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Multiple solutions for an anisotropic elliptic equation of Kirchhoff type in bounded domain

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Abstract

In this paper, we consider a class of anisotropic elliptic equations of Kirchhoff type

$$\begin{cases} -M\left(\sum\limits_{i=1}^{N}\int_\Omega \frac{1}{p_i(x)}|\partial_{x_i}u|^{p_i(x)}\,dx\right)\sum\limits_{i=1}^{N}\partial_{x_i}\left(|\partial_{x_i}u|^{p_i(x)-2}\partial_{x_i}u\right) = f(x,u)+h(x), & x\in\Omega,\\ u=0, \quad x\in\partial\Omega, \end{cases}$$

where $\Omega\subset\mathbb{R}^N$ $(N\geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$, $M(t)=a+bt^\tau$, $\tau>0$ is a positive constant, and p_i, $i=1,2,...,N$ are continuous functions on $\overline{\Omega}$ such that $2\leq p_i(x)<N$, $a>0$, $b\geq 0$. Under appropriate assumptions on f and h, we prove the existence of as least two weak solutions for the problem by using the Ekeland variational principle and the mountain pass theorem in critical point theory.

Keywords: anisotropic elliptic equations, Kirchhoff type equations, variable exponents, variational methods.
2010 MSC: 35J60; 35J62; 35J70.

1. Introduction

In this paper, we are interested in the existence of weak solutions for the following anisotropic elliptic equations of Kirchhoff type

$$\begin{cases} -M\left(\sum\limits_{i=1}^{N}\int_\Omega \frac{1}{p_i(x)}|\partial_{x_i}u|^{p_i(x)}\,dx\right)\sum\limits_{i=1}^{N}\partial_{x_i}\left(|\partial_{x_i}u|^{p_i(x)-2}\partial_{x_i}u\right) = f(x,u)+h(x), & x\in\Omega,\\ u=0, \quad x\in\partial\Omega, \end{cases} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$, $M(t) = a + bt^\tau$, $\tau > 0$ is a positive constant, and p_i, $i = 1, 2, ..., N$ are continuous functions on $\overline{\Omega}$ such that $2 \leq p_i(x) < N$, $a > 0$, $b \geq 0$, $h : \Omega \to \mathbb{R}$ is a measurable function, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Since the first equation in (1.1) contains an integral over Ω, it is no longer a pointwise identity; therefore it is often called nonlocal problem. This problem models several physical and biological systems, where u describes a process which depends on the average of itself, such as the population density, see [12]. Moreover, problem (1.1) is related to the stationary version of the Kirchhoff equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0 \tag{1.2}$$

presented by Kirchhoff in 1883, see [24]. This equation is an extension of the classical d'Alembert's wave equation by considering the effects of the changes in the length of the string during the vibrations. The parameters in (1.2) have the following meanings: L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density, and P_0 is the initial tension. Lions [25] has proposed an abstract framework for the Kirchhoff-type equations. Some important and interesting results can be found, for example [9, 11, 13, 15, 16, 17]. The $p(x)$-Laplacian possesses more complicated properties than the p-Laplacian operator, mainly due to the fact that it is not homogeneous. The study of various mathematical problems with variable exponent are interesting in applications and raise many difficult mathematical problems, [29].

Our main purpose is to consider the Kirchhoff type equation in a new setting corresponding to anisotropic spaces of Sobolev-type. More precisely, the $p(x)$-Laplace operator $\Delta_{p(x)}$ in the above papers is replaced with the non-homogeneous differential operator $\sum_{i=1}^{N} \partial_{x_i}\left(|\partial_{x_i} u|^{p_i(x)-2}\partial_{x_i} u\right)$. This is an anisotropic operator with a complicated structure, in which different space directions have different roles. Anisotropic type problems was firtly studied by Mihailescu et al. [26] and have received specific attention in recent decades, see [2, 3, 8, 10, 14, 19, 22, 28, 31]. In the nonlocal case of anisotropic type problems, we refer the readers to the papers [1, 5, 6, 7, 21]. Motivated by the results due to Afrouzi et al. [1], Avci et al. [5, 6], Dai et al. [17] and the ideas introduced in the papers [11, 27], we study the existence of solutions for problem (1.1) with perturbation h. Using the Ekeland variational principle and the mountain pass theorem in critical point theory, we prove that if the functions f and h satisfies some suitable conditions then problem (1.1) has at least two non-trivial weak solutions in a generalized Sobolev space described in the next section.

