An Extension of Angelov's Fixed Point Theorem in Uniform Spaces

Ljubomir P. Georgiev

*Department of Mathematics, University of Mining and Geology “St. I. Rilski”, 1700 Sofia, Bulgaria

Abstract

In this paper we establish an existence result for fixed points of mapping in a uniform space, which extends some previous theorems of V. G. Angelov [1].

Keywords: fixed point theorems, uniform space.

2010 MSC: 47H10, 47H09

1. Introduction and Preliminaries

We begin the present note recalling some basic notions from [1], [2].

Further on we denote by \((X, A)\) a Hausdorff sequentially complete uniform space, whose uniformity is generated by a saturated family \(A = \{\rho_\alpha : \alpha \in A\}\) of pseudo-metrics \(\rho_\alpha : X \times X \to [0, \infty), A\) being an index set.

Recall that a Hausdorff uniform space is called sequentially complete if any Cauchy sequence in it is convergent. The sequence \(\{x_n \in X\}_{n=1}^\infty\) is said to be Cauchy one if for every \(\varepsilon > 0\) and \(\alpha \in A\) there is a natural number \(n_0 \in \mathbb{N} := \{1, 2, 3, \ldots\}\) such that \(\rho_\alpha(x_m, x_n) < \varepsilon\) for all \(m, n \geq n_0\). The sequence \(\{x_n \in X\}_{n=1}^\infty\) is called convergent if there exists an element \(x \in X\) such that for every \(\varepsilon > 0\) and \(\alpha \in A\), there exists \(n_0 \in \mathbb{N}\) with \(\rho_\alpha(x, x_n) < \varepsilon\) for all \(n \geq n_0\). Let us point out that the uniform spaces and gauge spaces are equivalent notions [3].

Let \((\Phi) = \{\Phi_\alpha : \alpha \in A\}\) be a family of functions \(\Phi_\alpha(\cdot) : [0, \infty) \to [0, \infty)\) with the properties (for every fixed \(\alpha \in A\)):

\((\Phi1)\) \(\Phi_\alpha(\cdot)\) is non-decreasing and continuous from the right;

\((\Phi2)\) \(0 < \Phi_\alpha(t) < t, \forall t > 0.\) (It follows \(\Phi_\alpha(0) = 0.\))

Let \(j : A \to A\) be an arbitrary mapping of the index set into itself. The iterations can be defined inductively as follows: \(j^n(\alpha) = j(j^{n-1}(\alpha)), j^0(\alpha) = \alpha (n = 1, 2, 3, \ldots).\)
Definition 1.1. [1] Let $M$ be a subset of $X$ and $T : M \to M$ be a mapping. $T$ is said to be $(\Phi, j)$–contractive on $M$, if $\rho_\alpha(T(x), T(y)) \leq \Phi_\alpha(\rho_j(x, y))$ for every fixed $\alpha \in A$ and for every $x, y \in M$.

Remark 1.2. Recall that if $\Phi_\alpha \in (\Phi)$ and the function $\varphi_\alpha : (0, \infty) \to (0, \infty)$, defined by

$$\varphi_\alpha(t) = \frac{\Phi_\alpha(t)}{t} \quad \forall t \in (0, \infty),$$

is non-decreasing, then $\sum_{i=0}^{\infty} \Phi^i_\alpha(t) < \infty$ for any fixed $t \in (0, \infty)$

(where $\Phi^0_\alpha(t) = t$, $\Phi^i_\alpha(t) = \Phi_\alpha(\Phi^{i-1}_\alpha(t))$, $i \in \mathbb{N}$).

Indeed, for any fixed $l \in \mathbb{N}$ and $t > 0$ it follows by (\Phi 2) $\Phi_\alpha^l(t) = \Phi_\alpha(\Phi^{l-1}_\alpha(t)) < \Phi^{l-1}_\alpha(t)$. Therefore $\Phi_\alpha^l(t) < \Phi^{l-1}_\alpha(t) < \Phi^{l-2}(t) \ldots < \Phi_\alpha(t) < t$ and the inequalities

$$\frac{\Phi_\alpha^{l+1}(t)}{\Phi_\alpha^l(t)} = \varphi_\alpha\left(\frac{\Phi_\alpha^l(t)}{\Phi_\alpha(t)}\right) \leq \varphi_\alpha\left(\Phi^{l-1}_\alpha(t)\right) \leq \ldots \leq \varphi_\alpha(\Phi_\alpha(t)) \leq \varphi_\alpha(t) = \frac{\Phi_\alpha(t)}{t} < 1 \quad (l \in \mathbb{N})$$

are a sufficient condition for the convergence of $\sum_{i=0}^{\infty} \Phi^i_\alpha(t)$.

