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An Extention of Angelov's Fixed Point Theorem in Uniform Spaces

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Abstract

In this paper we establish an existence result for fixed points of mapping in a uniform space, which extends some previous theorems of V. G. Angelov [1].

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1. Introduction and Preliminaries

We begin the present note recalling some basic notions from [1], [2].

Further on we denote by $(X, \mathbf{A})$ a Hausdorff sequentially complete uniform space, whose uniformity is generated by a saturated family $\mathbf{A} = \{\rho_\alpha : \alpha \in A\}$ of pseudo-metrics $\rho_\alpha : X \times X \to [0, \infty)$, A being an index set.

Recall that a Hausdorff uniform space is called *sequentially complete* if any Cauchy sequence in it is convergent. The sequence $\{x_n \in X\}_{n=1}^{\infty}$ is said to be *Cauchy* one if for every $\varepsilon > 0$ and $\alpha \in A$ there is a natural number $n_0 \in \mathbb{N} := \{1, 2, 3, ...\}$ such that $\rho_\alpha(x_m, x_n) < \varepsilon$ for all $m, n \geq n_0$. The sequence $\{x_n \in X\}_{n=1}^{\infty}$ is called *convergent* if there exists an element $x \in X$ such that for every $\varepsilon > 0$ and $\alpha \in A$, there exists $n_0 \in \mathbb{N}$ with $\rho_\alpha(x, x_n) < \varepsilon$ for all $n \geq n_0$. Let us point out that the uniform spaces and gauge spaces are equivalent notions [3].

Let $(\Phi) = \{\Phi_\alpha : \alpha \in A\}$ be a family of functions $\Phi_\alpha(\cdot) : [0, \infty) \to [0, \infty)$ with the properties (for every fixed $\alpha \in A$):

(Φ1) $\Phi_\alpha(\cdot)$ is non-decreasing and continuous from the right;

(Φ2) $0 < \Phi_\alpha(t) < t, \ \forall t > 0$. (It follows $\Phi_\alpha(0) = 0$.)

Let $j : A \to A$ be an arbitrary mapping of the index set into itself. The iterations can be defined inductively as follows: $j^n(\alpha) = j(j^{n-1}(\alpha)), j^0(\alpha) = \alpha$ $(n = 1, 2, 3, \dots)$.

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Definition 1.1. [1] Let M be a subset of X and $T : M \to M$ be a mapping. T is said to be $(\Phi, j)-$ contractive on M, if $\rho_\alpha(T(x), T(y)) \le \Phi_\alpha(\rho_{j(\alpha)}(x, y))$ for every fixed $\alpha \in A$ and for every $x, y \in M$.

Remark 1.2. Recall that if $\Phi_\alpha \in (\Phi)$ and the function $\varphi_\alpha : (0, \infty) \to (0, \infty)$, defined by $\varphi_\alpha(t) = \dfrac{\Phi_\alpha(t)}{t}$ $\forall t \in (0, \infty)$, is non-decreasing, then $\sum\limits_{l=0}^{\infty} \Phi^l_\alpha(t) < \infty$ for any fixed $t \in (0, \infty)$ (where $\Phi^0_\alpha(t) = t$, $\Phi^l_\alpha(t) = \Phi_\alpha\left(\Phi^{l-1}_\alpha(t)\right), l \in \mathbb{N}$).

Indeed, for any fixed $l \in \mathbb{N}$ and $t > 0$ it follows by **(Φ2)** $\Phi^l_\alpha(t) = \Phi_\alpha\left(\Phi^{l-1}_\alpha(t)\right) < \Phi^{l-1}_\alpha(t)$. Therefore $\Phi^l_\alpha(t) < \Phi^{l-1}_\alpha(t) < \Phi^{l-2}_\alpha(t)... < \Phi_\alpha(t) < t$ and the inequalities

$$\frac{\Phi^{l+1}_\alpha(t)}{\Phi^l_\alpha(t)} = \frac{\Phi_\alpha\left(\Phi^l_\alpha(t)\right)}{\Phi^l_\alpha(t)} = \varphi_\alpha\left(\Phi^l_\alpha(t)\right) \le \varphi_\alpha\left(\Phi^{l-1}_\alpha(t)\right) \le ... \le \varphi_\alpha\left(\Phi_\alpha(t)\right) \le \varphi_\alpha(t) = \frac{\Phi_\alpha(t)}{t} < 1 \quad (l \in \mathbb{N})$$

are a sufficient condition for the convergence of $\sum\limits_{l=0}^{\infty} \Phi^l_\alpha(t)$.

