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# Norm-Attainability and Range-Kernel Orthogonality of Elementary Operators

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**Abstract**
Various aspects of elementary operators have been characterized by many mathematicians. In this paper, we consider norm-attainability and orthogonality of these operators in Banach spaces. Characterizations and generalizations of norm-attainability and orthogonality are given in details. We first give necessary and sufficient conditions for norm-attainability of Hilbert space operators then we give results on orthogonality of the range and the kernel of elementary operators when they are implemented by norm-attainable operators in Banach spaces.

**Keywords:** Range-Kernel orthogonality, Elementary operator
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## 1. Introduction

Studies on Hilbert space operators have been carried out for along period of time with nice results obtained. Norm-attainability is one of the aspects which has been given attention. Let $H$ be an infinite dimensional separable complex Hilbert space and $B(H)$ the algebra of all bounded linear operators on $H$. An operator $S \in B(H)$ is said to be norm-attainable if there exists a unit vector $x_0 \in H$ such that $\|Sx_0\| = \|S\|$. For an operator $S \in B(H)$ we define a numerical range by $W(S) = \{\langle Sx, x\rangle : x \in H, \|x\| = 1\}$ and the maximal numerical range by $W_0(S) = \{\beta \in \mathbb{C} : \langle Sx_n, x_n\rangle \to \beta, \text{ where } \|x_n\| = 1, \|Sx_n\| \to \|S\|\}$. The second aspect in consideration is orthogonality which is a concept that has been analyzed for quite a period of time. Benitez [1] described several types of orthogonality which have been studied in real normed spaces namely: Robert's orthogonality, Birkhoff's orthogonality, Orthogonality in the sense of James, Isoceles, Pythagoras, Carlsson, Diminnie, Area among others. Some of these orthogonalities are described as follows. For $x \in \mathscr{M}$ and $y \in \mathscr{N}$ where $\mathscr{M}$ and $\mathscr{N}$ are subspaces of $E$ which is a normed linear space, we have: (i). Roberts: $\|x-\lambda y\| = \|x+\lambda y\|, \forall, \lambda \in \mathbb{R}$; (ii). Birkhoff: $\|x+y\| \geq \|y\|$; (iii). Isosceles: $\|x-y\| = \|x+y\|$; (iv). Pythagorean: $\|x-y\|^2 = \|x\|^2 + \|y\|^2$; (v). $a$-Pythagorean: $\|x-ay\|^2 = \|x\|^2 + a^2\|y\|^2$, $a \neq 0$; (vi). Diminnie: $\sup\{f(x)g(y) - f(y)g(x) : f,\ g \in S'\} = \|x\|\|y\|$ where $S'$ denotes the unit sphere of the topological dual of $E$; (vii). Area: $\|x\|\|y\| = 0$ or they are linearly independent and such that $x, -x, y, -y$ divide the unit ball of their own plane (identified by $\mathbb{R}^2$) in four equal areas. In this paper we will consider the orthogonality of elementary operators when they are implemented by norm-attainable operators. Consider a normed space $\mathscr{A}$ and let $T_{A,B} : \mathscr{A} \to \mathscr{A}$. $T$ is called an elementary operator if it has the following representation: $T(X) = \sum_{i=1}^{n} A_i X B_i$, $\forall\, X \in \mathscr{A}$, where $A_i$, $B_i$ are fixed in $\mathscr{A}$. Let $\mathscr{A} = B(H)$. For $A, B \in B(H)$ we define the particular elementary operators: The left multiplication operator $L_A : B(H) \to B(H)$ by $L_A(X) = AX$, $\forall\, X \in B(H)$; the right multiplication operator $R_B : B(H) \to B(H)$ by $R_B(X) = XB$, $\forall\, X \in B(H)$; the generalized derivation (implemented by $A$, $B$) by $\delta_{A,B} = L_A - R_B$; the basic elementary operator(implemented by $A$, $B$) by $M_{A,\,B}(X) = AXB$, $\forall\, X \in B(H)$; the Jordan elementary operator(implemented by $A$, $B$) by $\mathscr{U}_{A,\,B}(X) = AXB + BXA$, $\forall\, X \in B(H)$; Regarding orthogonality involving elementary operators, Anderson[2] established the orthogonality of the range and kernel of normal derivations. Others who have also worked on orthogonality include: Kittaneh [3], Mecheri [4] among others. For details see [1-2, 4-31]. We shall investigate the orthogonality of the range and the kernel of several types of important elementary operators in Banach spaces. Anderson [2] in his investigations proved that if $N$ and $S$ are operators in $B(H)$ such that $N$ is normal and $NS = SN$ then for all $X \in B(H)$, $\|\delta_N(X)+S\| \geq \|S\|$. If $S$(above) is a Hilbert-Schmidt operator then Kittaneh [3] (see also the references therein) showed that $\|\delta_N(X)+S\|_2^2 = \|\delta_N(X)\|_2^2 + \|S\|_2^2$. We extend this study to the general Banach spaces.

## 2. Preliminaries

In this section, we give some preliminary results. We begin by the following proposition.