Fixed point theorems from [1], [2] guarantee an existence of fixed points of $(\Phi, j)$–contractive (or just $(\Phi)$–contractive) and $j$– non-expansive mappings under various conditions.

In this paper we extend the following result from [1]:

Theorem 1.3. (Angelov). Let the following conditions hold:

1) the operator $T : X \to X$ is $(\Phi)$–contractive;
2) for every $\alpha \in A$ there exists a function $\overline{\Phi}_\alpha \in (\Phi)$ such that

$$\sup \left\{ \Phi_\alpha^{j_n(\alpha)}(t) : n = 0, 1, 2, 3, \ldots \right\} \leq \overline{\Phi}_\alpha(t) \quad \text{and} \quad \frac{\overline{\Phi}_\alpha(t)}{t} \text{ is non-decreasing } (t > 0);$$
3) there exists an element $x_0 \in X$ such that for every $\alpha \in A$ there is $q(\alpha) > 0$:

$$\rho_j^{j_n(\alpha)}(x_0, T(x_0)) \leq q(\alpha) < \infty \quad (n = 0, 1, 2, 3, \ldots).$$

Then $T$ has at least one fixed point in $X$.

If, in addition, we suppose that

4) for every $\alpha \in A$ and $x, y \in X$ there exists $p = p(x, y, \alpha)$ such that

$$\rho_\alpha^p(x, y) \leq p(x, y, \alpha) < \infty \quad (k = 0, 1, 2, 3, \ldots),$$

then the fixed point of $T$ is unique.

2. Main results

Let $j_1 : A \to A$, $j_2 : A \to A$ be two mappings of the index set into itself.

In this paper we introduce the notion of $(\Phi, j_1, j_2)$–contractive mappings and we establish some fixed point results for such mappings in uniform spaces.

Introduce the subfamily $(\Psi) \subset (\Phi)$ of functions $\Phi_\alpha \in (\Phi)$ which are sub-additive, i.e.

$(\Phi 3)$ for every $\alpha \in A$ $(\forall \Phi_\alpha \in (\Psi))$ $: \Phi_\alpha(t_1 + t_2) \leq \Phi_\alpha(t_1) + \Phi_\alpha(t_2)$ for all $t_1, t_2 \in [0, \infty)$.

Definition 2.1. The mapping $T : X \to X$ is said to be $(\Phi, j_1, j_2)$–contractive on $X$, if for any fixed $\alpha \in A$ there is a function $\Phi_\alpha \in (\Psi)$ such that $\rho_\alpha(T(x), T(y)) \leq \frac{1}{2} \Phi_\alpha(\rho_{j_1}(x, y) + \rho_{j_2}(x, y))$ for every $x, y \in X$.

Define the mapping $S_1 : A \to j_1(A) \cup j_2(A)$ as follows: $S_1(\gamma) = \{j_1(\gamma), j_2(\gamma)\}$ for $\gamma \in A$. Introduce for any fixed index $\alpha \in A$ the following notations: $S^0(\alpha) \equiv \alpha_0 \equiv \alpha; S^1(\alpha) \equiv S_1(\alpha);$

$$S^n(\alpha) = \{\sigma^n = (\alpha_1, \ldots, \alpha_n) : k \in S_1(\alpha_{k-1}), k = 1, \ldots, n\} \quad \forall n \in \mathbb{N}, n > 1.$$}

Theorem 2.2. Let $(X, A)$ be a Hausdorff sequentially complete uniform space, whose uniformity is generated by a saturated family of pseudo-metrics $\mathbf{A} = \{\rho_\alpha(x, y) : \alpha \in A\}$, where $A$ is an index set. Let the mappings $j_1 : A \to A$ and $j_2 : A \to A$ be defined and $(\Psi) = \{\Phi_\alpha : \alpha \in A\}$ be the family of functions with properties $(\Phi 1)$ – $(\Phi 3)$. Let the following conditions hold:
1. The mapping $T : X \to X$ is $(\Phi, j_1, j_2)$-contractive on $X$.