Fixed point theorems from [1], [2] guarantee an existence of fixed points of $(\Phi, j)-$ contractive (or just $(\Phi)-$ *contractive*) and $j-$ *non-expansive* mappings under various conditions.

In this paper we extend the following result from [1]:

Theorem 1.3. (Angelov). *Let the following conditions hold:*

1) the operator $T : X \to X$ is $(\Phi)-$ contractive;

2) for every $\alpha \in A$ there exists a function $\overline{\Phi}_\alpha \in (\Phi)$ such that

$$\sup\left\{\Phi_{j^n(\alpha)}(t) : n = 0, 1, 2, 3, ...\right\} \le \overline{\Phi}_\alpha(t) \text{ and } \frac{\overline{\Phi}_\alpha(t)}{t} \text{ is non-decreasing } (t > 0);$$

3) there exists an element $x_0 \in X$ such that for every $\alpha \in A$ there is $q(\alpha) > 0$:

$$\rho_{j^n(\alpha)}(x_0, T(x_0)) \le q(\alpha) < \infty \, (n = 0, 1, 2, 3, ...).$$

Then T has at least one fixed point in X.

If, in addition, we suppose that

4) for every $\alpha \in A$ and $x, y \in X$ there exists $p = p(x, y, \alpha)$ such that

$$\rho_{j^k(\alpha)}(x, y) \le p(x, y, \alpha) < \infty \, (k = 0, 1, 2, 3, ...),$$

then the fixed point of T is unique.

2. Main results

Let $j_1 : A \to A$, $j_2 : A \to A$ be two mappings of the index set into itself.

In this paper we introduce the notion of $(\Phi, j_1, j_2)-$ contractive mappings and we establish some fixed point results for such mappings in uniform spaces.

Introduce the subfamily $(\Psi) \subset (\Phi)$ of functions $\Phi_\alpha \in (\Phi)$ which are sub-additive, i.e.
(Φ3) for every $\alpha \in A \, (\forall \, \Phi_\alpha \in \Psi) : \Phi_\alpha(t_1 + t_2) \le \Phi_\alpha(t_1) + \Phi_\alpha(t_2)$ for all $t_1, t_2 \in [0, \infty)$.

Definition 2.1. The mapping $T : X \to X$ is said to be (Φ, j_1, j_2) −contractive on X, if for any fixed $\alpha \in A$ there is a function $\Phi_\alpha \in (\Psi)$ such that $\rho_\alpha(T(x), T(y)) \le \dfrac{1}{2}\Phi_\alpha\big(\rho_{j_1(\alpha)}(x, y) + \rho_{j_2(\alpha)}(x, y)\big)$ for every $x, y \in X$.

Define the mapping $S_1 : A \to j_1(A) \cup j_2(A)$ as follows: $S_1(\gamma) = \{j_1(\gamma), j_2(\gamma)\}$ for $\gamma \in A$. Introduce for any fixed index $\alpha \in A$ the following notations: $S^0(\alpha) \equiv \alpha_0 \equiv \alpha; S^1(\alpha) \equiv S_1(\alpha)$;

$S^n(\alpha) = \{\sigma^n = (\alpha_1, ..., \alpha_n) : \alpha_k \in S_1(\alpha_{k-1}), \forall k = 1, ..., n\}$ for every $n \in \mathbb{N}$, $n > 1$.

Theorem 2.2. *Let $(X, \mathbf{A})$ be a Hausdorff sequentially complete uniform space, whose uniformity is generated by a saturated family of pseudo-metrics $\mathbf{A} = \{\rho_\alpha(x, y) : \alpha \in A\}$, where A is an index set. Let the mappings $j_1 : A \to A$ and $j_2 : A \to A$ be defined and $(\Psi) = \{\Phi_\alpha : \alpha \in A\}$ be the family of functions with properties* ***(Φ1)*** *–* ***(Φ3)****. Let the following conditions hold:*

1. The mapping $T : X \to X$ is (Φ, j_1, j_2) −contractive on X.

2. For every $\alpha \in A$ there is a function $\overline{\Phi}_\alpha \in (\Phi)$ such that $\frac{\overline{\Phi}_\alpha(t)}{t}$ is non-decreasing, $\Phi_\alpha(t) \leq \overline{\Phi}_\alpha(t)$, $\forall t > 0$ and for any fixed $k \in \mathbb{N}$ for all $\sigma^k = (\alpha_1, ..., \alpha_k) \in S^k(\alpha)$ the inequalities $\Phi_{\alpha_i}(t) \leq \overline{\Phi}_\alpha(t)$, $\forall t > 0$ are satisfied for all coordinates α_i of σ^k $(i = 1, ..., k)$.