**Proposition 2.1.** *Let $H$ be an infinite dimensional separable complex Hilbert space. Let $S \in B(H)$, $\beta \in W_0(S)$ and $\alpha > 0$. Then the following conditions hold:*
*(i). There exists $Z \in B(H)$ such that $\|S\| = \|Z\|$, with $\|S - Z\| < \alpha$.*
*(ii). There exists a vector $\eta \in H$, $\|\eta\| = 1$ such that $\|Z\eta\| = \|Z\|$ with $\langle Z\eta, \eta\rangle = \beta$.*

*Proof.* Let $\|S\| = 1$ and also that $0 < \alpha < 2$. Let $x_n \in H$ $(n = 1, 2, ...)$ be such that $\|x_n\| = 1$, $\|Sx_n\| \to 1$ and also $\lim_{n\to\infty}\langle Sx_n, x_n\rangle = \beta$. Let $S = GL$ be the polar decomposition of $S$. Here $G$ is a partial isometry and we write $L = \int_0^1 \beta dE_\beta$, the spectral decomposition of $L = (S^*S)^{\frac{1}{2}}$. Since $L$ is a positive operator with norm 1, for any $x \in H$ we have that $\|Lx_n\| \to 1$ as $n$ tends to $\infty$ and $\lim_{n\to\infty}\langle Sx_n, x_n\rangle = \lim_{n\to\infty}\langle GLx_n, x_n\rangle = \lim_{n\to\infty}\langle Lx_n, G^*x_n\rangle$. Now for $H = \overline{Ran(L)} \oplus KerL$, we can choose $x_n$ such that $x_n \in \overline{Ran(L)}$ for large $n$. Indeed, let $x_n = x_n^{(1)} \oplus x_n^{(2)}$, $n = 1, 2, ...$ Then we have that $Lx_n = Lx_n^{(1)} \oplus Lx_n^{(2)} = Lx_n^{(1)}$ and that $\lim_{n\to\infty}\|x_n^{(1)}\| = 1$, $\lim_{n\to\infty}\|x_n^{(2)}\| = 0$ since $\lim_{n\to\infty}\|Lx_n\| = 1$. Replacing $x_n$ with $\frac{x_n^{(1)}}{\|x_n^{(1)}\|}$, we get

$$\lim_{n\to\infty}\left\|L\frac{1}{\|x_n^{(1)}\|}x_n^{(1)}\right\| = \lim_{n\to\infty}\left\|S\frac{1}{\|x_n^{(1)}\|}x_n^{(1)}\right\| = 1, \lim_{n\to\infty}\left\langle S\frac{1}{\|x_n^{(1)}\|}x_n^{(1)}, \frac{1}{\|x_n^{(1)}\|}x_n^{(1)}\right\rangle = \beta$$

Next let $x_n \in \overline{RanL}$. Since $G$ is a partial isometry from $\overline{RanL}$ onto $\overline{RanS}$, we have that $\|Gx_n\| = 1$ and $\lim_{n\to\infty}\langle Lx_n, G^*x_n\rangle = \beta$. Since $L$ is a positive operator, $\|L\| = 1$ and for any $x \in H$, $\langle Lx, x\rangle \leq \langle x, x\rangle = \|x\|^2$. Replacing $x$ with $L^{\frac{1}{2}}x$, we get that $\langle L^2x, x\rangle \leq \langle Lx, x\rangle$, where $L^{\frac{1}{2}}$ is the positive square root of $L$. Therefore we have that $\|Lx\|^2 = \langle Lx, Lx\rangle \leq \langle Lx, x\rangle$. It is obvious that $\lim_{n\to\infty}\|Lx_n\| = 1$ and that $\|Lx_n\|^2 \leq \langle Lx_n, x_n\rangle \leq \|Lx_n\|^2 = 1$. Hence, $\lim_{n\to\infty}\langle Lx_n, x_n\rangle = 1 = \|L\|$. Moreover, Since $I - L \geq 0$, we have $\lim_{n\to\infty}\langle (I-L)x_n, x_n\rangle = 0$. thus $\lim_{n\to\infty}\|(I-L)^{\frac{1}{2}}x_n\| = 0$. Indeed, $\lim_{n\to\infty}\|(I-L)x_n\| \leq \lim_{n\to\infty}\|(I-L)^{\frac{1}{2}}\|.\|(I-L)^{\frac{1}{2}}x_n\| = 0$. For $\alpha > 0$, let $\gamma = [0, 1-\frac{\alpha}{2}]$ and let $\rho = (1-\frac{\alpha}{2}, 1]$. We have $L = \int_\gamma \mu dE_\mu + \int_\rho \mu dE_\mu = LE(\gamma) \oplus LE(\rho)$. Next we show that $\lim_{n\to\infty}\|E(\gamma)x_n\| = 0$. If there exists a subsequence $x_{n_i}, (i = 1, 2, ...,)$ such that $\|E(\gamma)x_{n_i}\| \geq \varepsilon > 0$, $(i = 1, 2, ...,)$, then since $\lim_{i\to\infty}\|x_{n_i} - Lx_{n_i}\| = 0$, it follows that

$$\lim_{i\to\infty}\|x_{n_i} - Lx_{n_i}\|^2 = \lim_{i\to\infty}(\|E(\gamma)x_{n_i} - LE(\gamma)x_{n_i}\|^2 + \|E(\rho)x_{n_i} - LE(\rho)x_{n_i}\|^2) = 0.$$

