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Finite Element Method for the Solution of a Time-Dependent Heat-Like Lane-Emden Equation

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Abstract

In this study, finite element method (FEM) with Galerkin Formula is applied to find the numerical solution of a time-dependent heat-like Lane-Emden equation. An example is solved to assess the accuracy of the method. The numerical results are obtained for different values (n) of equation. The results indicate that Galerkin method is effectively implemented. It is seen that results are compatible with exact solutions and consistent with other existing numerical methods.

1. Introduction

In this paper, we consider heat-type equation for physical problems

$$u_{xx}+\frac{r}{x}u_x+ag(x,t)u+h(x,t)=u_t, \tag{1.1}$$

for $0<x\leq L$, $0<t<T$, $r>0$, $a\in Z$, subject to the boundary conditions

$$u(0,t)=v(t), \quad u_x(0,t)=0.$$

where $g(x,t)y(u)+h(x,t)$ is nonlinear heat source, $u(x,t)$ is the temperature, and t is the dimensionless time variable.

Some researchers dealed with this type of models. The analytic solutions to several forms of the above problem were presented by [1], Wazwaz used the Adomian decomposition method [2]. Chowdhury, He and Noorani solved these problems using homotopy-perturbation and variational iteration methods, Momani applied the method to the time fractional heat-like equation with variable coefficient, Ucar applied non-polynomial spline method to this equation [3, 4, 5, 6, 7].

In this study, we construct so-called finite element approximations to solutions to time-dependent heat-like equations. The term ”finite element method” has come to be associated with using piecewise polynomials in one, two, and three dimensions together with so-called Rayleigh-Ritz method and its more general counter part, the Galerkin method, to approximate solutions to operator equations. In this study, we concentrate on Galerkin method with splines.

The paper is organized as follows: Galerkin method is described and solution of equation (1.1) is presented in Section 2 briefly. In Section 3 some numerical results that are illustrated using MATLAB programme are given to clarify the method. Concluding remarks are given in Section 4.

2. Galerkin method

A usual scalar product for two real valued functions $u(x)$ and $v(x)$ is defined by $<u,v>=\int_0^T u(x)v(x)dx$, $u(x)$ and $v(x)$ are orthogonal if $<u,v>=0$. And a norm associated with this scalar product is defined by

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$\|u\| = \sqrt{<u,u>} = (\int_0^T |u(x)|^2 dx)^{\frac{1}{2}}$. Let

i. $T_h : 0 = x_0 < x_1 < ... < x_M < x_{M+1} = 1$ be a partition of $(0,1)$, $h_j = x_j - x_{j-1}$
ii.$V_h = v : v$, continuous and piecewise linear function on T_h with $v(0) = v(1) = 0$
iii.$\{\varphi_j\}$, $j = 1,..M$ be a basis function for V_h where

$$\varphi_j(x) = \left\{ \begin{array}{cc} \frac{x-x_{j-1}}{h_j}, & x_{j-1} \le x \le x_j \\ \frac{x_{j+1}-x}{h_{j+1}}, & x_j \le x \le x_{j+1} \\ 0 \quad , & \text{otherwise} \end{array} \right\}.$$

Firstly, we should modify the equation (1.1) by applying finite difference to the right-hand side of the equality to solve it by using Galerkin method:

$$u_i'' + \frac{r}{x}u_i' + ag(x_i,t)u_i + h(x_i,t) = \frac{u_i - f(x_i)}{k}$$

where $f(x_i) = u(x_i, 0)$. From algebraic manipulations we obtain

$$ku_i'' + \frac{rk}{x}u_i' + akg(x_i,t)u_i - u_i + kh(x_i,t) + f(x_i) = 0. \tag{2.1}$$

