Conference Proceedings of Science and Technology, 1(1), 2018, 16–19

Conference Proceeding of 7th International Eurasian Conference on Mathematical Sciences and Applications (IECMSA-2018).

ISSN: 2651-544X

http://dergipark.gov.tr/cpost

Helicoidal Surfaces Which Have the Timelike Axis in Minkowski Space with Density

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Abstract: In this paper, we study the prescribed curvature problem in manifold with density. We consider the Minkowski 3-space with a positive density function. For a given plane curve and an axis in the plane in Minkowski 3-space, a helicoidal surface can be constructed by the plane curve under helicoidal motions around the axis. Also we give examples of helicoidal surface with weighted Gaussian curvature.

Keywords: Helicoidal, Manifold with density, Minkowski space, Weighted curvature.

1 Introduction

Helicoidal surface is a natural generalization of a rotation surface. A few works have been done with helicoidal surafaces under some given certain conditions [1–4]. Recently, the popular question is whether a helicoidal surface can be constructed when its curvatures are prescribed. Several researchers worked on this problem and obtained useful results. Firstly, Baikoussis et. al have studied helicoidal surfaces with prescribed mean and Gaussian curvature in $\mathbb{R}^3$ [5]. Then, Beneki et. al [6] and Ji et. al [7] have studied the similar work in $\mathbb{R}^3_1$. This problem is extended to manifolds with density. Helicoidal surfaces with prescribed mean and Gaussian curvature in $\mathbb{R}^3$ with density have been studied by Dae Won Yoon et. al [8]. Furthermore, Yıldız et. al have constructed the type I^+ helicoidal surfaces with prescribed weighted curvatures in $\mathbb{R}^3_1$ with density [9].

A manifold with a positive density function ψ used to weight the volume and the hypersurface area. In terms of the underlying Riemannian volume dV_0 and area dA_0, the new, weighted volume and area are given by $dV = \psi dV_0$ and $dA = \psi dA_0$, respectively. One of the most important examples of manifolds with density, with applications to probability and statistics, is Gauss space with density $\psi = e^{a\left(-x^2-y^2-z^2\right)}$ for $a \in \mathbb{R}$, $(x, y, z) \in \mathbb{R}^3$ [10]. For more details on manifolds with density, see [10–16].

In the Minkowski $3-$space with density e^{φ}, the weighted Gaussian curvature is given with

$$G_{\varphi} = G - \triangle\varphi$$

where G is the Gaussian curvature of the surface and $\triangle$ is the Laplacian operator [17].

In this paper, we study helicoidal surfaces which have the timelike axis in the Minkowski $3-$space R^3_1 with density e^{φ}, where $\varphi = x^2$. Firstly, we construct a helicodial surface with prescribed weighted Gaussian curvature. Finally, we give examples to illustrate.

2 Preliminaries

The Minkowski $3-$space $\mathbb{R}^3_1$ is the real vector space $\mathbb{R}^3$ provided with the standart flat metric given by

$$ds^2 = -dx^2 + dy^2 + dz^2$$

where (x, y, z) is a rectangular coordinate system of $\mathbb{R}^3_1$.

For a given plane curve and an axis in the plane in $\mathbb{R}^3_1$, a helicoidal surface can be constructed by the plane curve under helicoidal motions g_t : $\mathbb{R}^3_1 \to \mathbb{R}^3_1$, $t \in \mathbb{R}$ around the axis. So, a helicoidal surface is non-degenerate and invariant under g_t, $t \in \mathbb{R}$ for which one parameter subgroup of rigid motions is in $\mathbb{R}^3_1$. There exist four kinds of helicoidal surfaces in $\mathbb{R}^3_1$ which are defined by Beneki et. al [6] and these are called type I, type II, type III , type IV. In this study, type III^+ is considered which has the timelike axis of revolution and the profile curve in $xy-$plane. In addition, the helicoidal surface is called type III^+ since the discriminant of the first fundamental form $u^2\left(1 - g'^2\right) - c^2$ is positive [6].

Let γ be a C^2-curve on $xy-$plane of type $\gamma(u) = (g(u), u, 0)$ where $u \in I$ for an open interval $I \subset \mathbb{R} - \{0\}$. By using helicoidal motion on γ, we can obtain the helicoidal as

$$X(u,v) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos v & -\sin v \\ 0 & \sin v & \cos v \end{bmatrix} \begin{bmatrix} g(u) \\ u \\ 0 \end{bmatrix} + \begin{bmatrix} cv \\ 0 \\ 0 \end{bmatrix}$$

with $x-$axis and a pitch $c \in \mathbb{R}$. So the parametric equation can be given in the form

$$X(u,v) = (g(u) + cv, u\cos v, u \sin v).$$

It is straightforward to see that the Gaussian curvature G is

$$G = \frac{u^3 g' g'' - c^2}{\left[u^2(1-g'^2) - c^2\right]^2}$$

where $u^2\left(1-g'^2\right) - c^2 > 0$ [6]. We assume that M is the surface in $\mathbb{R}^3_1$ with density e^{φ}, where $\varphi = x^2$. By considering density function, we can calculate the weighted Gaussian curvature G_φ as

$$G_\varphi = \frac{u^3 g' g'' - c^2}{\left(u^2(1-g'^2) - c^2\right)^2} - 2. \tag{1}$$