2. Anisotropic variable exponent Sobolev spaces

We recall in what follows some definitions and basic properties of the generalized Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ where Ω is an open subset of $\mathbb{R}^N$. In that context, we refer to the books [18, 29], the paper of Kováčik and Rákosník [23]. Set

$$C_+(\overline{\Omega}) := \{h;\ h \in C(\overline{\Omega}), h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any $h \in C_+(\overline{\Omega})$ we define

$$h^+ = \sup_{x\in\overline{\Omega}} h(x) \text{ and } h^- = \inf_{x\in\overline{\Omega}} h(x).$$

For any $p(x) \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \left\{u : \text{a measurable real-valued function such that} \int_\Omega |u(x)|^{p(x)}\,dx < \infty\right\}.$$

We recall the following so-called *Luxemburg norm* on this space defined by the formula

$$|u|_{p(x)} = \inf\left\{\mu > 0;\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \leq 1\right\}.$$

Variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many respects: they are Banach spaces, the Hölder inequality holds, they are reflexive if and only if $1 < p^- \leq p^+ < \infty$ and continuous functions are dense if $p^+ < \infty$. The inclusion between Lebesgue spaces also generalizes naturally: if $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents so that $p_1(x) \leq p_2(x)$ a.e. $x \in \Omega$ then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$. We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ the Hölder inequality

$$\left|\int_\Omega uv\,dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right)|u|_{p(x)}|v|_{p'(x)}$$

holds true.

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the *modular* of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)}\,dx.$$

If $u \in L^{p(x)}(\Omega)$ and $p^+ < \infty$ then the following relations hold

$$|u|_{p(x)}^{p^-} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^+} \tag{2.1}$$

provided $|u|_{p(x)} > 1$ while

$$|u|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^-} \tag{2.2}$$

provided $|u|_{p(x)} < 1$ and

$$|u_n - u|_{p(x)} \to 0 \ \Leftrightarrow\ \rho_{p(x)}(u_n - u) \to 0. \tag{2.3}$$

If $p \in C_+(\overline{\Omega})$ the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$, consisting of functions $u \in L^{p(x)}(\Omega)$ whose distributional gradient ∇u exists almost everywhere and belongs to $[L^{p(x)}(\Omega)]^N$, endowed with the norm

$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)},$$

is a separable and reflexive Banach space. The space of smooth functions are in general not dense in $W^{1,p(x)}(\Omega)$, but if the exponent $p \in C_+(\overline{\Omega})$ is logarithmic Hölder continuous, that is,

$$|p(x) - p(y)| \leq -\frac{M}{\log(|x-y|)}, \quad \forall x, y \in \Omega, \quad |x-y| \leq \frac{1}{2},$$

then the smooth functions are dense in $W^{1,p(x)}(\Omega)$ and so we can define the space $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_{p(x)} = |\nabla u|_{p(x)}$$

by the $p(x)$-Poincaré inequality. We point out that the above norm is equivalent with the following norm

$$\|u\|_{p(x)} = \sum_{i=1}^N |\partial_{x_i} u|_{p(x)},$$

provied that $p(x) \geq 2$ for all $x \in \overline{\Omega}$. The space $\left(W_0^{1,p(x)}(\Omega), \|.\|_{p(x)}\right)$ is a separable and Banach space. We note that if $s \in C_+(\overline{\Omega})$ and $s(x) < p^*(x)$ for all $x \in \overline{\Omega}$ then the embedding

$$W_0^{1,p(x)}(\Omega) \hookrightarrow L^{s(x)}(\Omega)$$

is compact and continuous, where $p^*(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$ or $p^*(x) = \infty$ if $p(x) > N$.

We introduce a natural generalization of the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ that will enable us to study problem (1.1) with sufficient accuracy. Define $\overrightarrow{p}:\overline{\Omega}\to\mathbb{R}^N$ the vectorial function $\overrightarrow{p}(x)=(p_1(x),p_2(x),..,p_N(x))$, the components $p_i\in C(\overline{\Omega})$, $i\in\{1,2,...,N\}$ are logarithmic Hölder continuous, that is, there exists $M>0$ such that $|p_i(x)-p_i(y)|\le -\frac{M}{\log(|x-y|)}$ for any $x,y\in\Omega$ with $|x-y|\le\frac{1}{2}$ and $i\in\{1,2,...,N\}$. We introduce the anisotropic variable exponent Sobolev space, $W_0^{1,\overrightarrow{p}(x)}(\Omega)$, as the colsure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{\overrightarrow{p}(x)}=\sum_{i=1}^N|\partial_{x_i}u|_{p_i(x)}.$$

Then $W_0^{1,\overrightarrow{p}(x)}(\Omega)$ is a reflexive and separable Banach space, see [26]. In the case when p_i are all constant functions the resulting anisotropic space is denoted by $W_0^{1,\overrightarrow{p}}(\Omega)$, where $\overrightarrow{p}$ is the constant vector $(p_1,p_2,...,p_N)$. The theory of such spaces has been developed in [10, 22]. Let us introduce $\overrightarrow{P}_+,\overrightarrow{P}_-\in\mathbb{R}^N$ and $P_+^+,P_-^+,P_+^-,P_-^-\in\mathbb{R}^+$ as

$$\overrightarrow{P}_+=(p_1^+,p_2^+,...,p_N^+),\quad \overrightarrow{P}_-=(p_1^-,p_2^-,...,p_N^-)$$

and

$$P_+^+=\max\{p_1^+,p_2^+,...,p_N^+\},\quad P_-^+=\max\{p_1^-,p_2^-,...,p_N^-\},$$
$$P_+^-=\min\{p_1^+,p_2^+,...,p_N^+\},\quad P_-^-=\min\{p_1^-,p_2^-,...,p_N^-\}.$$

Throughout this paper we assume that

$$\sum_{i=1}^N\frac{1}{p_i^-}>1 \tag{2.4}$$

and define $P_-^*\in\mathbb{R}$ and $P_{-,\infty}\in\mathbb{R}^+$ by

$$P_-^*=\frac{N}{\sum_{i=1}^N\frac{1}{p_i^-}-1},\quad P_{-,\infty}=\max\{P_-^+,P_-^*\}.$$

We recall that if $s\in C_+(\overline{\Omega})$ satisfies $1<s(x)<P_{-,\infty}$ for all $x\in\overline{\Omega}$ then the embedding $W_0^{1,\overrightarrow{p}(x)}(\Omega)\hookrightarrow L^{s(x)}(\Omega)$ is compact, see for example [26, Theorem 1].