Hence, we have that

$$\lim_{i\to\infty}\|E(\gamma)x_{n_i} - LE(\gamma)x_{n_i}\|^2 = 0.$$

Now it is clear that

$$\|E(\gamma)x_{n_i} - LE(\gamma)x_{n_i}\| \geq \|E(\gamma)x_{n_i}\| - \|LE(\gamma)\|.\|E(\gamma)x_{n_i}\| \geq (I - \|LE(\gamma)\|)\|E(\gamma)x_{n_i}\| \geq \frac{\alpha}{2}\varepsilon > 0.$$

This is a contradiction. Therefore, $\lim_{n\to\infty}\|E(\gamma)x_n\| = 0$. Since $\lim_{n\to\infty}\langle Lx_n, x_n\rangle = 1$, we have that $\lim_{n\to\infty}\langle LE(\rho)x_n, E(\rho)x_n\rangle = 1$ and $\lim_{n\to\infty}\langle E(\rho)x_n, G^*E(\rho)x_n\rangle = \beta$. It is easy to see that

$$\lim_{n\to\infty}\|E(\rho)x_n\| = 1, \lim_{n\to\infty}\left\langle L\frac{E(\rho)x_n}{\|E(\rho)x_n\|}, \frac{E(\rho)x_n}{\|E(\rho)x_n\|}\right\rangle = 1$$

and

$$\lim_{n\to\infty}\left\langle L\frac{E(\rho)x_n}{\|E(\rho)x_n\|}, G^*\frac{E(\rho)x_n}{\|E(\rho)x_n\|}\right\rangle = \beta$$

Replacing $x$ with $\frac{E(\rho)x_n}{\|E(\rho)x_n\|}$, we can assume that $x_n \in E(\rho)H$ for each $n$ and $\|x_n\| = 1$. Let $J = \int_\gamma \mu dE_\mu + \int_\rho \mu dE_\mu = J_1 \oplus E(\rho)$. Then it is evident that $\|J\| = \|S\| = \|L\| = 1, Jx_n = x_n$, and $\|J - L\| \leq \frac{\alpha}{2}$. If we can find a contraction $V$ such that $V - G \leq \frac{\alpha}{2}$ and $\|Vx_n\| = 1$ and $\langle Vx_n, x_n\rangle = \beta$, for a large $n$ then letting $Z = VJ$, we have that $\|Zx_n\| = \|VJx_n\| = 1$, and that

$$\langle Zx_n, x_n\rangle = \langle VJx_n, x_n\rangle = \langle Vx_n, x_n\rangle = \beta,$$

$$\|S - Z\| = \|GL - VJ\| \leq \|GL - GJ\| + \|GJ - VJ\| \leq \|G\|.\|L - J\| + \|G - V\|.\|J\| \leq \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha.$$

Lastly, we now construct the desired contraction $V$. Clearly, $\lim_{n\to\infty}\langle x_n, G^* x_n\rangle = \beta$, because $\lim_{n\to\infty}\langle Lx_n, G^* x_n\rangle = \beta$ and $\lim_{n\to\infty}\|x_n - Lx_n\| = 0$. Let $Gx_n = \phi_n x_n + \varphi_n y_n,\ (y_n \perp x_n,\ \|y_n\| = 1)$ then $\lim_{n\to\infty}\phi_n = \beta$, because $\lim_{n\to\infty}\langle Gx_n, x_n\rangle = \lim_{n\to\infty}\langle x_n, G^* x_n\rangle = \beta$ but $\|Gx_n\|^2 = |\phi_n|^2 + |\varphi_n|^2 = 1$, so we have that $\lim_{n\to\infty}|\varphi_n| = \sqrt{1 - |\beta|}^2$. Now without loss of generality, there exists an integer $M$ such that $|\phi_M - \beta| < \frac{\alpha}{8}$. Choose $\varphi_M^0$ such that $|\varphi_M^0| = \sqrt{1 - |\beta|}^2,\ \ |\varphi_M - \varphi_M^0| < \frac{\alpha}{8}$. We have that

$$Gx_M = \phi_M x_M + \varphi_M y_M - \beta x_M + \beta x_M - \varphi_M^0 y_M + \varphi_M^0 y_M = (\phi - \beta)x_M + (\varphi_M - \varphi_M^0)y_M + \beta x_M + \varphi_M^0 y_M.$$

Let

$$q_M = \beta x_M + \varphi_M^0 y_M, Gx_M = (\phi - \beta)x_M + (\varphi_M - \varphi_M^0)y_M + q_M.$$

Suppose that $y \perp x_M$, then

$$\langle Gx_M, Gy\rangle = (\phi - \beta)\langle x_M, Gy\rangle + (\varphi_M - \varphi_M^0)\langle y_M, Gy\rangle + \langle q_M, Gy\rangle = 0,$$

because $G^*G$ is a projection from $H$ to $RanL$. It follows that

$$|\langle q_M, Gy\rangle| \leq |\phi_M - \beta|.\|y\| + |\varphi_M - \varphi_M^0|.\|y\| \leq \frac{\alpha}{4}\|y\|$$