3. There exists an element $x_0 \in X$ such that for every $\alpha \in A$ there exists a constant $q_\alpha = q(\alpha) > 0$ such that $\rho_\alpha(x_0, T(x_0)) \leq q_\alpha$ and for any fixed $k \in \mathbb{N}$ for all $\sigma^k = (\alpha_1, ..., \alpha_k) \in S^k(\alpha)$ the inequalities $\rho_{\alpha_m}(x_0, T(x_0)) \leq q_\alpha$ are satisfied for all coordinates α_m of σ^k $(m = 1, ..., k)$.

Then T has at least one fixed point in X.

Theorem 2.3. *If to the conditions of Theorem 2.2 we add the following assumption for the set X:*

4. for any fixed $\alpha \in A$ there exists $p_\alpha : X \times X \to (0, \infty)$ such that $\rho_\alpha(x, y) \leq p_\alpha(x, y)$ for all $(x, y) \in X \times X$ and for any fixed $k \in \mathbb{N}$ for all $\sigma^k = (\alpha_1, ..., \alpha_k) \in S^k(\alpha)$ the inequalities $\rho_{\alpha_n}(x, y) \leq p_\alpha(x, y)$ are satisfied for all $(x, y) \in X \times X$ and for all coordinates α_l of σ^k $(l = 1, ..., k)$, then the fixed point of T is unique.

Proof. (*of Theorem 2.2*) Begin with $x_0 \in X$, we define the sequence $\{x_n : n = 0, 1, 2, ...\}$, $x_n = T^n(x_0)$, where $T^0 \equiv Id$ and $T^n(\cdot) = T\left(T^{n-1}(\cdot)\right)$ for $n \in \mathbb{N}$. If $x_{n'} = x_{n'-1}$ for some $n' \in \mathbb{N}$ then $x_{n'-1}$ is a fixed point of T. Consequently we may assume $x_n \neq x_{n-1}, \forall n \in \mathbb{N}$.

Let $\alpha \in A$ be any fixed index. Define the sequence $\{c_n^\alpha\}_{n=0}^\infty$: $c_n^\alpha = \rho_\alpha(x_n, x_{n+1})$ $(n = 0, 1, 2, ...)$. For any fixed $n \in \mathbb{N}$ let $\sigma^n = (\alpha_1, ..., \alpha_n)$ be an arbitrary element of $S^n(\alpha)$.
Define $c_k^{\alpha_{n-k}} = \rho_{\alpha_{n-k}}(x_k, x_{k+1})$ for every $k = 1, ..., n$ (with $\alpha_0 \equiv \alpha$ and $c_n^{\alpha_0} \equiv c_n^\alpha$). It follows:

$$\begin{aligned} c_1^{\alpha_{n-1}} &= \rho_{\alpha_{n-1}}(x_1, x_2) \leq \frac{1}{2}\Phi_{\alpha_{n-1}}(\rho_{j_1(\alpha_{n-1})}(x_0, x_1) + \rho_{j_2(\alpha_{n-1})}(x_0, x_1)) \\ &\leq \frac{1}{2}\Phi_{\alpha_{n-1}}(\rho_{j_1(\alpha_{n-1})}(x_0, x_1)) + \frac{1}{2}\Phi_{\alpha_{n-1}}(\rho_{j_2(\alpha_{n-1})}(x_0, x_1)) \\ &\leq \frac{1}{2}\overline{\Phi}_\alpha(\rho_{j_1(\alpha_{n-1})}(x_0, x_1)) + \frac{1}{2}\overline{\Phi}_\alpha(\rho_{j_2(\alpha_{n-1})}(x_0, x_1)) \leq \frac{1}{2} \cdot 2\overline{\Phi}_\alpha(q_\alpha) = \overline{\Phi}_\alpha(q_\alpha). \end{aligned}$$

