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Research article

Second centralizers and autocommutator subgroups of automorphisms

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Abstract

In 1994, Hegarty introduced the notion of $K(G)$ and $L(G)$, the autocommutator and autocentral subgroups of G, respectively. He proved that if $G/L(G)$ is finite, then so is $K(G)$ and for the converse he showed that the finiteness of $K(G)$ and $Aut(G)$ gives that $G/L(G)$ is also finite. In the present article, we construct a precise upper bound for the order of the autocentral factor group $G/L(G)$, when $K(G)$ is finite and $Aut(G)$ is finitely generated. In 2012, Endimioni and Moravec showed that if the centralizer of an automorphism α of a polycyclic group G is finite, then $L(G)$ and $G/K(G)$ are both finite. Finally, we show that if in a 2-auto-Engel polycyclic group G, there exist two automorphisms α_1 and α_2 such that $C_G(\alpha_1, \alpha_2) = \{g \in G | [g, \alpha_1, \alpha_2] = 1\}$ is finite, then $L_2(G)$ and $G/K_2(G)$ are both finite.

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1. Introduction and preliminary results

In 1904, I. Schur [11] showed that the finiteness of the central factor group of a given group implies that the derived subgroup is also finite. B. H. Neumann in 1951 [7] proved that the converse of Schur's result holds, for finitely generated groups. In 2005, K. Podoski and B. Szegedy [9] showed that for a given group G, if $[\gamma_2(G) : \gamma_2(G) \cap Z(G)] = n$ then $[G : Z_2(G)] \leq n^{2\log_2 n}$. Note that the converse of Schur's theorem is not true in general. As a counterexample, consider the infinite extra special p-groups, for any odd prime number p. P. Niroomand [8] generalized Neumann's theorem as follows: If $\gamma_2(G)$ is finite and $G/Z(G)$ is finitely generated, then $[G : Z(G)] \leq |\gamma_2(G)|^{d(G/Z(G))}$, where $d(X)$ is the minimal number of generators of a group X. Also in 2010, B. Sury [12] proved that if the set of commutator elements S of a group G is finite and $G/Z(G)$ is finitely generated, then $[G : Z(G)] \leq |S|^{d(G/Z(G))}$. D. Gumber et.al [3] handled the case $G/Z(G) = \langle x_1Z(G), \dots, x_nZ(G)\rangle$ such that $[x_i, G]$ is finite, for all $1 \leq i \leq n$ and showed that $[G : Z(G)] \leq \prod_{i=1}^{n} |[x_i, G]|$ and $\gamma_2(G)$ is finite.

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For an element $x \in G$ and automorphism α of G, $[x, \alpha] = x^{-1}x^{\alpha}$ is the *autocommutator* of x and α. So one may define

$$K(G) = \langle [x, \alpha] | x \in G, \ \alpha \in Aut(G) \rangle,$$

and

$$L(G) = \{x \in G | [x, \alpha] = x^{-1}\alpha(x) = 1, \quad for\ all\ \alpha \in Aut(G)\},$$

which are called the *autocommutator* and *autocentral* subgroups of G, respectively. Clearly both subgroups are characteristic so that $K(G)$ contains the derived subgroup and $L(G)$ is contained in the centre of G (see also [4,6] for more information).

In [5], P. Hegarty proved that if $G/L(G)$ is finite, then so is $K(G)$. Also he showed that $G/L(G)$ is finite, whenever both $K(G)$ and $Aut(G)$ are finite. In section 2, we construct an upper bound for the autocentral factor group of G, assuming its autocommutator subgroup is finite and the automorphism group of G is finitely generated.

For any automorphism α of the group G, we define

$$C_G(\alpha) = \{x \in G | [x, \alpha] = 1\},$$

$$[G, \alpha] = \langle [x, \alpha] | x \in G \rangle,$$

which are the centralizer of α in G and the commutator subgroup of G and α. Clearly $[G, \alpha]$ is a subgroup of $[G, Aut(G)] = K(G)$ and $C_G(\alpha)$ contains the autocentre of G.