3. Main result

In this section, we will state and prove the main result of the paper. Let us assume that $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a continuous function satisfying the following conditions:

(F1) f satisfies the subcritical growth condition

$$|f(x,t)|\le C(1+|t|^{q(x)-1}),\quad \forall(x,t)\in\Omega\times\mathbb{R},$$

where $P_+^+<q^-\le q^+<P_-^*$ and C is a positive constant;

(F2) $f(x,t)=o(|t|^{P_+^+-1})$, $t\to 0$, uniformly a.e. $x\in\Omega$;

(F3) There exists a constant $\mu>P_+^+(\tau+1)$ such that

$$\mu F(x,t):=\mu\int_0^t f(x,s)\,ds\le f(x,t)t$$

for any $x\in\Omega$ and $t\in\mathbb{R}$;

(F4) $\inf_{\{x\in\Omega;\ |t|=1\}} F(x,t) > 0$.

Definition 3.1. A function $u \in W_0^{1,\overrightarrow{p}(x)}(\Omega)$ is said to be a weak solution of problem (1.1) if

$$\left[a + b\left(\sum_{i=1}^{N}\int_\Omega \frac{1}{p_i(x)}|\partial_{x_i}u|^{p_i(x)}\,dx\right)^{\tau}\right]\sum_{i=1}^{N}\int_\Omega |\partial_{x_i}u|^{p_i(x)-2}\partial_{x_i}u\partial_{x_i}v\,dx - \int_\Omega f(x,u)v\,dx$$
$$-\int_\Omega h(x)v\,dx = 0$$

for all $v \in W_0^{1,\overrightarrow{p}(x)}(\Omega)$.

Theorem 3.2. *Suppose that $h \in L^{\frac{P_+^+}{P_+^+-1}}(\Omega)$ and $h \not\equiv 0$ and the condition (2.4) hold. Let (F1)-(F4) hold, then there exists $\delta > 0$ such that problem (1.1) has at least two different non-trivial weak solutions when $|h|_{\frac{P_+^+}{P_+^+-1}} < \delta$.*

Note that in [17], Dai et al. studied the existence and multiplicity of solutions for a class of $p(x)$-Kirchhoff type problems using variational methods. Some extensions for the anisotropic case can be found in the papers [1, 5, 6], in which the authors obatined at least one solution or infinitely many solutions for problem (1.1) with $h \equiv 0$. The purpose of this paper is to obtain at least two non-trival weak solutions for (1.1) using the mountain pass theorem [4] combined with the Ekeland variational principle [20].

Let us denote by X the anisotropic variable exponent Sobolev space $W_0^{1,\overrightarrow{p}(x)}(\Omega)$ and consider the energy functional $J : X \to \mathbb{R}$ given by the formula

$$J(u) = a\sum_{i=1}^{N}\int_\Omega \frac{1}{p_i(x)}|\partial_{x_i}u|^{p_i(x)}\,dx + \frac{b}{\tau+1}\left(\sum_{i=1}^{N}\int_\Omega \frac{1}{p_i(x)}|\partial_{x_i}u|^{p_i(x)}\,dx\right)^{\tau+1}$$
$$-\int_\Omega F(x,u)\,dx - \int_\Omega h(x)u\,dx.$$

Then by the hypothesis (F1) and the continuous embeddings, we can show that that the functional J is well-defined on X and $J \in C^1(X,\mathbb{R})$ with the derivative given by

$$J'(u)(v) = \left[a + b\left(\sum_{i=1}^{N}\int_\Omega \frac{1}{p_i(x)}|\partial_{x_i}u|^{p_i(x)}\,dx\right)^{\tau}\right]\sum_{i=1}^{N}\int_\Omega |\partial_{x_i}u|^{p_i(x)-2}\partial_{x_i}u\cdot\partial_{x_i}v\,dx$$
$$-\int_\Omega f(x,u)v\,dx - \int_\Omega h(x)v\,dx$$

for all $u, v \in X$. Hence, we can find weak solutions of problem (1.1) as the critical points of the functional J in the space X.