2. For every $\alpha \in A$ there is a function $\Phi_\alpha \in (\Phi)$ such that $\frac{\Phi_\alpha(t)}{t}$ is non-decreasing, $\Phi_\alpha(t) \leq \Phi_\alpha(t)$, $\forall t > 0$ and for any fixed $k \in \mathbb{N}$ for all $\sigma^k = (\alpha_1, ..., \alpha_k) \in S^k(\alpha)$ the inequalities $\Phi_\alpha(t) \leq \Phi_\alpha(t)$, $\forall t > 0$ are satisfied for all coordinates $\alpha_i$ of $\sigma^k (i = 1, ..., k)$.

3. There exists an element $x_0 \in X$ such that for every $\alpha \in A$ there exists a constant $q_\alpha = q(\alpha) > 0$ such that $\rho_\alpha(x_0,T(x_0)) \leq q_\alpha$ and for any fixed $k \in \mathbb{N}$ for all $\sigma^k = (\alpha_1, ..., \alpha_k) \in S^k(\alpha)$ the inequalities $\rho_\alpha(x_0,T(x_0)) \leq q_\alpha$ are satisfied for all coordinates $\alpha_m$ of $\sigma^k (m = 1, ..., k)$.

Then $T$ has at least one fixed point in $X$.

**Theorem 2.3.** If to the conditions of Theorem 2.2 we add the following assumption for the set $X$:

4. for any fixed $\alpha \in A$ there exists $p_\alpha : X \times X \to (0, \infty)$ such that $p_\alpha(x,y) \leq p_\alpha(x,y)$ for all $(x,y) \in X \times X$ and for any fixed $k \in \mathbb{N}$ for all $\alpha^k = (\alpha_1, ..., \alpha_k) \in S^k(\alpha)$ the inequalities $p_\alpha(x,y) \leq p_\alpha(x,y)$ are satisfied for all $(x,y) \in X \times X$ and for all coordinates $\alpha_1$ of $\sigma^k (l = 1, ..., k)$, then the fixed point of $T$ is unique.

Proof. (of Theorem 2.2) Begin with $x_0 \in X$, we define the sequence $\{x_n : n = 0, 1, 2, \ldots\}$, $x_n = T^n(x_0)$, where $T^0 \equiv I$ and $T^n(\cdot) = T(T^{n-1}(\cdot))$ for $n \in \mathbb{N}$. If $x_n = x_{n+1}$ for some $n' \in \mathbb{N}$ then $x_{n+1}$ is a fixed point of $T$. Consequently we may assume $x_n \neq x_{n+1}, \forall n \in \mathbb{N}$.

Let $\alpha \in A$ be any fixed index. Define the sequence $\{c_\alpha^n\}_{n=0}^\infty$; $c_\alpha^n = \rho_\alpha(x_n,x_{n+1}) (n = 0, 1, 2, ...).$ For any fixed $n \in \mathbb{N}$ let $\sigma^n = (\alpha_1, ..., \alpha_n)$ be an arbitrary element of $S^n(\alpha)$.

Define $c_{\alpha}^{n-k} = \rho_\alpha(x_k,x_{k+1})$ for every $k = 1, ..., n$ (with $\alpha_0 \equiv \alpha$ and $c_\alpha^{n0} \equiv c_\alpha^n$). It follows:

$$c_\alpha^{n-1} = \rho_\alpha(x_1,x_2) \leq \frac{1}{2} \Phi_\alpha(\rho_{j_1}(x_0,x_1) + \rho_{j_2}(x_0,x_1))$$

$$\leq \frac{1}{2} \Phi_\alpha(\rho_{j_1}(x_0,x_1)) + \frac{1}{2} \Phi_\alpha(\rho_{j_2}(x_0,x_1)) \leq \frac{1}{2} \Phi_\alpha(\rho_{j_1}(x_0,x_1)) + \frac{1}{2} \Phi_\alpha(\rho_{j_2}(x_0,x_1)) \leq \frac{1}{2} \Phi_\alpha(q_\alpha) = \Phi_\alpha(q_\alpha).$$