Now the Galerkin method for the equation (2.1) is formulated as follows:
Find the approximate solution $U(x) \in V_h$ such that

$$\int_0^1 W\left(kU'' + \frac{rk}{x}U' + akg(x,t)U - U + kh(x,t) + f(x)\right)dx = 0, \quad (\forall W(x) \in V_h)$$

so we get

$$\int_0^1 \left(kWU'' + \frac{rk}{x}WU' + (akg(x,t)-1)WU\right)dx = -\int_0^1 (Wkh(x,t) + Wf(x))\,dx.$$

$$\int_0^1 WU''dx = \left[[U'W]\right]_0^1 - \int_0^1 W'U'dx$$

and since $W(0) = W(1) = 0$ for $W(x) \in V_h$ we get

$$\int_0^1 \left(-kW'U' + \frac{rk}{x}WU' + (akg(x,t)-1)WU\right)dx = -\int_0^1 (Wkh(x,t) + Wf(x))dx. \tag{2.2}$$

We may find the approximate solution $U(x) \in V_h$ by using basis functions $\varphi_j(x)$ as

$$U(x) = \sum_{j=1}^{M} c_j\varphi_j(x), \quad U'(x) = \sum_{j=1}^{M} c_j\varphi_j'(x), \quad W(x) = \sum_{i=1}^{M} s_i\varphi_i(x).$$

If we use these identities in equation (2.2), then we get

$$\int_0^1 \left[\sum_{i=1}^{M} s_i\varphi_i'(x)\sum_{j=1}^{M} kc_j\varphi_j'(x) + \sum_{i=1}^{M} s_i\varphi_i(x)\sum_{j=1}^{M}\frac{rk}{x}c_j\varphi_j'(x) + \sum_{i=1}^{M} s_i\varphi_i(x)\sum_{j=1}^{M}(akg(x,t)-1)c_j\varphi_j(x)\right]dx$$
$$= -\int_0^1 \left[\sum_{i=1}^{M} s_i\varphi_i(x)(f(x) + kh(x,t))\right]dx$$

$$\sum_{i=1}^{M} s_i \int_0^1 \sum_{j=1}^{M} c_j\left(k\varphi_i'(x)\varphi_j'(x) + \frac{rk}{x}\varphi_i(x)\varphi_j'(x) + (akg(x,t)-1)\varphi_i(x)\varphi_j(x)\right)dx = -\sum_{i=1}^{M} s_i\int_0^1 \varphi_i(x)\left(f(x) + kh(x,t)\right)dx.$$

For $|i-j| > 1$ we have $\int_0^1 \varphi_j'\varphi_i dx = 0$ and $\int_0^1 \varphi_j\varphi_i dx = 0$, since if so then we have that φ_j and φ_i have non-overlapping supports.
The method is described in matrix form in the following way:
for $i = 2, \quad j = 1,..,M$

$$\alpha_{21} = \int_h^{2h}\left(k\varphi_2'(x)\varphi_1'(x) + \frac{rk}{x}\varphi_2(x)\varphi_1'(x) + (akg(x,t)-1)\varphi_2(x)\varphi_1(x)\right)dx$$
$$\alpha_{22} = \int_h^{3h}\left(k\varphi_2'(x)\varphi_2'(x) + \frac{rk}{x}\varphi_2(x)\varphi_2'(x) + (akg(x,t)-1)\varphi_2(x)\varphi_2(x)\right)dx$$
$$\alpha_{23} = \int_{2h}^{3h}\left(k\varphi_2'(x)\varphi_3'(x) + \frac{rk}{x}\varphi_2(x)\varphi_3'(x) + (akg(x,t)-1)\varphi_2(x)\varphi_3(x)\right)dx$$

for $i = n, \quad j = 1,..,M$

$$\alpha_{n(n-1)} = \int_{(n-1)h}^{nh}\left(k\varphi_n'(x)\varphi_{n-1}'(x) + \frac{rk}{x}\varphi_n(x)\varphi_{n-1}'(x) + (akg(x,t)-1)\varphi_n(x)\varphi_{n-1}(x)\right)dx$$

$$\alpha_{nn}=\int_{(n-1)h}^{(n+1)h}\left(k\varphi_n'(x)\varphi_n'(x)+\frac{rk}{x}\varphi_n(x)\varphi_n'(x)+(akg(x,t)-1)\varphi_n(x)\varphi_n(x)\right)dx$$