If we suppose that

$$Gy = \phi q_M + y^0,\ (y^0 \perp q_M,)$$

then $y^0$ is uniquely determined by $y$. Hence we can define $V$ as follows$V : x_M \to q_M,\ y \to y^0,\ \phi x_M + \varphi_M y \to \phi q_M + \varphi_M y^0$, with both $\phi, \varphi$ being complex numbers. $V$ is a linear operator. We prove that $V$ is a contraction. Now,

$$\|Vx_M\|^2 = \|q_M\|^2 = |\beta|^2 = |\varphi_M^0|^2 = 1,$$

$$\|Vy\|^2 = \|Gy\|^2 - |\phi y|^2 \leq \|Gy\|^2 \leq \|y\|^2.$$

It follows that

$$\|V\phi\|^2 = \|\phi\|^2\|Vx_M\|^2 + |\varphi|^2\|Vy\|^2 \leq |\phi|^2 + |\varphi|^2 = 1,$$

for each $x \in H$ satisfying that $x = \phi x_M + \varphi_M y,\ \ \|x\| = 1,\ x_M \perp y$, which is equivalent to that $V$ is a contraction. From the definition of $V$, we can show that

$$\|Gx_M - Vx_M\|^2 = |\phi - \beta|^2 + |\varphi_M - \varphi_M^0|^2 \leq \frac{2\alpha^2}{16} = \frac{1}{8}\alpha^2.$$

If $y \perp x_M,\ \|y\| \leq 1$ then obtain

$$\|Gy - Vy\| = |\phi|\|Vx_M\| = |\langle Gy, Vx_M\rangle| = |\langle q_M, Gy\rangle| < \frac{\alpha}{4}.$$

Hence for any $x \in H,\ x = \phi x_M + \varphi_M y,\ \ \|x\| = 1$,

$$\|Gx - Vx\|^2 = \|\phi(G - V)x_M + \varphi(G - V)y\|^2 = |\phi|^2\|(G - V)x_M\|^2 + |\varphi|^2\|(G - V)y\|^2 < |\phi|^2.\frac{\alpha^2}{16} + |\varphi|^2.\frac{\alpha^2}{16} < \frac{\alpha^2}{8},$$

which implies that $\|(G - V)x\| < \frac{\alpha}{2},\ \ \|x\| = 1$, and hence $\|(G - V)\| < \frac{\alpha}{2}$. Let $Z = VJ$. Then $Z$ is what we want. □

The next result gives the conditions for norm-attainability of an inner derivation. We give the following proposition.

**Proposition 2.2.** *Let $H$ be an infinite dimensional separable complex Hilbert space and $S \in B(H)$. $\delta_S$ is norm-attainable if there exists a vector $\zeta \in H$ such that $\|\zeta\| = 1,\ \|S\zeta\| = \|S\|,\ \langle S\zeta, \zeta\rangle = 0$.*

*Proof.* For any $x$ satisfying that $x \perp \{\zeta, S\zeta\}$, define $X$ as follows $X : \zeta \to \zeta,\ S\zeta \to -S\zeta,\ x \to 0$. Since $X$ is a bounded operator on $H$ and

$$\|X\zeta\| = \|X\| = 1, \|SX\zeta - XS\zeta\| = \|S\zeta - (-S\zeta)\| = 2\|S\zeta\| = 2\|S\|.$$

It follows that $\|\delta_S\| = 2\|S\|$ via the result in [30, Theorem 1], because $\langle S\zeta, \zeta\rangle = 0 \in W_0(S)$. Hence we have that $\|SX - XS\| = 2\|S\| = \|\delta_S\|$. Therefore, $\delta_S$ is norm-attainable. □

The next result gives the conditions for norm-attainability of a generalized derivation. We give the following proposition.

**Proposition 2.3.** *Let $H$ be an infinite dimensional separable complex Hilbert space. Let $S, T \in B(H)$. If there exists vectors $\zeta, \eta \in H$ such that $\|\zeta\| = \|\eta\| = 1,\ \|S\zeta\| = \|S\|,\ \|T\eta\| = \|T\|$ and $\frac{1}{\|S\|}\langle S\zeta, \zeta\rangle = -\frac{1}{\|T\|}\langle T\eta, \eta\rangle$, then $\delta_{S,T}$ is norm-attainable.*

*Proof.* By linear dependence of vectors, if $\eta$ and $T\eta$ are linearly dependent, i.e.,$T\eta = \phi\|T\|\eta$, then it is true that $|\phi| = 1$ and $|\langle T\eta, \eta\rangle| = \|T\|$. It follows that $|\langle S\zeta, \zeta\rangle| = \|S\|$ which implies that $S\zeta = \varphi\|S\|\zeta$ and $|\varphi| = 1$. Hence $\left\langle \frac{S\zeta}{\|S\|}, \zeta \right\rangle = \varphi = -\left\langle \frac{T\eta}{\|T\|}, \eta \right\rangle = -\phi$. Defining $X$ as $X : \eta \to \zeta,\ \{\eta\}^{\perp} \to 0$, we have $\|X\| = 1$ and $(SX - XT)\eta = \varphi(\|S\| + \|T\|)\zeta$, which implies that