Lemma 3.3. *Assume that the conditions (F1) and (F3) hold and $h \in L^{\frac{P_+^+}{P_+^+-1}}(\Omega)$. Then there exist some constants $\rho, \alpha, \delta > 0$ such that $J(u) \geq \alpha$ for all $u \in X$ with $\|u\|_{\overrightarrow{p}(x)} = \rho$ when $|h|_{\frac{P_+^+}{P_+^+-1}} < \delta$.*

Proof. Since the embeddings $X \hookrightarrow L^{P_+^+}(\Omega)$ and $X \hookrightarrow L^{q(x)}(\Omega)$ are continuous, there exist constants $C_1, C_2 > 0$ such that

$$|u|_{P_+^+} \leq C_1\|u\|_{\overrightarrow{p}(x)}, \quad |u|_{q(x)} \leq C_2\|u\|_{\overrightarrow{p}(x)}. \tag{3.1}$$

Let $0 < \epsilon C_1^{P_+^+} < \frac{a}{2P_+^+ N^{P_+^+-1}}$, where C_1 is given by (3.1). From (F2) and (F3), we have

$$F(x,t) \le \epsilon |t|^{P_+^+} + C(\epsilon)|t|^{q(x)}, \quad \forall (x,t) \in \Omega \times \mathbb{R}. \tag{3.2}$$

Let $u \in X$ with $\|u\|_{\overrightarrow{p}(x)} < 1$ sufficiently small. For such an element u we get $|\partial_{x_i} u|_{p_i(x)} < 1$ for all $i = 1, 2, ..., N$. Using (2.1) and some simple computations, we obtain

$$\begin{aligned}
\sum_{i=1}^{N} \int_\Omega |\partial_{x_i} u|^{p_i(x)}\, dx &\ge \sum_{i=1}^{N} |\partial_{x_i} u|_{p_i(x)}^{p_i^+} \\
&\ge \sum_{i=1}^{N} |\partial_{x_i} u|_{p_i(x)}^{P_+^+} \\
&\ge N \left(\frac{\sum_{i=1}^N |\partial_{x_i} u|_{p_i(x)}}{N} \right)^{P_+^+} \\
&= \frac{\|u\|_{\overrightarrow{p}(x)}^{P_+^+}}{N^{P_+^+-1}}.
\end{aligned} \tag{3.3}$$

From (3.1)-(3.3), applying the Hölder inequality we have

$$\begin{aligned}
J(u) &= a \sum_{i=1}^{N} \int_\Omega \frac{1}{p_i(x)} |\partial_{x_i} u|^{p_i(x)}\, dx + \frac{b}{\tau+1} \left(\sum_{i=1}^{N} \int_\Omega \frac{1}{p_i(x)} |\partial_{x_i} u|^{p_i(x)}\, dx \right)^{\tau+1} \\
&\quad - \int_\Omega F(x,u)\, dx - \int_\Omega h(x) u\, dx \\
&\ge \frac{a}{P_+^+ N^{P_+^+-1}} \|u\|_{\overrightarrow{p}(x)}^{P_+^+} - \epsilon \int_\Omega |u|^{P_+^+}\, dx - C(\epsilon) \int_\Omega |u|^{q(x)}\, dx - |h|_{\frac{P_+^+}{P_+^+-1}} |u|_{P_+^+} \\
&\ge \frac{a}{P_+^+ N^{P_+^+-1}} \|u\|_{\overrightarrow{p}(x)}^{P_+^+} - \epsilon C_1^{P_+^+} \|u\|_{\overrightarrow{p}(x)}^{P_+^+} - C(\epsilon) C_2^{q^-} \|u\|_{\overrightarrow{p}(x)}^{q^-} - C_1 |h|_{\frac{P_+^+}{P_+^+-1}} \|u\|_{\overrightarrow{p}(x)} \\
&\ge \left(\frac{a}{2P_+^+ N^{P_+^+-1}} \|u\|_{\overrightarrow{p}(x)}^{P_+^+-1} - C(\epsilon) C_2^{q^-} \|u\|_{\overrightarrow{p}(x)}^{q^- -1} - C_1 |h|_{\frac{P_+^+}{P_+^+-1}} \right) \|u\|_{\overrightarrow{p}(x)},
\end{aligned}$$

where $C_2 > 0$ is given by (3.1). Consider the function $\gamma_1 : [0, +\infty) \to \mathbb{R}$ is given by the formula

$$\gamma_1(s) = \frac{a}{2P_+^+ N^{P_+^+-1}} s^{P_+^+-1} - C(\epsilon) C_2^{q^-} s^{q^- -1}.$$

From the definition of γ_1 and the fact that $q^- > P_+^+$, there exists a constant $s = \rho > 0$ such that $\gamma_1(\rho) = \max_{s \in [0,+\infty)} \gamma_1(s) > 0$. Taking $\delta = \frac{1}{2C_1}\gamma_1(\rho) > 0$, it then follows that, if $|h|_{\frac{P_+^+}{P_+^+-1}} < \delta$ we can choose α and $\rho > 0$ such that $J(u) \ge \alpha > 0$ for all $u \in X$ with $\|u\|_{\overrightarrow{p}(x)} = \rho$. $\square$

Lemma 3.4. *Assume that the conditions (F3) and (F4) hold. Then there exists a function $e \in X$ with $\|e\|_{\overrightarrow{p}(x)} > \rho$ such that $J(e) < 0$, where ρ is given by Lemma 3.3.*