Therefore $c_1^{\alpha_{n-1}} \leq \overline{\Phi}_\alpha(q_\alpha)$ for any $\alpha_{n-1} \in S_1(\alpha_{n-2})$. By induction, for every choice of $\sigma^n \in S^n(\alpha)$ and its coordinates α_{n-k}, we prove:

$$c_k^{\alpha_{n-k}} \leq \overline{\Phi}_\alpha^k(q_\alpha) \, \forall k = 1, ..., n-1.$$

In fact, such estimates are valid when $k = 1$, as we have already proven. Suppose that the above inequalities are valid for all $k \leq m < n-1$. Then for $k = m+1$ we obtain:

$$\begin{aligned} c_{m+1}^{\alpha_{n-m-1}} &= \rho_{\alpha_{n-m-1}}(x_{m+1}, x_{m+2}) \leq \frac{1}{2}\Phi_{\alpha_{n-m-1}}(\rho_{j_1(\alpha_{n-m-1})}(x_m, x_{m+1}) + \rho_{j_2(\alpha_{n-m-1})}(x_m, x_{m+1})) \\ &\leq \frac{1}{2}\Phi_{\alpha_{n-m-1}}(\rho_{j_1(\alpha_{n-m-1})}(x_m, x_{m+1})) + \frac{1}{2}\Phi_{\alpha_{n-m-1}}(\rho_{j_2(\alpha_{n-m-1})}(x_m, x_{m+1})) \\ &\leq \frac{1}{2}\overline{\Phi}_\alpha(\rho_{j_1(\alpha_{n-m-1})}(x_m, x_{m+1})) + \frac{1}{2}\overline{\Phi}_\alpha(\rho_{j_2(\alpha_{n-m-1})}(x_m, x_{m+1})) \leq \overline{\Phi}_\alpha\left(c_m^{\alpha^*_{n-m}}\right), \end{aligned}$$

where $\alpha^*_{n-m} \in S_1(\alpha_{n-m-1})$ is such that

$$c_m^{\alpha^*_{n-m}} = \rho_{\alpha^*_{n-m}}(x_m, x_{m+1}) = \max\{\rho_{j_1(\alpha_{n-m-1})}(x_m, x_{m+1}), \rho_{j_2(\alpha_{n-m-1})}(x_m, x_{m+1})\}.$$

It follows by assumption that $c_m^{\alpha^*_{n-m}} \leq \overline{\Phi}_\alpha^m(q_\alpha)$. Therefore $c_{m+1}^{\alpha_{n-m-1}} \leq \overline{\Phi}_\alpha\left(\overline{\Phi}_\alpha^m(q_\alpha)\right) = \overline{\Phi}_\alpha^{m+1}(q_\alpha)$.

For c_n^α we obtain as follows:

$$\begin{aligned} c_n^\alpha &= \rho_\alpha(x_n, x_{n+1}) \leq \frac{1}{2}\Phi_\alpha(\rho_{j_1(\alpha)}(x_{n-1}, x_n) + \rho_{j_2(\alpha)}(x_{n-1}, x_n)) \\ &\leq \frac{1}{2}\overline{\Phi}_\alpha\left(\rho_{j_1(\alpha)}(x_{n-1}, x_n)\right) + \frac{1}{2}\overline{\Phi}_\alpha\left(\rho_{j_2(\alpha)}(x_{n-1}, x_n)\right) \leq \overline{\Phi}_\alpha\left(\rho_{\alpha_1^*}(x_{n-1}, x_n)\right), \end{aligned}$$

where $\alpha_1^* \in S_1(\alpha)$ is such that

$$\rho_{\alpha_1^*}(x_{n-1}, x_n) = \max\{\rho_{\alpha_1}(x_{n-1}, x_n) : \alpha_1 \in S_1(\alpha)\} = \max\{\rho_{j_1(\alpha)}(x_{n-1}, x_n), \rho_{j_2(\alpha)}(x_{n-1}, x_n)\}.$$

Hence $c_n^\alpha \le \overline{\Phi}_\alpha\left(\rho_{\alpha_1^*}(x_{n-1},x_n)\right) = \overline{\Phi}_\alpha\left(c_{n-1}^{\alpha_1^*}\right) \le \overline{\Phi}_\alpha(\overline{\Phi}_\alpha^{n-1}(q_\alpha))$ (since $\alpha_1^* \in S_1(\alpha)$).