$$\|SX - XT\| = \|(SX - XT)\eta\| = \|S\| + \|T\|.$$

By [3], it follows that $\|SX - XT\| = \|S\| + \|T\| = \|\delta_{S,T}\|$. That is $\delta_{S,T}$ is norm-attainable. If $\eta$ and $T\eta$ are linearly independent, then $\left|\left\langle \frac{T\eta}{\|T\|}, \eta \right\rangle\right| < 1$, which implies that $\left|\left\langle \frac{S\zeta}{\|S\|}, \zeta \right\rangle\right| < 1$. Hence $\zeta$ and $S\zeta$ are also linearly independent. Let us redefine $X$ as follows: $X : \eta \to \zeta,\ \frac{T\eta}{\|T\|} \to -\frac{S\zeta}{\|S\|},\ x \to 0$, where $x \in \{\eta, T\eta\}^{\perp}$. We show that $X$ is a partial isometry. Let $\frac{T\eta}{\|T\|} = \left\langle \frac{T\eta}{\|T\|}, \eta \right\rangle \eta + \tau h,\ \|h\| = 1,\ h \perp \eta$. Since $\eta$ and $T\eta$ are linearly independent, $\tau \neq 0$. So we have that

$$X\frac{T\eta}{\|T\|} = \left\langle \frac{T\eta}{\|T\|}, \eta \right\rangle X\eta + \tau Xh = -\left\langle \frac{S\zeta}{\|S\|}, \zeta \right\rangle \zeta + \tau Xh,$$

which implies that

$$\left\langle X\frac{T\eta}{\|T\|}, \zeta \right\rangle = -\left\langle \frac{S\zeta}{\|S\|}, \zeta \right\rangle + \tau\langle Xh, \zeta\rangle = -\left\langle \frac{S\zeta}{\|S\|}, \zeta \right\rangle.$$

It follows then that $\langle Xh, \zeta\rangle = 0$ i.e., $Xh \perp \zeta (\zeta = X\eta)$. Hence we have that

$$\left\| \left\langle \frac{S\zeta}{\|S\|}, \zeta \right\rangle \zeta \right\|^2 + \|\tau Xh\|^2 = \left\| X\frac{T\eta}{\|T\|} \right\|^2 = \left| \left\langle \frac{T\eta}{\|T\|}, \eta \right\rangle \right|^2 + |\tau|^2 = 1,$$

which implies that $\|Xh\| = 1$. Now it is evident that $X$ a partial isometry and $\|(SX - XT)\zeta\| = \|SX - XT\| = \|S\| + \|T\|$, which is equivalent to $\|\delta_{S,T}(X)\| = \|S\| + \|T\|$. By Proposition 2.2 and [28], $\|\delta_{S,T}\| = \|S\| + \|T\|$. Hence $\delta_{S,T}$ is norm-attainable. □

The next result is a consequence of Proposition 2.2 and 2.3. It gives the necessary and sufficient conditions for norm-attainability of a basic elementary operator.

**Corollary 2.4.** *Let $S, T \in B(H)$ If both $S$ and $T$ are norm-attainable then the basic elementary operator $M_{S,\,T}$ is also norm-attainable.*

*Proof.* For any pair (S,T) it is known that $\|M_{S,\,T}\| = \|S\|\|T\|$. We can assume that $\|S\| = \|T\| = 1$. If both $S$ and $T$ are norm-attainable, then there exists unit vectors $\zeta$ and $\eta$ with $\|S\zeta\| = \|T\eta\| = 1$. We can therefore define an operator $X$ by $X = \langle\cdot, T\eta\rangle\zeta$. Clearly, $\|X\| = 1$. Therefore, we have $\|SXT\| \geq \|SXT\eta\| = \|\|T\eta\|^2 S\zeta\| = 1$. Hence, $\|M_{S,\,T}(X)\| = \|SXT\| = 1$, that is $M_{S,\,T}$ is also norm-attainable. □

In the next section, we dedicate our work to orthogonality of elementary operators on Banach spaces. From this point henceforth, all the elementary operators are implemented by norm-attainable operators unless otherwise stated. First we note that $\Omega$ denotes the algebra of all norm-attainable operators. In fact $\Omega$ is a Banach algebra. Let $T:\Omega\to\Omega$ be defined by $T(X)=\sum_{i=1}^{n}A_iXB_i,\ \forall\, X\in\Omega$, where $A_i,\ B_i$ are fixed in $\Omega$. We define the range of $T$ by $RanT=\{Y\in\Omega : Y=T(X),\ \forall\, X\in\Omega\}$, and the Kernel of $T$ by $KerT=\{X\in\Omega : T(X)=0,\ \forall\, X\in\Omega\}$. It is known [4] that for any of the examples of the elementary operators defined in Section 1 (inner derivation, generalized derivation, basic elementary operator, Jordan elementary operator), the following implications hold for a general bounded linear operator $T$ on a normed linear space $W$ , i.e. $RanT\perp KerT\Rightarrow \overline{RanT}\cap KerT=\{0\}\Rightarrow RanT\cap KerT=\{0\}$. Here $\overline{RanT}$ denotes the closure of the range of $T$ and $KerT$ denotes the kernel of $T$ and $RanT\perp KerT$ means $RanT$ is orthogonal to the Kernel of $T$ in the sense of Birkhoff. Let $A\in\Omega$. The algebraic numerical range $V(A)$ of $A$ is defined by: $V(A)=\{f(A): f\in\Omega'$ and $\|f\|=f(I)=1\}$ where $\Omega'$ is the dual space of $\Omega$ and $I$ is the identity element in $\Omega$. If $V(A)\subseteq\mathbb{R}$, then $A$ is called a Hermitian element. Given two Hermitian elements $S$ and $R$, such that $SR=RS$ then $D=S+Ri$ is called normal [29].