Proof. For each $x \in \Omega$ and $t \in \mathbb{R}$, let us define the function $\gamma_2(s) = s^{-\mu} F(x, st) - F(x,t)$ for all $s \ge 1$. Then we deduce from (F3) that

$$\gamma_2'(s) = s^{-\mu-1} \left(f(x,st)st - \mu F(x,st) \right) \ge 0, \quad \forall s \ge 1$$

of the function γ_2 is increasing on $[1,+\infty)$ and $\gamma_2(\tau) \geq \gamma_2(1) = 0$ for all $s \in [1,+\infty)$. Hence,

$$F(x, st) \geq s^{\mu} F(x,t), \quad \forall x \in \Omega, \quad t \in \mathbb{R}, \quad s \geq 1. \tag{3.4}$$

Let $\varphi \in C_0^{\infty}(\Omega)$, $\varphi \geq 0$ and $\varphi \not\equiv 0$, we have

$$\begin{aligned}
J(s\varphi) &= a \sum_{i=1}^{N} \int_\Omega \frac{1}{p_i(x)} |\partial_{x_i} s\varphi|^{p_i(x)}\, dx + \frac{b}{\tau+1} \left(\sum_{i=1}^{N} \int_\Omega \frac{1}{p_i(x)} |\partial_{x_i} s\varphi|^{p_i(x)}\, dx \right)^{\tau+1} \\
&\quad - \int_\Omega F(x, s\varphi)\, dx - \int_\Omega h(x) s\varphi\, dx \\
&\leq a s^{P_+^+} \sum_{i=1}^{N} \int_\Omega \frac{1}{p_i(x)} |\partial_{x_i} \varphi|^{p_i(x)}\, dx + \frac{b s^{P_+^+(\tau+1)}}{\tau+1} \left(\sum_{i=1}^{N} \int_\Omega \frac{1}{p_i(x)} |\partial_{x_i} \varphi|^{p_i(x)}\, dx \right)^{\tau+1} \\
&\quad - s^{\mu} \int_\Omega F(x, \varphi)\, dx - s \int_\Omega h(x)\varphi\, dx \\
&\to -\infty,
\end{aligned}$$

as $s \to +\infty$ since $\mu > P_+^+(\tau+1) > P_+^+$. Therefore, there exists a constant $s_0 > 0$ such that $\|s_0\varphi\|_{\overrightarrow{p}(x)} > \rho$ and $J(s_0\varphi) < 0$. Let $e = s_0\varphi_0$ the proof of the lemma is complete. □

Lemma 3.5. *Assume that the conditions (F1)-(F3) hold. Then the functional J satisfies the Palais-Smale condition.*

Proof. Let $\{u_n\} \subset X$ be such that

$$J(u_n) \to \overline{c} > 0, \quad J'(u_n) \to 0 \text{ in } X^*. \tag{3.5}$$

We will prove that $\{u_n\}$ is bounded in X. Indeed, assume by contradiction that $\|u_n\|_{\overrightarrow{p}(x)} \to +\infty$ as $n \to \infty$. By (3.1) and the condition (F3), applying the Hölder inequality we deduce for n large enough that $\|u_n\|_{\overrightarrow{p}(x)} > 1$ and

$$\begin{aligned}
&\overline{c} + 1 + \|u_n\|_{\overrightarrow{p}(x)} \\
&\geq J(u_n) - \frac{1}{\mu} J'(u_n)(u_n) \\
&= a \sum_{i=1}^{N} \int_\Omega \frac{1}{p_i(x)} |\partial_{x_i} u_n|^{p_i(x)}\, dx + \frac{b}{\tau+1} \left(\sum_{i=1}^{N} \int_\Omega \frac{1}{p_i(x)} |\partial_{x_i} u_n|^{p_i(x)}\, dx \right)^{\tau+1} \\
&\quad - \int_\Omega F(x, u_n)\, dx - \int_\Omega h(x) u_n\, dx - \frac{a}{\mu} \sum_{i=1}^{N} \int_\Omega |\partial_{x_i} u_n|^{p_i(x)}\, dx \\
&\quad - \frac{b}{\mu} \left(\sum_{i=1}^{N} \int_\Omega \frac{1}{p_i(x)} |\partial_{x_i} u_n|^{p_i(x)}\, dx \right)^{\tau} \times \left(\sum_{i=1}^{N} \int_\Omega |\partial_{x_i} u_n|^{p_i(x)}\, dx \right) \\
&\quad + \frac{1}{\mu} \int_\Omega f(x, u_n) u_n\, dx + \frac{1}{\mu} \int_\Omega h(x) u_n\, dx \\
&\geq a \left(\frac{1}{P_+^+} - \frac{1}{\mu} \right) \sum_{i=1}^{N} \int_\Omega |\partial_{x_i} u_n|^{p_i(x)}\, dx + b \left(\frac{1}{\tau+1} - \frac{P_+^+}{\mu} \right) \left(\sum_{i=1}^{N} \int_\Omega \frac{1}{p_i(x)} |\partial_{x_i} u_n|^{p_i(x)}\, dx \right)^{\tau+1} \\
&\quad + \int_\Omega \left(\frac{1}{\mu} f(x, u_n) u_n - F(x, u_n) \right) dx - \left(1 - \frac{1}{\mu} \right) \int_\Omega h(x) u_n\, dx
\end{aligned}$$