$$\alpha_{n(n+1)}=\int_{nh}^{(n+1)h}\left(k\varphi_n'(x)\varphi_{n+1}'(x)+\frac{rk}{x}\varphi_n(x)\varphi_{n+1}'(x)+(akg(x,t)-1)\varphi_n(x)\varphi_{n+1}(x)\right)dx$$

for $i=M-1,\ j=1,..,M$

$$\alpha_{(M-1)(M-2)}=\int_{(M-2)h}^{(M-1)h}\left(k\varphi_{M-1}'(x)\varphi_{M-2}'(x)+\frac{rk}{x}\varphi_{M-1}(x)\varphi_{M-2}'(x)+(akg(x,t)-1)\varphi_{M-1}(x)\varphi_{M-2}(x)\right)dx$$

$$\alpha_{(M-1)(M-1)}=\int_{(M-2)h}^{(M)h}\left(k\varphi_{M-1}'(x)\varphi_{M-1}'(x)+\frac{rk}{x}\varphi_{M-1}(x)\varphi_{M-1}'(x)+(akg(x,t)-1)\varphi_{M-1}(x)\varphi_{M-1}(x)\right)dx$$

$$\alpha_{(M-1)M}=\int_{(M-1)h}^{Mh}\left(k\varphi_{M-1}'(x)\varphi_{M}'(x)+\frac{rk}{x}\varphi_{M-1}(x)\varphi_{M}'(x)+(akg(x,t)-1)\varphi_{M-1}(x)\varphi_{M}(x)\right)dx$$

so we get the matrices

$$A=\begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0\\ \alpha_{21} & \alpha_{22} & \alpha_{23} & 0 & \dots & 0 & 0\\ 0 & \alpha_{32} & \alpha_{33} & \alpha_{34} & 0 & \dots & 0\\ . & . & . & . & . & . & .\\ . & . & . & . & . & . & .\\ . & . & . & . & . & . & .\\ 0 & \dots & 0 & 0 & \alpha_{(M-1)(M-2)} & \alpha_{(M-1)(M-1)} & \alpha_{(M-1)M}\\ . & . & . & . & 0 & 0 & 1 \end{bmatrix},$$

$$B=\begin{bmatrix} u(0,t)=e^{\sin t}\\ \int_h^{3h}\varphi_2(x)\left(f(x)+kh(x,t)\right)\\ \int_{2h}^{4h}\varphi_3(x)\left(f(x)+kh(x,t)\right)\\ .\\ .\\ .\\ \int_{(M-1)h}^{(M+1)h}\varphi_M(x)\left(f(x)+kh(x,t)\right)\\ u(1,t)=e^{1+\sin t} \end{bmatrix},$$

$$C=[c_1,c_2,...,c_M]'.$$

$$AC=B. \tag{2.3}$$

Finally the approximate solution U is obtained by solving C from equation (2.3) using Matlab programme.

3. Numerical example

In this section, we test our scheme on an example. We consider the numerical results obtained by applying the scheme discussed above to the following equation

$$u''+\frac{2}{x}u'-(6+4x^2-\cos t)u=u_t \quad 0<x<1, \quad t>0$$

with initial condition

$$u(x,0)=e^{x^2}$$

and boundary conditions

$$u(0,t)=e^{\sin t}, \quad u_x(0,t)=0.$$

The exact solution of the above problem is $u(x,t)=e^{x^2+\sin t}$. The problem is solved by using the scheme above in this paper. The maximum absolute errors are listed in Table 1. Also, numerical results given by scheme are shown in Figure 1.

Table 1: Maximum absolute errors, $k=0.01$

n	Spline method[3]	Galerkin method
11	6.8863e-03	2.3842e-03
21	1.9090e-03	6.7890e-04
41	7.8276e-04	1.8533e-04
61	5.9650e-04	7.6553e-05
121	5.2565e-05	3.8863e-05

4. Conclusion

In this paper, finite element method with Galerkin formula is applied for the numerical solution of the heat-like time-dependent Lane-Emden equation and the maximum absolute errors have shown in Table 1, which shows that this method approximate the exact solution very well. The implementation of the present method is more computational than the existing methods.

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