## 3. Main results

**Proposition 3.1.** *Let $A,B,C\in\Omega$ with $CB=I$ ($I$ is an identity element of $\Omega$). Then for a generalized derivation $\delta_{A,B}=AX-XB$ and an elementary operator $\Theta_{A,B}(X)=AXB-X$, $R_B(\overline{Ran\delta_{A,C}}\cap Ker\delta_{A,C})=\overline{Ran\Theta_{A,B}}\cap Ker\Theta_{A,B}$. Moreover, if $\overline{Ran\delta_{A,C}}\cap Ker\delta_{A,C}=\{0\}$ then $\overline{Ran\Theta_{A,B}}\cap Ker\Theta_{A,B}=\{0\}$.*

*Proof.* First, we prove that if $CB=I$ then $R_B\delta_{A,C}=\Theta_{A,B}$. To see this, $\forall\, X\in\Omega$, $R_B\delta_{A,C}(X)=AXB-XCB=AXB-X=\Theta_{A,B}$. Suppose that $P\in R_B(\overline{Ran\delta_{A,C}}\cap Ker\delta_{A,C})$. Now, it is a fact that the uniform norm assigns to real- or complex-valued continuous bounded operator $R_B$ defined on any set $\Omega$ the nonnegative number $\|R_B\|_\infty=\sup\{\|R_B(X)\| : X\in\Omega\}$. Since $R_B\delta_{A,C}=\Theta_{A,B}$ and $R_B$ is continuous for the uniform norm, then $P\in\overline{Ran\Theta_{A,B}}\cap Ker\Theta_{A,B}$. Conversely, since $R_C$ is continuous for the uniform norm, then by the same argument we prove that if $P\in R_B(\overline{Ran\Theta_{A,B}}\cap Ker\Theta_{A,B})$ then $P\in R_B(\overline{Ran\delta_{A,C}}\cap Ker\delta_{A,C})$. □

It is important to note the following. Let $A,B,C\in\Omega$ with $CB=I$ ($I$ is an identity element of $\Omega$). Then $R_B(\overline{Ran\delta_{A,C}}\cap Ker\delta_{A,C})=Ran\Theta_{A,B}\cap Ker\Theta_{A,B}$. Indeed, since $Ran\Theta_{A,B}\subseteq\overline{Ran\Theta_{A,B}}$, then by Proposition 3.1, the equality holds.

**Proposition 3.2.** *Let $S$ and $R$ be Hermitian elements. Then $\delta_{S,R}$ is also Hermitian.*

*Proof.* From [22], it is known that if $X$ is a Banach space then $V(\delta_{S,R})=V(S)-V(R)$ for all $S,R\in B(X)$. Therefore, $V(\delta_{S,R})\subseteq V(L_S)-V(L_R)=V(S)-V(R)\subseteq\mathbb{R}$. □

**Corollary 3.3.** *If $D$ and $E$ are normal elements in $\Omega$ then $\delta_{D,E}$ is also normal.*

*Proof.* Assume $D=S+Ri$ and $E=T+Ui$ where $S,R,T,U$ are Hermitian elements in $\Omega$ such that $SR=RS$ and $TU=UT$. Then $\delta_{D,E}=\delta_{S,T}+i\delta_{R,U}$ with $\delta_{S,T}\delta_{R,U}=\delta_{R,U}\delta_{S,T}$. Since $S,R,T,U$ are Hermitian, then by Proposition 3.2 $\delta_{R,U}$ and $\delta_{S,T}$ are Hermitian and so is $\delta_{D,E}$. □

**Remark 3.4.** *([22]) Let $X$ be a Banach space and $T\in B(X)$. If $T$ is a normal operator, then $RanT\perp KerT$. Moreover, if $D$ and $E$ are normal elements in $\Omega$ then $Ran\delta_{D,E}\perp Ker\delta_{D,E}$. Indeed, assume that $D$ and $E$ are normal elements in $\Omega$. Then by Corollary 3.3, $\delta_{D,E}$ is normal and by Proposition 3.2 $Ran\delta_{D,E}\perp Ker\delta_{D,E}$.*

**Corollary 3.5.** *If $A,B\in\Omega$ are normal and there exists $C\in\Omega$ such that $BC=I$ then $\overline{Ran\Theta_{A,C}}\cap Ker\Theta_{A,C}=\{0\}$.*

*Proof.* If $A,B\in\Omega$ are normal and self-adjoint elements, then by Corollary 3.3, $Ran\delta_{A,B}\perp Ker\delta_{A,B}$. This implies that $\overline{Ran\delta_{A,B}}\cap Ker\delta_{A,B}=\{0\}$. Using Proposition 2.2, we conclude that $\overline{Ran\Theta_{A,C}}\cap Ker\Theta_{A,C}=\{0\}$. □

The next theorem gives a stronger result on power sequences of operators $A^n,B^n\in\Omega$ for all $n\in\mathbb{N}$.