$$\geq a\left(\frac{1}{P_+^+}-\frac{1}{\mu}\right)\sum_{i=1}^N\int_\Omega|\partial_{x_i}u_n|^{p_i(x)}\,dx-\left(1-\frac{1}{\mu}\right)|h|_{\frac{P_+^+}{P_+^+-1}}|u|_{P_+^+}$$

since $\mu > P_+^+(\tau+1) > P_+^+$. For each $i \in \{1,2,...,N\}$ and n we define

$$\alpha_{i,n}=\begin{cases}P_+^+, & \text{if } |\partial_{x_i}u_n|_{p_i(x)}<1,\\ P_-^-, & \text{if } |\partial_{x_i}u_n|_{p_i(x)}>1.\end{cases}\tag{3.6}$$

Using (3.6), we infer that for n large enough,

$$\begin{aligned}&\overline{c}+1+\|u_n\|_{\overrightarrow{p}(x)}\\
&\geq a\left(\frac{1}{P_+^+}-\frac{1}{\mu}\right)\sum_{i=1}^N\int_\Omega|\partial_{x_i}u_n|^{p_i(x)}\,dx-\left(1-\frac{1}{\mu}\right)|h|_{\frac{P_+^+}{P_+^+-1}}|u|_{P_+^+}\\
&\geq a\left(\frac{1}{P_+^+}-\frac{1}{\mu}\right)\sum_{i=1}^N|\partial_{x_i}u_n|_{p_i(x)}^{\alpha_{i,n}}-C_1\left(1-\frac{1}{\mu}\right)|h|_{\frac{P_+^+}{P_+^+-1}}\|u\|_{\overrightarrow{p}(x)}\\
&\geq a\left(\frac{1}{P_+^+}-\frac{1}{\mu}\right)\sum_{i=1}^N|\partial_{x_i}u_n|_{p_i(x)}^{P_-^-}-a\left(\frac{1}{P_+^+}-\frac{1}{\mu}\right)\sum_{\{i:\ \alpha_{i,n}=P_+^+\}}\left(|\partial_{x_i}u_n|_{p_i(x)}^{P_-^-}-|\partial_{x_i}u_n|_{p_i(x)}^{P_+^+}\right)\\
&\quad-C_1\left(1-\frac{1}{\mu}\right)|h|_{\frac{P_+^+}{P_+^+-1}}\|u_n\|_{\overrightarrow{p}(x)}\\
&\geq a\left(\frac{1}{P_+^+}-\frac{1}{\mu}\right)\frac{1}{N^{P_-^-}}\|u_n\|_{\overrightarrow{p}(x)}^{P_-^-}-aN\left(\frac{1}{P_+^+}-\frac{1}{\mu}\right)-C_1\left(1-\frac{1}{\mu}\right)|h|_{\frac{P_+^+}{P_+^+-1}}\|u_n\|_{\overrightarrow{p}(x)},\end{aligned}$$

where $\mu > P_+^+(\tau+1) > P_+^+ > 1$.

Dividing by $\|u\|_{\overrightarrow{p}(x)}^{P_-^-}$ in the above inequality and passing to the limit as $n\to\infty$, we obtain a contradiction. This follows that the sequence $\{u_n\}$ is bounded in X.

Now, since the Banach space X is reflexive, there exists $u \in X$ such that passing to a subsequence, still denoted by $\{u_n\}$, it converges weakly to u in X and converges strongly to u in the space $L^{q(x)}(\Omega)$. Using the condition (F1) and the Hölder inequality, we have

$$\begin{aligned}\left|\int_\Omega f(x,u_n)(u_n-u)\,dx\right|&\leq\int_\Omega|f(x,u_n)||u_n-u|\,dx\\
&\leq C\int_\Omega(1+|u_n|^{q(x)-1})|u_n-u|\,dx\\
&\leq C\Big(1+||u_n|^{q(x)-1}|_{\frac{q(x)}{q(x)-1}}\Big)|u_n-u|_{q(x)}\\
&\to 0 \text{ as } n\to\infty,\end{aligned}$$

which yields

$$\lim_{m\to\infty}\int_\Omega f(x,u_n)(u_n-u)\,dx=0.\tag{3.7}$$

Moreover, we have

$$\left|\int_\Omega h(x)(u_n-u)\,dx\right|\leq\int_\Omega|h(x)||u_n-u|\,dx$$

$$\leq |h|_{\frac{P_+^+}{P_+^+-1}} |u_n - u|_{P_+^+}$$
$$\to 0 \text{ as } n \to \infty,$$

Therefore, since $\{u_n\}$ converges weakly to u in X, by (3.5) we have $J'(u_n)(u_n - u) \to 0$ as $n \to \infty$ which leads to

$$\lim_{n\to\infty} \left[a + b\left(\sum_{i=1}^N \int_\Omega \frac{1}{p_i(x)} |\partial_{x_i} u_n|^{p_i(x)}\, dx\right)^\tau\right] \sum_{i=1}^N \int_\Omega |\partial_{x_i} u_n|^{p_i(x)-2} \partial_{x_i} u_n (\partial_{x_i} u_n - \partial_{x_i} u)\, dx = 0$$

or

$$\lim_{n\to\infty} \sum_{i=1}^N \int_\Omega |\partial_{x_i} u_n|^{p_i(x)-2} \partial_{x_i} u_n (\partial_{x_i} u_n - \partial_{x_i} u)\, dx = 0.$$