The n^{th} autocommutator subgroup and n^{th} autocentre of G can be defined recursively, as follows:

$$K_n(G) = \langle [x, \alpha_1, ..., \alpha_n] = [[x, \alpha_1, ..., \alpha_{n-1}], \alpha_n] | x \in G, \ \alpha_i \in Aut(G) \rangle,$$

$$L_n(G) = \{x \in G | [x, \alpha_1, ..., \alpha_n] = 1, \quad for\ all\ \alpha_i \in Aut(G)\}.$$

So we obtain the following series

$$G = K_0(G) \supseteq K(G) = K_1(G) \supseteq K_2(G) \supseteq \cdots \supseteq K_n(G) \supseteq \cdots ,$$

$$1 = L_0(G) \subseteq L_1(G) = L(G) \subseteq \cdots \subseteq L_n(G) \subseteq \cdots .$$

One observes that $K_n(G)$ and $L_n(G)$ are characteristic subgroups in G such that $K_n(G)$ contains the $(n+1)^{st}$ term of the lower central series $\gamma_{n+1}(G)$, and $L_n(G)$ is contained in the n^{th} centre of G, $Z_n(G)$ (see [6] for more information).

2. Upper bound for the autocentral factor groups

In this section, we construct an upper bound for the autocentral factor group of a given group, when the autocommutator subgroup is finite and the automorphism group is finitely generated.

The following results of Hegarty [5] are needed for our further studies.

Theorem 2.1.
(a) [5, Theorem 1.1] *If $[G : L(G)]$ is finite, then so are $K(G)$ and $Aut(G)$.*
(b) [5, Theorem 1.2] *If $K(G)$ and $Aut(G)$ are both finite, then so is $[G : L(G)]$.*

Note that Fournelle's example in [2] shows that the conclusion of part(b) is not true if the finiteness condition on $Aut(G)$ is removed.

In the following, we extend this theorem of [5] to the case, where $Aut(G)$ is finitely generated and construct a sharp upper bound for the order of $G/L(G)$, provided $K(G)$ is finite.

Theorem 2.2. *Let the autocommutator subgroup $K(G)$ of a given group G be finite of order m, and d be the minimal number of generators of $Aut(G)$. Then*

$$|G/L(G)| \leq m^d.$$

Proof. Let $\{\alpha_1, \dots, \alpha_d\}$ be a minimal set of generators of automorphism group of a given group G. Define the following map

$$f : G/L(G) \longrightarrow \underbrace{K(G) \times K(G) \times \cdots \times K(G)}_{d-times},$$

given by $xL(G) \longmapsto ([x^{-1}, \alpha_1], \dots, [x^{-1}, \alpha_d])$, for all $x \in G$ and $\alpha_1, \dots, \alpha_d \in Aut(G)$. The map f is well-defined, for if $x_1L(G) = x_2L(G)$ for any $x_1, x_2 \in G$, then $x_1^{-1}x_2 \in L(G)$ and hence $[x_1^{-1}x_2, \alpha_i] = 1$, for all $1 \le i \le d$. So it follows that $[x_1^{-1}, \alpha_i] = [x_2^{-1}, \alpha_i]$. To show the injectivity of f assume $f(x_1L(G)) = f(x_2L(G))$, which implies that $[x_1^{-1}x_2, \alpha_i] = 1$, for all $1 \le i \le d$. Hence $x_1^{-1}x_2 \in L(G)$, which shows that f is injective. Therefore

$$|G/L(G)| \le m^d.$$

□

The above map need not be onto. For example, the automorphism group of D_8 is isomorphic to itself. Hence $L(D_8) = \mathbb{Z}_2$ and $K(D_8) = \mathbb{Z}_4$, which imply that $|K(D_8)|^{d(D_8)} = 16$, while $|D_8/L(D_8)| = 4$.

It should be noted that this bound is the best possible one, as the following example shows:

Let $\mathbb{Z}_2$ and $\mathbb{Z}_4$ be the cyclic groups of orders 2 and 4, respectively. Then clearly $L(\mathbb{Z}_2) = L(\mathbb{Z}_4) = \mathbb{Z}_2$, $K(\mathbb{Z}_2) = 1$ and $K(\mathbb{Z}_4) = \mathbb{Z}_2$. Hence, in the both cases the bounds are attained.

3. The second centralizers and autocommutator subgroups in auto-Engel groups

An element x of a group G is called right n-auto-Engel if $[x, \underbrace{\alpha, \dots, \alpha}_{n-times}] = 1$, for all $\alpha \in Aut(G)$. The element x is said to be left n-auto-Engel, whenever $[\alpha, \underbrace{x, \dots, x}_{n-times}] = 1$. A group G is called n-auto-Engel if $[x, \underbrace{\alpha, \dots, \alpha}_{n-times}] = [\alpha, \underbrace{x, \dots, x}_{n-times}] = 1$, for all $\alpha \in Aut(G)$ and $x \in G$. (see also [10] for more details).