**Theorem 3.6.** *Let $A,B\in\Omega$ be normal and self-adjoint with $C\in\Omega$ such that $BC=I$ and $\|C\|\leq 1$. If $\|A\|\leq 1$ and $\|B\|\leq 1$ for all $n\in\mathbb{N}$ then $Ran\delta_{A,B}\perp Ker\delta_{A,B}$.*

*Proof.* It is well known [2] that $A^nX-XB^n=\Sigma_{i=0}^{n-1}A^{n-i-1}(AX-XB)B^i$ and

$$A^nX-XB^n-\Sigma_{i=0}^{n-1}A^{n-i-1}(AX-XB-Y)B^i=nYB^{n-1},$$

where $Y\in Ker\delta_{A,B}$. Multiplying this equality by $C^{n-1}$ we obtain

$$A^nXC^{n-1}-XB-\Sigma_{i=0}^{n-1}A^{n-i-1}(AX-XB-Y)B^iC^{n-1}=nYB^{n-1}C^{n-1}$$

which is equivalent to

$$nY = A^nXC^{n-1} - XB - \Sigma_{i=0}^{n-1}A^{n-i-1}(AX - XB - Y)B^iC^{n-1}.$$

Now, the assumption that $BC = I$ with $\|C\| \leq 1$ and $\|B\| \leq 1$ implies that $\|C^n\| = \|B^n\| = 1$, for all $n \in \mathbb{N}$. This shows that dividing both sides by $n$ and taking norms we obtain

$$\|Y\| \leq \frac{1}{n}\{\|A^n\|\|X\|\|C\|^{n-1} + \|X\|\|B\|\} + \frac{1}{n}\Sigma_{i=0}^{n-1}\|A\|^{n-i-1}\|AX - XB - Y\|\|B\|^i\|C\|^{n-1}$$

$= \frac{1}{n}\{\|A^n\|\|X\| + \|X\|\} + \frac{1}{n}\Sigma_{i=0}^{n-1}\|A\|^{n-i-1}\|AX - XB - Y\|$.
Hence $\|Y\| \leq \frac{2}{n}\|X\| + \frac{1}{n}\Sigma_{i=0}^{n-1}\|AX - XB - Y\|$. Taking limits as $n \to \infty$, we obtain that $\|Y\| \leq \|AX - XB - Y\|$. Therefore, $Ran\delta_{A,B} \perp Ker\delta_{A,B}$. □

The following theorem from Kittaneh [3] gives a general orthogonality condition for linear operators. The proof is omitted.

**Theorem 3.7** (3). *Let be $\Omega$ a normed algebra with the norm $\|.\|$ satisfying $\|XY\| \leq \|X\|\|Y\|$ for all $X$, $Y \in \Omega$ and let $\delta : \Omega \to \Omega$ be a linear map with $\|\delta\| \leq 1$. If $\delta(Y) = Y$ for some $Y \in \Omega$, then $\|\delta(X) - X + Y\| \geq \|Y\|$, for all $X \in \Omega$.*

We utilize the Theorem 3.7 to prove some results for general elementary operators. Let $T : \Omega \to \Omega$ be an elementary operator defined by $T(X) = \sum_{i=1}^n A_iXB_i$, $\forall X \in \Omega$. Now suppose that $T(Y) = Y$ for some $Y \in \Omega$. If $\|T\| \leq 1$, then $\|T(X) - X + Y\| \geq \|Y\|$, for all $X \in \Omega$. The following theorem follows immediately.

**Theorem 3.8.** *Suppose that $T(Y) = Y$ for some normal self-adjoint $Y \in \Omega$. If $\|\sum_{i=1}^n A_iA_i^*\|^{\frac{1}{2}} \|\sum_{i=1}^n B_i^*B_i\|^{\frac{1}{2}} \leq 1$, then $\|T(X) - X + Y\| \geq \|Y\|$, for all $X \in \Omega$.*

*Proof.* We only need to show that $\|T\| \leq 1$. Let $Z_1 = [A_1, ..., A_n]$ and $Z_2 = [B_1, ..., B_n]^T$. Taking $Z_1Z_1^*$ and $Z_2^*Z_2$ shows that $\|Z_1\| = \|\sum_{i=1}^n A_iA_i^*\|^{\frac{1}{2}}$ and $\|Z_2\| = \|\sum_{i=1}^n B_i^*B_i\|^{\frac{1}{2}}$. From [16], it is known that $T(X) = Z_1(X \otimes I_n)Z_2$, where $I_n$ is the identity of $M_n(\mathbb{C})$. Therefore it follows that $\|T(X)\| \leq \|Z_1\|\|Z_2\|\|X\|$. Hence $\|T\| \leq 1$. □