Combining this with the fact that $\{u_n\}$ converges weakly to u in X, we get

$$\lim_{m\to\infty} \sum_{i=1}^N \int_\Omega \Big(|\partial_{x_i} u_m|^{p_i(x)-2} \partial_{x_i} u_m - |\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u\Big)(\partial_{x_i} u_m - \partial_{x_i} u)\, dx = 0. \tag{3.8}$$

Next, we apply the following inequality (see [30])

$$(|\xi|^{r-2}\xi - |\eta|^{r-2}\eta) \cdot (\xi - \eta) \geq 2^{-r}|\xi - \eta|^r, \quad \xi, \eta \in \mathbb{R}^N, \tag{3.9}$$

valid for all $r \geq 2$. Relations (3.8) and (3.9) show actually $\{u_m\}$ converges strongly to u in X. Thus, the functional J satisfies the Palais-Smale condition. □

Lemma 3.6. *Assume that the conditions (F1)-(F4) hold. Then, there exists a function $\psi \in X$, $\psi \not\equiv 0$ such that $J(\tau\psi) < 0$ for all $\tau > 0$ small enough.*

Proof. For any $(x, t) \in \Omega \times \mathbb{R}$, set $\gamma_3(\tau) = F(x, \tau^{-1}t)\tau^\mu$, $\tau \geq 1$. By (F3), we have

$$\begin{aligned}
\gamma_3'(\tau) &= f(x, \tau^{-1}t)\left(-\frac{t}{\tau^2}\right)\tau^\mu + F(x, \tau^{-1}t)\mu\tau^{\mu-1} \\
&= t^{\mu-1}[\mu F(x, \tau^{-1}t) - \tau^{-1} t f(x, \tau^{-1}t)] \\
&\leq 0
\end{aligned}$$

so, $\gamma_3(t)$ is nonincreasing. Thus, for any $|t| \geq 1$, we have $\gamma_3(1) \geq \gamma_3(|t|)$, that is

$$F(x, t) \geq F(x, |t|^{-1}t)|t|^\mu \geq C_3|t|^\mu, \tag{3.10}$$

where $C_3 = \inf_{x\in\Omega, |t|=1} F(x, t) > 0$ by (F4). From (F2), there exists a constant $\eta > 0$ such that

$$\left|\frac{f(x,t)t}{|t|^{P_+^+}}\right| = \left|\frac{f(x,t)}{|t|^{P_+^+-1}}\right| \leq 1, \tag{3.11}$$

for all $x \in \Omega$ and all $0 < |t| \leq \eta$. By (F1), for all $x \in \Omega$ and all $\eta \leq |t| \leq 1$, there exists $C_4 > 0$ such that

$$\left|\frac{f(x,t)t}{|t|^{P_+^+}}\right| \leq \frac{C(1 + |t|^{q(x)-1})|t|}{|t|^{P_+^+}} \leq C_4. \tag{3.12}$$

From (3.11) and (3.12), we deduce that

$$f(x, t)t \geq -(C_4 + 1)|t|^{P_+^+}$$

for all $x \in \Omega$ and all $|t| \in [0,1]$. Using the equality $F(x,t) = \int_0^1 f(x,st)t\,ds$, we obtain

$$F(x,t) \geq -\frac{1}{P_+^+}(C_4+1)|t|^{P_+^+} \tag{3.13}$$

for all $x \in \Omega$ and all $|t| \in [0,1]$. Taking $C_5 = \frac{1}{P_+^+}(C_4+1) + C_3$, we then get from (3.10) and (3.13) that

$$F(x,t) \geq C_3|t|^{\mu} - C_5|t|^{P_+^+} \tag{3.14}$$

for all $x \in \Omega$ and all $t \in \mathbb{R}$.

We now prove that there exists a function $\psi \in X$ such that $J(s\psi) < 0$ for all $s > 0$ small enough. Let $\psi \in C_0^\infty(\Omega)$ be such that

$$\int_\Omega h(x)\psi(x)\,dx > 0,$$

then by (3.14) we have

$$\begin{aligned} J(s\psi) &= a\sum_{i=1}^N \int_\Omega \frac{1}{p_i(x)}|\partial_{x_i} s\psi|^{p_i(x)}\,dx + \frac{b}{\tau+1}\left(\sum_{i=1}^N \int_\Omega \frac{1}{p_i(x)}|\partial_{x_i} s\psi|^{p_i(x)}\,dx\right)^{\tau+1} \\ &\quad - \int_\Omega F(x,s\psi)\,dx - \int_\Omega h(x)s\psi\,dx \\ &\leq as^{P_-^-}\sum_{i=1}^N \int_\Omega \frac{1}{p_i(x)}|\partial_{x_i}\psi|^{p_i(x)}\,dx + \frac{bs^{P_-^-(\tau+1)}}{2}\left(\sum_{i=1}^N \int_\Omega \frac{1}{p_i(x)}|\partial_{x_i}\psi|^{p_i(x)}\,dx\right)^{\tau+1} \\ &\quad - C_5 s^{\mu}\int_\Omega |\psi|^{\mu}\,dx + C_3 s^{P_+^+}\int_\Omega |\psi|^{P_+^+}\,dx - s\int_\Omega h(x)\psi\,dx \\ &< 0, \end{aligned}$$