Thus we obtain the inequality $c_n^\alpha \le \overline{\Phi}_\alpha^n(q_\alpha)$, which is valid for any fixed index $\alpha \in A$ and for every $n \in \mathbb{N}$. Consequently for every fixed $m = 0,1,2,...$ and $p \in \mathbb{N}$ we obtain:

$$\rho_\alpha(x_m,x_{m+p}) \le \sum_{k=0}^{p-1} c_{m+k}^\alpha \le \sum_{k=0}^{p-1} \overline{\Phi}_\alpha^{m+k}(q_\alpha) = \Psi_{m+p} - \Psi_m,$$

where $\Psi_k = \sum_{l=0}^{k-1} \overline{\Phi}_\alpha^l(q_\alpha)$ is the k-th partial sum of the series $\sum_{l=0}^{\infty} \overline{\Phi}_\alpha^l(q_\alpha)$, which is convergent, in view of Remark 1.2.

Therefore for any $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that $\forall m \ge N_0$: $\rho_\alpha(x_m,x_{m+p}) \le \Psi_{m+p} - \Psi_m < \varepsilon$ for every $p \in \mathbb{N}$, i.e. $\{x_n = T^n(x_0) : n = 0,1,2,...\}$ is a Cauchy sequence in X. In view of the sequential completeness of $(X,\mathbf{A})$ there exists $x \in X$: $\rho_\alpha(x_n,x) \underset{n\to\infty}{\to} 0$, $\forall \alpha \in A$.

The right-continuity of Φ_α and the inequalities

$$\begin{aligned}\rho_\alpha(x,T(x)) &\le \rho_\alpha(x,T^{n+1}(x_0)) + \rho_\alpha(T^{n+1}(x_0),T(x)) \le \\ &\le \rho_\alpha(x,x_{n+1}) + \frac{1}{2}\Phi_\alpha\left(\rho_{j_1(\alpha)}(x_n,x) + \rho_{j_2(\alpha)}(x_n,x)\right) (n \in \mathbb{N})\end{aligned}$$

imply $\rho_\alpha(x,T(x)) = 0\ \forall \alpha \in A$, that is $x = T(x)$. Theorem 2.2 is thus proved. □

Proof. (*of Theorem 2.3*) Let $(x,y) \in X \times X$ be an arbitrary fixed pair. Let $\alpha \in A$ be any fixed index. Denote by $d_n^\alpha = d_n^\alpha(x,y) = \rho_\alpha(T^n(x),T^n(y))$ $(n = 0,1,2,...)$. For any fixed $n \in \mathbb{N}$ let $\sigma^n = (\alpha_1,...,\alpha_n)$ be an arbitrary element of $S^n(\alpha)$.

Define $d_k^{\alpha_{n-k}} = d_k^{\alpha_{n-k}}(x,y) = \rho_{\alpha_{n-k}}(T^k(x),T^k(y))$ for every $k = 1,...,n$ (with $\alpha_0 \equiv \alpha$ and $d_n^{\alpha_0} \equiv d_n^\alpha$). It follows:

$$\begin{aligned}d_1^{\alpha_{n-1}} &= \rho_{\alpha_{n-1}}(T(x),T(y)) \le \frac{1}{2}\Phi_{\alpha_{n-1}}(\rho_{j_1(\alpha_{n-1})}(x,y) + \rho_{j_2(\alpha_{n-1})}(x,y)) \\ &\le \frac{1}{2}\Phi_{\alpha_{n-1}}(\rho_{j_1(\alpha_{n-1})}(x,y)) + \frac{1}{2}\Phi_{\alpha_{n-1}}(\rho_{j2(\alpha_{n-1})}(x,y)) \le \overline{\Phi}_\alpha(\rho_{\alpha_n^*}(x,y)),\end{aligned}$$

where $\alpha_n^* \in S_1(\alpha_{n-1})$ is such that $\rho_{\alpha_n^*}(x,y) = \max\{\rho_{j_1(\alpha_{n-1})}(x,y), \rho_{j_2(\alpha_{n-1})}(x,y)\}$.

