcal{A}}\|_\chi = \limsup_n \sum_k |\tilde{\mathfrak{a}}_{nk}|$$

holds.

Corollary 3.

1. $L_{\mathcal{A}}$ *is compact for* $\mathcal{A} \in (\ell_\infty(T), \ell_\infty)$ *if*

$$\lim_n \sum_k |\tilde{\mathfrak{a}}_{nk}| = 0.$$

2. $L_{\mathcal{A}}$ *is compact for* $\mathcal{A} \in (\ell_\infty(T), c)$, *if and only if*

$$\lim_n \sum_k |\tilde{\mathfrak{a}}_{nk} - \tilde{\alpha}_k| = 0.$$

3. $L_{\mathcal{A}}$ *is compact for* $\mathcal{A} \in (\ell_\infty(T), c_0)$ *if and only if*

$$\lim_n \sum_k |\tilde{\mathfrak{a}}_{nk}| = 0.$$

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