The next lemmas of [10] will be used frequently in our further investigations.

Lemma 3.1. [10, Lemma 3.2] *Let x be a right 2-auto-Engel element and α, β and γ be arbitrary automorphisms of a given group G. Then*

(1) *x is a left 2-auto-Engel element;*
(2) *$x^{Aut(G)} = \langle x^\alpha : \alpha \in Aut(G)\rangle$ is abelian and its elements are right (so left) 2-auto-Engel elements;*
(3) $[x, \alpha, \beta] = [x, \beta, \alpha]^{-1}$;
(4) $[x, [\alpha, \beta]] = [x, \alpha, \beta]^2$;
(5) $[x, [\alpha, \beta], \gamma] = 1$.

Lemma 3.2. [10, Lemma 3.4] *Let G be a 2-auto-Engel group. Then the following properties hold, for all $x, y \in G$, $\alpha \in Aut(G)$ and $n \in \mathbb{Z}$.*

(1) $[x, x^\alpha] = 1$;
(2) $[x, \alpha^n] = [x, \alpha]^n = [x^n, \alpha]$;
(3) $[x^\alpha, y] = [x, y^\alpha]$;
(4) $[\alpha, x, y] = [\alpha, y, x]^{-1}$.

Definition 3.3. For any automorphisms α_1 and α_2 of a group G , we call

$$C_G(\alpha_1, \alpha_2) = \{x \in G | [x, \alpha_1, \alpha_2] = 1\}$$

the *second centralizer* of α_1 and α_2. Clearly, if G is abelian, then $C_G(\alpha_1, \alpha_2)$ is a subgroup of G.

Lemma 3.4. *Let G be a 2-auto-Engel group, α_1 and α_2 be arbitrary automorphisms of G. Then $C_G(\alpha_1, \alpha_2)$ and $[G, \alpha_1, \alpha_2] = \langle [x, \alpha_1, \alpha_2] | x \in G \rangle$ are both normal subgroups of G.*

Proof. For any elements $x_1, x_2 \in C_G(\alpha_1, \alpha_2)$, then by Lemma 3.1(3) and Lemma 3.2(2), we have

$$\begin{aligned}[x_1 x_2^{-1}, \alpha_1, \alpha_2] &= \Big[[x_1, \alpha_1]^{x_2^{-1}} [x_2^{-1}, \alpha_1], \alpha_2\Big] \\ &= \Big[[x_1, \alpha_1][x_1, \alpha_1, x_2^{-1}][x_2^{-1}, \alpha_1], \alpha_2\Big] \\ &= \Big[[x_1, \alpha_1, x_2^{-1}][x_2^{-1}, \alpha_1], \alpha_2\Big] \\ &= [x_1, \alpha_1, x_2^{-1}, \alpha_2]^{[x_2^{-1}, \alpha_1]} \\ &= [x_1, \alpha_1, \phi_{x_2^{-1}}, \alpha_2]^{[x_2^{-1}, \alpha_1]} \\ &= \left(\Big[x_1, \alpha_1, \alpha_2, \phi_{x_2^{-1}}\Big]^{-1}\right)^{[x_2^{-1}, \alpha_1]} = 1,\end{aligned}$$

where $\phi_{x_2^{-1}}$ denotes the inner automorphism induced by x_2^{-1}. Therefore, $x_1 x_2^{-1} \in C_G(\alpha_1, \alpha_2)$ and hence it is a subgroup.

Now, to prove the normality of $C_G(\alpha_1, \alpha_2)$, take any element $x \in C_G(\alpha_1, \alpha_2)$ and $g \in G$, then by Lemma 3.1(3) we have

$$\begin{aligned}[x^g, \alpha_1, \alpha_2] &= [x[x, g], \alpha_1, \alpha_2] \\ &= \Big[[x, \alpha_1]^{[x,g]} [x, g, \alpha_1], \alpha_2\Big] \\ &= [[x, \alpha_1][x, \alpha_1, [x, g]][x, g, \alpha_1], \alpha_2] \\ &= [[x, \alpha_1, [x, g]]\, [x, g, \alpha_1], \alpha_2] \\ &= [[x, \alpha_1, [x, g]]\,, \alpha_2]^{[x,g,\alpha_1]}\, [x, g, \alpha_1, \alpha_2] \\ &= \Big[x, \alpha_1, \phi_{[x,g]}, \alpha_2\Big]^{[x,g,\alpha_1]} [x, \phi_g, \alpha_1, \alpha_2] \\ &= (\Big[x, \alpha_1, \alpha_2, \phi_{[x,g]}\Big]^{-1})^{[x,g,\alpha_1]}\, [x, \alpha_1, \alpha_2, \phi_g] = 1,\end{aligned}$$

and hence $C_G(\alpha_1, \alpha_2)$ is normal in G.