3 Helicoidal Surfaces with Prescribed Gaussian Curvature

Let's solve the ordinary differential equation (1), which is second-order nonlinear ordinary differantial equation. If we take

$$\Psi = \frac{-u^2 {g'}^2 - c^2}{\left(u^2(1-g'^2) - c^2\right)} \tag{2}$$

then we obtain

$$G_\varphi = -\frac{1}{2u}\Psi' - 2$$

equivalently,

$$\Psi' = -2uG_\varphi - 4u. \tag{3}$$

The general solution of the equation (3) becomes

$$\Psi = -2u^2 - 2\int uG_\varphi du + c_1 \tag{4}$$

where $c_1 \in \mathbb{R}$. Combining the equation (2) and the equation (4), we get

$$u^2\left(-1 - 2u^2 - 2\int uG_\varphi du + c_1\right) {g'}^2(u) = \left(u^2 - c^2\right)\left(-2u^2 - 2\int uG_\varphi du + c_1\right) + c^2.$$

It follows that

$$g(u) = \mp \int \frac{1}{u}\left[\frac{\left(u^2 - c^2\right)\left(-2u^2 - 2\int uG_\varphi du + c_1\right) + c^2}{-1 - 2u^2 - 2\int uG_\varphi du + c_1}\right]^{\frac{1}{2}} du + c_2 \tag{5}$$

where $c_2 \in \mathbb{R}$.

Conversely, for a given $c \in \mathbb{R}$ and a smooth function $G_\varphi(u)$ defined on an open interval $I \subset \mathbb{R}^+$ and an arbitrary $u_0 \in I$, there exists an open subinterval $I' \subset I$ containing u_0 and an open interval $J \subset \mathbb{R}$ containing

$$\hat{c}_1 = \left(2 + 2u^2 + 2\int uG_\varphi du\right)(u_0)$$

such that

$$F(u, c_1) = -1 - 2u^2 - 2\int uG_\varphi du > 0$$

is defined on $I' \times J$ and it is easily seen F is positive. Thus, two-parameter family of the curves can be given as

$$\gamma(u, G_\varphi(u), c, c_1, c_2) = \left(\mp \int \frac{1}{u}\left[\frac{\left(u^2 - c^2\right)\left(-2u^2 - 2\int uG_\varphi du + c_1\right) + c^2}{-1 - 2u^2 - 2\int uG_\varphi du + c_1}\right]^{\frac{1}{2}} du + c_2, u, 0\right)$$

where $(u, c_1) \in I' \times J$; $c_2 \in \mathbb{R}$, $c \in \mathbb{R}$ and G_φ is smooth function.

Therefore, we have proved the following theorem.

Theorem 1. *Let $\gamma(u)$ be a profile curve of the helicoidal surface given with $X(u,v) = (g(u) + cv, u\cos v, u\sin v)$ in $\mathbb{R}^3_1$ with density e^{x^2} and $G_\varphi(u)$ be the weighted Gaussian curvature at $(g(u), u, 0)$. Then, there exists two-parameter family of the helicoidal surface given by the curves*

$$\gamma(u, G_\varphi(u), c, c_1, c_2) = \left(\mp \int \frac{1}{u} \left[\frac{(u^2 - c^2)(4u^2 - 2\int uG_\varphi du + c_1) + c^2}{-1 + 4u^2 - 2\int uG_\varphi du + c_1} \right]^{\frac{1}{2}} du + c_2, u, 0 \right)$$

here, c_1 and c_2 are constants. Conversely, for a given smooth function $G_\varphi(u)$, one can obtain the two-parameter family of curves $\gamma(u, G_\varphi(u), c, c_1, c_2)$ being the two-parameter family of helicoidal surfaces, accepting $G_\varphi(u)$ as the weighted Gaussian curvature c as a pitch.

Example Consider a helicoidal surface with the weighted Gaussian curvature

$$G_\varphi(u) = -\frac{8}{15} - \frac{u^2}{3} + \frac{\arctan(\sqrt{15}u)}{30\sqrt{15}u}$$

in R^3_1 with density e^{x^2}. By using the equation (5), we obtain

$$g(u) = 4u$$

for $c = 1, c_1 = 0, c_2 = 0$ and the parametrization of the surface as follows

$$X(u,v) = (4u + v, u\cos v, u\sin v).$$

The figure of the surface of the domain

$$\begin{cases} 2 < u < 5 \\ -10 < v < 10 \end{cases}$$

is given in Figure 1.

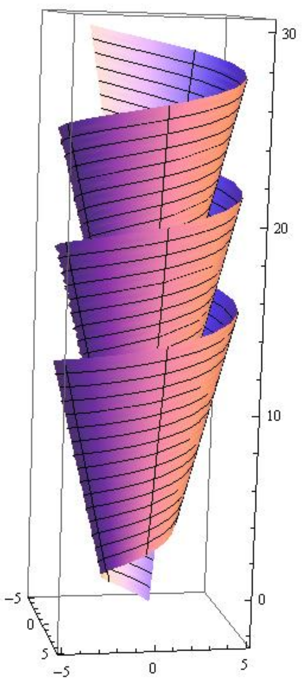

Fig. 1: The helicoidal surface with the weighted Gaussian curvature

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