Therefore $c_\alpha^{n-k} \leq \Phi_\alpha(q_\alpha)$ for any $\alpha_{n-k} \in S_1(\alpha_{n-2})$. By induction, for every choice of $\sigma^n \in S^n(\alpha)$ and its coordinates $\alpha_{n-k}$, we prove:

$$c_\alpha^{n-k} \leq \Phi_\alpha(q_\alpha) \forall k = 1, ..., n - 1.$$
Hence \( c_n^\alpha \leq \Phi_\alpha \left( \rho_{\alpha^*_n} (x_{n-1}, x_n) \right) = \Phi_\alpha \left( c_{n-1}^\alpha \right) \) (since \( \alpha^*_n \in S_1 (\alpha) \)).

Thus we obtain the inequality \( c_n^\alpha \leq \Phi_\alpha (q_n) \), which is valid for any fixed index \( \alpha \in A \) and for every \( n \in \mathbb{N} \). Consequently for every fixed \( m = 0, 1, 2, \ldots \) and \( p \in \mathbb{N} \) we obtain:

\[
\rho_\alpha (x_m, x_{m+p}) \leq \sum_{k=0}^{p-1} c_{m+k}^\alpha \leq \sum_{k=0}^{p-1} \Phi_\alpha (q_{\alpha}) = \Psi_{m+p} - \Psi_m,
\]

where \( \Psi_k = \sum_{l=0}^{k-1} \Phi_\alpha (q_m) \) is the \( k \)-th partial sum of the series \( \sum_{l=0}^{\infty} \Phi_\alpha (q_m) \), which is convergent, in view of Remark 12.

Therefore for any \( \varepsilon > 0 \) there exists \( N_0 \in \mathbb{N} \) such that \( \forall m \geq N_0: \rho_\alpha (x_m, x_{m+p}) \leq \Psi_{m+p} - \Psi_m < \varepsilon \) for every \( p \in \mathbb{N} \), i.e. \( \{ x_n = T^n (x_0) : n = 0, 1, 2, \ldots \} \) is a Cauchy sequence in \( X \). In view of the sequential completeness of \( (X, A) \) there exists \( x \in X: \rho_\alpha (x_m, x) \rightarrow 0, \forall \alpha \in A \).

The right-continuity of \( \Phi_\alpha \) and the inequalities

\[
\rho_\alpha (x, T(x)) \leq \rho_\alpha (x, T^{n+1}(x_0)) + \rho_\alpha (T^{n+1}(x_0), T(x)) \leq \rho_\alpha (x, x_{n+1}) + \frac{1}{2} \Phi_\alpha \left( \rho_{\alpha_1} (x_{n+1}, x) + \rho_{\alpha_2} (x_{n+1}, x) \right) (n \in \mathbb{N})
\]

imply \( \rho_\alpha (x, T(x)) = 0 \forall \alpha \in A \), that is \( x = T(x) \). Theorem 2.2 is thus proved.

\[ \square \]

**Proof. (of Theorem 2.3)** Let \((x, y) \in X \times X\) be an arbitrary fixed pair. Let \( \alpha \in A \) be any fixed index. Denote by \( d_{n}^\alpha = d_n^\alpha (x, y) = \rho_\alpha (T^n (x), T^n (y)) \) \((n = 0, 1, 2, \ldots)\). For any fixed \( n \in \mathbb{N} \) let \( \sigma^n = (\alpha_1, \ldots, \alpha_n) \) be an arbitrary element of \( S^n (\alpha) \).

Define \( d_{k}^{n-k} = d_{k}^{n-k} (x, y) = \rho_\alpha (T^k (x), T^k (y)) \) for every \( k = 1, \ldots, n \) (with \( \alpha_0 = \alpha \) and \( d_{n}^\alpha = d_{n}^\alpha \)). It follows:

\[
d_{k}^{n-1} = \rho_{\alpha_{n-1}} (T(x), T(y)) \leq \frac{1}{2} \Phi_{\alpha_{n-1}} (\rho_{\alpha_1} (x_{n-1}, x) + \rho_{\alpha_2} (x_{n-1}, x)) \leq \frac{1}{2} \Phi_{\alpha_{n-1}} (\rho_{\alpha_1} (x_{n-1}, x)) + \frac{1}{2} \Phi_{\alpha_{n-1}} (\rho_{\alpha_2} (x_{n-1}, x)) \leq \Phi_\alpha (\rho_{\alpha_{n}^*} (x_{n}))
\]

where \( \alpha_{n}^* \in S_1 (\alpha_{n-1}) \) is such that \( \rho_\alpha (x_{n}) = \max \{ \rho_{\alpha_1} (x_{n-1}, x), \rho_{\alpha_2} (x_{n-1}, x) \} \).