For all $[x, \alpha_1, \alpha_2] \in [G, \alpha_1, \alpha_2]$ and $g \in G$, we have

$$[x, \alpha_1, \alpha_2]^g = [x, \alpha_1, \alpha_2][x, \alpha_1, \alpha_2, g].$$

Now, it is enough to show that $[x, \alpha_1, \alpha_2, g] \in [G, \alpha_1, \alpha_2]$. Easily by Lemmas 3.1(3) and 3.2(2), we write

$$\begin{aligned}[x, \alpha_1, \alpha_2, g] = [x, \alpha_1, \alpha_2, \phi_g] &= [x, \alpha_1, \phi_g, \alpha_2]^{-1} \\ &= \Big[[x, \phi_g, \alpha_1]^{-1}, \alpha_2\Big]^{-1} \\ &= [x, g, \alpha_1, \alpha_2] \in [G, \alpha_1, \alpha_2].\end{aligned}$$

□

The following lemma is useful for our further studies.

Lemma 3.5. (1) *If α_1 and α_2 are any automorphisms of a 2-auto-Engel group G, then $C_G(\alpha_i) \leq C_G(\alpha_1, \alpha_2)$ for $i = 1, 2$ and $[G, \alpha_1, \alpha_2] \leq [G, \alpha_i]$.*

(2) *If N is α_1 and α_2-invariant subgroup of a finite abelian group G, then $|C_{G/N}(\bar{\alpha_1}, \bar{\alpha_2})| \leq |C_G(\alpha_1, \alpha_2)|$, where $\bar{\alpha_1}$ and $\bar{\alpha_2}$ are induced automorphisms by α_1 and α_1, respectively.*

Proof. Part (1) follows easily by the above discussion.

(2) By Definition 3.3,

$$\begin{aligned} C_{G/N}(\bar{\alpha_1}, \bar{\alpha_2}) &= \{xN \in G/N | [xN, \bar{\alpha_1}, \bar{\alpha_2}] = N\} \\ &= \{xN \in G/N | [x, \alpha_1, \alpha_2] \in N\}. \end{aligned}$$

Now, we define the map $\theta : G \longrightarrow N$ given by $x \longmapsto [x, \alpha_1, \alpha_2]$, for all $x \in G$. Clearly θ is a homomorphism and hence $G/\ker\theta \cong \theta(G) \leq N$. Therefore $|C_{G/N}(\bar{\alpha_1}, \bar{\alpha_2})| \leq |G/N| \leq |\ker\theta| = |C_G(\alpha_1, \alpha_2)|$. □

Corollary 3.6. *Let α_1 and α_2 be automorphisms of a 2-auto-Engel polycyclic group G such that α_1 is of order 2 and $C_G(\alpha_1, \alpha_2)$ is finite, then the derived autocommutator subgroup $[G, \alpha_1, \alpha_2]'$ is also finite.*

Proof. The proof follows by using Lemma 3.3 and [1, Theorem 1]. □

Corollary 3.7. *Let α_1 and α_2 be two automorphisms of a 2-auto-Engel soluble group G such that α_1 is a fixed point free automorphism of order 2, then $[G, \alpha_1, \alpha_2]$ is abelian.*

Proof. As G is soluble, Theorem 6 of [1] says that; for any fixed point free automorphism α_1 of order 2, the subgroup $[G, \alpha_1]$ is abelian. Now, by Lemma 3.5(1), $[G, \alpha_1, \alpha_2]$ must be abelian. □

G. Endimioni and P. Moravec in [1], proved the following result.

Theorem 3.8. [1, Theorem 2] *Let α be an automorphism of a polycyclic group G. If $C_G(\alpha)$ is finite, then so is $G/[G, \alpha]$.*

In our terminology, we have the following corollary of the above theorem.

Corollary 3.9. *Let α be an automorphism of a polycyclic group G. If $C_G(\alpha)$ is finite, then so are $L(G)$ and $G/K(G)$.*

If we assume additionally in Theorem 3.8 that the group G is 2-auto-Engel, then we are able to show that the finiteness of $C_G(\alpha_1, \alpha_2)$ implies that $G/[G, \alpha_1, \alpha_2]$ is also finite (see Theorem 3.12 below). It is equivalent to saying that, if $C_G(\alpha_1, \alpha_2)$ is finite, then so are $L_2(G)$ and $G/K_2(G)$.