for all $s > 0$ small enough since $\mu > P_+^+(\tau+1)$. □

Proof of Theorem 3.2. By Lemmas 3.3-3.5, there exists $\delta > 0$ such that for if $|h|_{\frac{P_+^+}{P_+^+-1}} < \delta$, all assumptions of the mountain pass theorem by Ambrosetti-Rabinowitz [4] hold. Then, there exists a critical point $u_1 \in X$ of the functional J, i.e. $J'(u_1) = 0$ and thus, problem (1.1) has a nontrivial weak solution $u_1 \in X$ with positive energy $J(u_1) = \overline{c} > 0$. We will show the existence of the second non-trivial weak solution $u_2 \in X$ and $u_2 \neq u_1$ by using the Ekeland variational principle.

Indeed, by Lemma 3.3, it follows that on the boundary of the ball centered at the origin and of radius ρ in X, denoted by $B_\rho(0)$, we have

$$\overline{c} = \inf_{u \in \partial B_\rho(0)} J(u) > 0.$$

On the other hand, by Lemma 3.3 again, the functional J is bounded from below on $B_\rho(0)$. Moreover, by Lemma 3.6, there exists $\varphi \in X$ such that $J(\tau\varphi) < 0$ for all $\tau > 0$ small enough. It follows that

$$-\infty < \underline{c} = \inf_{u \in \overline{B}_\rho(0)} J(u) < 0.$$

Let us choose $\epsilon > 0$ such that $0 < \epsilon < \inf_{u \in \partial B_\rho(0)} J(u) - \inf_{u \in \overline{B}_\rho(0)} J(u)$. Applying the Ekeland variational principle in [20] to the functional $J : \overline{B}_\rho(0) \to \mathbb{R}$, it follows that there exists $u_\epsilon \in \overline{B}_\rho(0)$ such that

$$\begin{aligned} J(u_\epsilon) &< \inf_{u \in \overline{B}_\rho(0)} J(u) + \epsilon, \\ J(u_\epsilon) &< J(u) + \epsilon\|u - u_\epsilon\|_{\overrightarrow{p}(x)}, \quad u \neq u_\epsilon, \end{aligned}$$

then, we have $J(u_\epsilon) < \inf_{u\in\partial B_\rho(0)} J(u)$ and thus, $u_\epsilon \in B_\rho(0)$.

Now, we define the functional $I : \overline{B}_\rho(0) \to \mathbb{R}$ by $I(u) = J(u) + \epsilon\|u - u_\epsilon\|_{\overrightarrow{p}(x)}$. It is clear that u_ϵ is a minimum point of I and thus

$$\frac{I(u_\epsilon + \tau v) - I(u_\epsilon)}{t} \geq 0$$

for all $\tau > 0$ small enough and all $v \in B_\rho(0)$. The above information shows that

$$\frac{J(u_\epsilon + \tau v) - J(u_\epsilon)}{\tau} + \epsilon\|v\|_{\overrightarrow{p}(x)} \geq 0.$$

Letting $\tau \to 0^+$, we deduce that

$$\left\langle J'(u_\epsilon), v\right\rangle \geq -\epsilon\|v\|_{\overrightarrow{p}(x)}.$$

It should be noticed that $-v$ also belongs to $B_\rho(0)$, so replacing v by $-v$, we get

$$\left\langle J'(u_\epsilon), -v\right\rangle \geq -\epsilon\| - v\|_{\overrightarrow{p}(x)}$$

or

$$\left\langle J'(u_\epsilon), v\right\rangle \leq \epsilon\|v\|_{\overrightarrow{p}(x)},$$

which helps us to deduce that $\|J'(u_\epsilon)\|_{X^*} \leq \epsilon$. Therefore, there exists a sequence $\{u_n\} \subset B_\rho(0)$ such that

$$J(u_n) \to \underline{c} = \inf_{u\in\overline{B}_\rho(0)} J(u) < 0 \text{ and } J'(u_n) \to 0 \text{ in } X^* \text{ as } n \to \infty. \tag{3.15}$$

From Lemma 3.5, the sequence $\{u_n\}$ converges strongly to some u_2 as $n \to \infty$. Moreover, since $J \in C^1(X, \mathbb{R})$, by (3.15) it follows that $J'(u_2) = 0$. Thus, u_2 is a non-trivial weak solution of problem (1.1) with negative energy $J(u_2) = \underline{c} < 0$.

Finally, we point out the fact that $u_1 \neq u_2$ since $J(u_1) = \overline{c} > 0 > \underline{c} = J(u_2)$. The proof of Theorem 3.2 is complete. □

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