Therefore \( d_{k}^{n-1} \leq \Phi_\alpha (p_{\alpha} (x, y)) \) for \( \alpha_{n-1} \in S_1 (\alpha_{n-2}) \).

By induction, for every choice of \( \sigma^n \in S^n (\alpha) \) and its coordinates \( \alpha_{n-k} \), we prove:

\[
d_{k}^{n-k} \leq \Phi_\alpha (p_{\alpha} (x, y)) \forall k = 1, 2, \ldots, n - 1.
\]

Indeed, we have just obtained that the estimates are valid for \( k = 1 \). If we suppose that the above inequalities are satisfied for all \( k \leq m < n - 1 \), then for \( k = m + 1 \) we obtain:

\[
d_{m+1}^{n-m-1} = \rho_{\alpha_{m-1}} (T^{m+1}(x), T^{m+1}(y)) \leq \frac{1}{2} \Phi_{\alpha_{m-1}} (\rho_{\alpha_1} (x_{m-1}, T^{m}(x))) + \frac{1}{2} \Phi_{\alpha_{m-1}} (\rho_{\alpha_2} (x_{m-1}, T^{m}(y))) \leq \Phi_\alpha (d_{m+1}^{n-m})
\]

where \( \alpha_{n-m}^* \in S_1 (\alpha_{n-m-1}) \) is such that

\[
d_{m-1}^{n-m} = \rho_{\alpha_{m-1}} (T^{m}(x), T^{m}(y)) = \max \{ \rho_{\alpha_1} (T^{m}(x), T^{m}(y)), \rho_{\alpha_2} (x_{m-1}, T^{m}(x)), \rho_{\alpha_2} (x_{m-1}, T^{m}(y)) \}
\]

In particular, \( d_{m-1}^{n-m} \leq \Phi_\alpha (p_{\alpha} (x, y)) \). It follows \( d_{m+1}^{n-m} \leq \Phi_\alpha (\rho_{\alpha} (x, y)) = \Phi_{\alpha} (p_{\alpha} (x, y)) \). This completes the induction. Finally,

\[
d_{n}^{n} = \rho_{\alpha} (T^{n}(x), T^{n}(y)) \leq \frac{1}{2} \Phi_{\alpha} (\rho_{\alpha_1} (T^{n-1}(x), T^{n-1}(y)) + \rho_{\alpha_2} (T^{n-1}(x), T^{n-1}(y))) \leq \Phi_{\alpha} (\rho_{\alpha_n^*} (T^{n-1}(x), T^{n-1}(y))),
\]


131
where \( \alpha_1^* \in S_1(\alpha) \) is such that \( \rho_{\alpha_1^*}(x, y) = \max\{\rho_{j_1(\alpha)}(x, y), \rho_{j_2(\alpha)}(x, y)\} \), and consequently
\[
\rho_{\alpha_1^*}(T^{n-1}(x), T^{n-1}(y)) = d_{\rho_{\alpha_1^*}}^{n-1}(p_{\alpha}(x, y)).
\]

The properties of the function \( \Phi_\alpha \in (\Phi) \) guarantee that \( \Phi_\alpha^n(t) \to 0 \) for any fixed \( t \in [0, \infty) \). Thus, if we suppose that there exist two elements \( x \neq y \) of \( X \), for which \( x = T(x) \) and \( y = T(y) \), then for every index \( \alpha \in A \)
\[
\rho_\alpha(T^n(x), T^n(y)) = d_\alpha^n \leq \Phi_\alpha^n(p_\alpha(x, y))
\]
for all \( n \in \mathbb{N} \), which implies \( \rho_\alpha(x, y) = 0 \) for every \( \alpha \in A \).

The obtained contradiction proves Theorem 2.3.

References