Remark 3.10. We shall make frequently use of the following well-known fact, without further references: if H is a finitely generated characteristic subgroup of a given group A and H_1 is a normal subgroup of H of finite index, then H_1 contains a characteristic subgroup H_2 of A of finite index in H.

The following technical lemmas are needed in proving our main result.

Lemma 3.11. *Let A be a finitely generated abelian group and α_1 and α_2 be any automorphisms of A, with $C_A(\alpha_1, \alpha_2)$ is finite. Then A contains a characteristic subgroup of finite index, such that its elements are of the form $[a, \alpha_1, \alpha_2]$, for some $a \in A$.*

Proof. Define the map $\theta : A \longrightarrow A$ given by $a \longmapsto [a, \alpha_1, \alpha_2]$, for all $a \in A$. The map θ is a homomorphism, as A is abelian. Clearly, the kernel of θ is $C_A(\alpha_1, \alpha_2)$, which is finite by the assumption. Then A and $\theta(A)$ have the same torsion-free rank and hence the index of $\theta(A)$ in A is finite. Clearly, the elements of $\theta(A)$ are of the required form. Now, by the above Remark and taking $H = A$, as A is finitely generated and $\theta(A)$ is a normal subgroup of finite index in A, then $\theta(A)$ contains a characteristic subgroup of finite index in A such that its elements are also of the form $[a, \alpha_1, \alpha_2]$, for some $a \in A$. □

Lemma 3.12. *Let G be an abelian group and K be a finitely generated characteristic subgroup of G. Let α_1 and α_2 be two automorphisms of G such that $C_G(\alpha_1, \alpha_2)$ is finite. Then $C_{G/K}(\bar{\alpha_1}, \bar{\alpha_2})$ is also finite, where $\bar{\alpha_1}$ and $\bar{\alpha_2}$ are automorphisms of G/K induced by the automorphisms α_1 and α_2, respectively.*

Proof. The result is trivially true, when the group G is finite. Otherwise, we assume that K is finite and put $C_{G/K}(\bar{\alpha_1}, \bar{\alpha_2}) = C/K$, then we define the map $\theta : C \longrightarrow K$ given by $c \longmapsto [c, \alpha_1, \alpha_2]$, for all $c \in C$. Clearly $\theta(c_1) = \theta(c_2)$ if and only if $c_1c_2^{-1} \in C_G(\alpha_1, \alpha_2)$, which implies that C is finite. Hence $|C| \leq |K| \times |C_G(\alpha_1, \alpha_2)|$. Therefore $\left|C_{G/K}(\bar{\alpha_1}, \bar{\alpha_2})\right| = [C : K] \leq |C_G(\alpha_1, \alpha_2)| < \infty$, and hence the claim. Now, we assume that K is finitely generated. Since $|C_K(\alpha_1, \alpha_2)| \leq |C_G(\alpha_1, \alpha_2)| < \infty$, Lemma 3.11 gives that K contains a characteristic subgroup K_0 of finite index such that for all $k_0 \in K_0$, there is a unique $y \in K$ where $k_0 = [y, \alpha_1, \alpha_2]$. Let $\alpha_1{}'$ and $\alpha_2{}'$ be the automorphisms of G/K_0 induced by α_1 and α_2, respectively. We show that $C_{G/K_0}(\alpha_1{}', \alpha_2{}')$ is finite. To prove this, take an element $xK_0 \in C_{G/K_0}(\alpha_1{}', \alpha_2{}')$, then we have $[x, \alpha_1, \alpha_2] \in K_0$. Assume $[x, \alpha_1, \alpha_2] = [y, \alpha_1, \alpha_2]$, for some $y \in K$. It follows that $[xy^{-1}, \alpha_1, \alpha_2] = 1$. Now, since the index of K_0 in K is finite, we may assume that $K = t_1K_0 \cup \cdots \cup t_nK_0$. So $y = t_ik'$, for some $k' \in K_0$ and hence $[x(t_ik')^{-1}, \alpha_1, \alpha_2] = 1$. Therefore $x(t_ik')^{-1} \in C_G(\alpha_1, \alpha_2)$ and so $xK_0 = ut_iK_0$, for some $u \in C_G(\alpha_1, \alpha_2)$. Therefore the group $C_{G/K_0}(\alpha_1{}', \alpha_2{}')$ is finite. Now, assume $\bar{\alpha_1}$ and $\bar{\alpha_2}$ are induced by the automorphisms α_1' and α_2' on G/K. As K/K_0 is finite, we conclude that the group $C_{G/K}(\bar{\alpha_1}, \bar{\alpha_2})$ is finite by the first part of the lemma. □