Now, we consider the orthogonality of Jordan elementary operators. We later consider the necessary and sufficient conditions for their normality. At this juncture a type of norm, called the unitarily invariant norm comes in handy. A unitarily invariant norm is any norm defined on some two-sided ideal of $B(H)$ and $B(H)$ itself which satisfies the following two conditions. For unitary operators $U, V \in B(H)$ the equality $|||UXV||| = |||X|||$ holds, and $|||X||| = s_1(X)$, for all rank one operators $X$. It is proved that any unitarily invariant norm depends only on the sequence of singular values. Also, it is known that the maximal ideal, on which $|||UXV|||$ has sense, is a Banach space with respect to that unitarily invariant norm. Among all unitarily invariant norms there are few important special cases. The first is the Schatten $p$-norm ($p \geq 1$) defined by $\|X\|_p = (\Sigma_{j=1}^{+\infty}s_j(X)^p)^{1/p}$ on the set $\mathscr{C}_p = \{X \in B(H) : \|X\|_p < +\infty\}$. For $p = 1, 2$ this norm is known as the nuclear norm (Hilbert-Schmidt norm) and the corresponding ideal is known as the ideal of nuclear (Hilbert-Schmidt) operators. The ideal $\mathscr{C}_2$ is also interesting for another reason. Namely, it is a Hilbert space with respect to the $\|.\|_2$ norm. The other important special case is the set of so-called Ky Fan norms $\|X\|_k = \Sigma_{j=1}^k s_j(X)$. The well-known Ky Fan dominance property asserts that the condition $\|X\|_k \leq \|Y\|$ for all $k \geq 1$ is necessary and sufficient for the validity of the inequality $|||X||| \leq |||Y|||$ in all unitarily invariant norms. For further details refer to [20].We state the following theorem from [20] on orthogonality.

**Theorem 3.9.** *Let $A, B \in B(H)$ be normal operators, such that $AB = BA$, and let $\mathscr{U}(X) = AXB - BXA$. Furthermore, suppose that $A^*A + B^*B > 0$. If $S \in Ker\mathscr{U}$, then $|||\mathscr{U}(X) + S||| \geq |||S|||$.*

We extend Theorem 3.9 to distinct operators $A, B, C, D \in B(H)$ in the theorem below.

**Theorem 3.10.** *Let $A, B, C, D \in B(H)$ be normal operators, such that $AC = CA$, $BD = DB$, $AA^* \leq CC^*$, $B^*B \leq D^*D$. For an elementary operator $\mathscr{U}(X) = AXB - CXD$ and $S \in B(H)$ satisfying $ASB = CSD$, then $\|\mathscr{U}(X) + S\| \geq \|S\|$, for all $X \in B(H)$.*

*Proof.* From $AA^* \leq CC^*$ and $B^*B \leq D^*D$, let $A = CU$, and $B = VD$, where $U, V$ are contractions. So we have $AXB - CXD = CUXVD - CXD = C(UXV - X)D$. Assume $C$ and $D^*$ are injective, $ASB = CSD$ if and only if $USV = S$. Moreover, $C$ and $U$ commute. Indeed from $A = CU$ we obtain $AC = CUC$. Therefore, $C(A - UC) = 0$. Thus since $C$ is injective $A = CU$. Similarly, $D$ and $V$ commute. So, $\|\mathscr{U}(X) + S\| = \|[AXB - CXD] + S\| = \|[U(CXD)V - CXD] + S\| \geq \|S\|, \forall X \in B(H)$. Now, under the condition of Theorem 3.10, $A$ and $C$ have operator matrices $A = \begin{pmatrix} 0 & 0 \\ 0 & A_0 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 0 \\ 0 & C_0 \end{pmatrix}$ with respect to the space decomposition $H = \overline{\mathscr{R}(C)} \oplus \mathscr{N}(C)$, respectively. Here, $A_0$ is a normal operator on $\overline{\mathscr{R}(C)}$ and $C_0$ is an injective and

normal operator on $\overline{\mathscr{R}(C)}$. $B$ and $D$ have operator matrices $B = \begin{pmatrix} B_0 & 0 \\ 0 & 0 \end{pmatrix}$ and $D = \begin{pmatrix} D_0 & 0 \\ 0 & 0 \end{pmatrix}$ with respect to the space decomposition $H = \overline{\mathscr{R}(D)} \oplus \mathscr{N}(D)$, respectively. Here, $B_0$ is a normal operator on $\overline{\mathscr{R}(D)}$ and $D_0$ is an injective and normal operator on $\overline{\mathscr{R}(D)}$. $X$ and $S$ have operator matrices $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$ and $S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$ which are as operator from the space decomposition $H = \overline{\mathscr{R}(D)} \oplus \mathscr{N}(D)$ into the space decomposition $H = \overline{\mathscr{R}(C)} \oplus \mathscr{N}(C)$, respectively.
In this case, $\mathscr{U}(X) = AXB - CXD = \begin{pmatrix} A_0X_{11}B_0 - C_0X_{11}D_0 & 0 \\ 0 & 0 \end{pmatrix}$ and $A_0S_{11}B_0 - C_0S_{11}D_0 = 0$. Therefore, $\|A_0X_{11}B_0 - C_0X_{11}D_0 + S_{11}\| \geq \|S_{11}\|$.
Hence,

$$\begin{aligned} \|\mathscr{U}(X) + S\| &= \left\| \begin{pmatrix} A_0X_{11}B_0 - C_0X_{11}D_0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \right\| \\ &= \left\| \begin{pmatrix} A_0X_{11}B_0 - C_0X_{11}D_0 + S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \right\| \\ &\geq \left\| \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \right\|. \end{aligned}$$

□

The result in Theorem 3.10 can be generalized Banach spaces and other complex spaces like operator spaces and function spaces.

## 4. Conclusion

We conclude this paper by remarking that these results can be extended to give more results on generalized finite operators in terms of orthogonality and norm-attainability in C*-algebras.

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