Therefore $d_1^{\alpha_{n-1}} \le \overline{\Phi}_\alpha(p_\alpha(x,y))$ for $\alpha_{n-1} \in S_1(\alpha_{n-2})$.

By induction, for every choice of $\sigma^n \in S^n(\alpha)$ and its coordinates α_{n-k}, we prove:

$$d_k^{\alpha_{n-k}} \le \overline{\Phi}_\alpha^k(p_\alpha(x,y)) \forall k = 1,2,...,n-1.$$

Indeed, we have just obtained that the estimates are valid for $k = 1$. If we suppose that the above inequalities are satisfied for all $k \le m < n-1$, then for $k = m+1$ we obtain:

$$\begin{aligned}d_{m+1}^{\alpha_{n-m-1}} &= \rho_{\alpha_{n-m-1}}(T^{m+1}(x),T^{m+1}(y)) \\ &\le \frac{1}{2}\Phi_{\alpha_{n-m-1}}(\rho_{j_1(\alpha_{n-m-1})}(T^m(x),T^m(y))) + \frac{1}{2}\Phi_{\alpha_{n-m-1}}(\rho_{j_2(\alpha_{n-m-1})}(T^m(x),T^m(y))) \\ &\le \overline{\Phi}_\alpha\left(d_m^{\alpha_{n-m}^*}\right),\end{aligned}$$

where $\alpha_{n-m}^* \in S_1(\alpha_{n-m-1})$ is such that

$$d_m^{\alpha_{n-m}^*} = \rho_{\alpha_{n-m}^*}(T^m(x),T^m(y)) = \max\{\rho_{j_1(\alpha_{n-m-1})}(T^m(x),T^m(y)), \rho_{j_2(\alpha_{n-m-1})}(x_m,x_{m+1})\}$$

In particular, $d_m^{\alpha_{n-m}^*} \le \overline{\Phi}_\alpha^m(p_\alpha(x,y))$. It follows $d_{m+1}^{\alpha_{n-m-1}} \le \overline{\Phi}_\alpha\left(\overline{\Phi}_\alpha^m(p_\alpha(x,y))\right) = \overline{\Phi}_\alpha^{m+1}(p_\alpha(x,y))$. This completes the induction. Finally,

$$\begin{aligned}d_n^\alpha &= \rho_\alpha(T^n(x),T^n(y)) \le \frac{1}{2}\Phi_\alpha(\rho_{j_1(\alpha)}(T^{n-1}(x),T^{n-1}(y)) + \rho_{j_2(\alpha)}(T^{n-1}(x),T^{n-1}(y))) \\ &\le \frac{1}{2}\Phi_\alpha(\rho_{j_1(\alpha)}(T^{n-1}(x),T^{n-1}(y))) + \frac{1}{2}\Phi_\alpha(\rho_{j_2(\alpha)}(T^{n-1}(x),T^{n-1}(y))) \le \overline{\Phi}_\alpha(\rho_{\alpha_1^*}(T^{n-1}(x),T^{n-1}(y))),\end{aligned}$$

where $\alpha_1^* \in S_1(\alpha)$ is such that $\rho_{\alpha_1^*}(x,y) = \max\{\rho_{j_1(\alpha)}(x,y), \rho_{j_2(\alpha)}(x,y)\}$, and consequently $\rho_{\alpha_1^*}(T^{n-1}(x), T^{n-1}(y)) = d_{n-1}^{\alpha_1^*} \le \overline{\Phi}_\alpha^{n-1}(p_\alpha(x,y))$. Therefore $d_n^\alpha \le \overline{\Phi}_\alpha^n(p_\alpha(x,y))$.

The properties of the function $\overline{\Phi}_\alpha \in (\Phi)$ guarantee that $\overline{\Phi}_\alpha^n(t) \underset{n\to\infty}{\to} 0$ for any fixed $t \in [0,\infty)$. Thus, if we suppose that there exist two elements $x \ne y$ of X, for which $x = T(x)$ and $y = T(y)$, then for every index $\alpha \in A$ $\rho_\alpha(x,y) = \rho_\alpha(T^n(x), T^n(y)) \le \overline{\Phi}_\alpha^n(p_\alpha(x,y))$ for all $n \in \mathbb{N}$, which implies $\rho_\alpha(x,y) = 0$ for every $\alpha \in A$. The obtained contradiction proves Theorem 2.3. □

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