Note that in the preceding lemma, in the case where K is finitely generated, one can not hope the inequality $\left|C_{G/K}(\bar{\alpha_1}, \bar{\alpha_2})\right| \leq |C_G(\alpha_1, \alpha_2)|$, if we take $G = \mathbb{Z}$, $K = 2\mathbb{Z}$ and $\alpha_1 = \alpha_2 = \alpha : x \longmapsto -x$, then we have $|C_G(\alpha, \alpha)| = 1$, while $|C_{G/K}(\bar{\alpha}, \bar{\alpha})| = 2$.

Now, we are able to prove the following result.

Theorem 3.13. *Let G be a polycyclic 2-auto-Engel group and α_1, α_2 be two automorphisms of G. If $C_G(\alpha_1, \alpha_2)$ is finite, then so is $G/[G, \alpha_1, \alpha_2]$.*

Proof. We proceed by induction on the derived length d of $[G, \alpha_1, \alpha_2]$. If $d = 0$, then $[G, \alpha_1, \alpha_2] = 1$. This implies that $C_G(\alpha_1, \alpha_2) = G$ and hence G is finite, which gives the result. Put $K = [G, \alpha_1, \alpha_2]^{(d-1)}$and it can be checked that K is finitely generated abelian. Now we can apply Lemma 3.12 and get the finiteness of $C_{G/K}(\bar{\alpha_1}, \bar{\alpha_2})$, where $\bar{\alpha_1}$ and $\bar{\alpha_2}$ are induced automorphisms by α_1 and α_2. Using induction hypothesis, we have $[G/K : [G/K, \bar{\alpha_1}, \bar{\alpha_2}]]$ is finite, which implies the finiteness of $G/[G, \alpha_1, \alpha_2]$, as required. □

The above theorem gives our main result of this section.

Corollary 3.14. *Let G be a polycyclic 2-auto-Engel group with two automorphisms α_1 and α_2 such that $C_G(\alpha_1, \alpha_2)$ is finite. Then $L_2(G)$ and $G/K_2(G)$ are both finite.*

The following technical lemma is needed to prove our final result.

Lemma 3.15. *Let x and y be arbitrary elements of a group G. Then for any automorphism α of G, $[y, [\phi_x, \alpha]] = [y, [x, \alpha]]$.*

Proof.

$$\begin{aligned}
[y, [\phi_x, \alpha]] &= y^{-1}y^{[\phi_x, \alpha]} = y^{-1}y^{(\phi_x^{-1}\alpha^{-1}\phi_x\alpha)} \\
&= y^{-1}(y^{x^{-1}})^{\alpha^{-1}\phi_x\alpha} = y^{-1}((y^{\alpha^{-1}})^{(x^{-1})^{\alpha^{-1}}})^{\phi_x\alpha} \\
&= y^{-1}((y^{\alpha^{-1}})^{(x^{-1})^{\alpha^{-1}}}x)^{\alpha} = y^{-1}y^{x^{-1}x^{\alpha}} = y^{-1}y^{[x,\alpha]} \\
&= [y, [x, \alpha]]
\end{aligned}$$

□

Corollary 3.16. *Let G be a 2-auto-Engel polycyclic group and $C_G(\alpha_1, \alpha_2)$ is finite, for any automorphisms α_1 and α_2 of G. Then $G/Z(G)$ is finite.*

Proof. Let $x, y \in G$, $\alpha, \beta \in Aut(G)$ and ϕ_x be the inner automorphism induced by x. As G is 2-auto-Engel group, by Lemma 3.1(5), $[y, [\phi_x, \alpha], \beta] = 1$ and Lemma 3.14 gives $[y, [x, \alpha], \beta] = 1$. Now using Lemma 3.1(3) $[x, \alpha, y, \beta] = 1$ and so $[x, \alpha, \beta, y] = 1$. This shows that $K_2(G)$ is contained in the centre of G. Hence the result is obtained by corollary 3.13. □

Open problem: Let G be an n-auto-Engel polycyclic group with $C_G(\alpha_1, \alpha_2, ..., \alpha_n)$ is finite, for any automorphisms $\alpha_1, \alpha_2, ..., \alpha_n$ of the group G. Are $L_n(G)$ and $G/K_n(G)$ finite?

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