Results in Nonlinear Analysis **1** (2018) No. 3, 107-115
Available online at www.nonlinear-analysis.com

Results in Nonlinear Analysis
ISSN 2636-7556

Peer Reviewed Scientific Journal

Using a hybrid technique for the analytical solution of a coupled system of two-dimensional BurgerŠs equations,

Atta Ullah[a], Kamal Shah[a]

[a] *Department of Mathematics, University of Malakand, Dir(L), Khyber Pakhtunkhwa, Pakistan.*

Abstract

Our aim in this paper is to investigate the analytical solutions of two dimensional non- linear system of Burger's equations. We present two problems for the efficiency of the proposed hybrid techniques form by coupling Laplace transform with Adomian polynomials method. The proposed method is named as Laplace transform Adomian decomposition method (LTADM). Two test examples are solved by the considered method to derive efficiency of the proposed method.

Keywords: Laplace transform, Adomian decomposition method, System of Burger's equations.
2010 MSC: 26A33 34A08 35B40.

1. Introduction and Preliminaries

Different types of methods have been presented for finding the solutions of linear and non- linear problems such as Fourier transform method(FTM) [12], homotopy analysis method(HAM) [3], Adomian decomposition method(ADM) [7], the Exponential function method (EFM) [8], variational iteration method (VIM) [6], etc. In present work, we use LTADM to find analytical solutions to non- linear system of Burger's equations. Usually the proposed coupled system arises in the study of fluid flow. This type of problems is used to model various kinds of phenomena such as turbulence and the approximate theory of flow through a shock wave traveling in a viscus fluid [2]. The Burger's equations contains Reynolds's number which was introduced by Stokes in 1851, but the Reynolds's number was named by Arnold Sommerfeld in 1908. The mentioned number is used to help portend flow shape in different fluid flow situations. At low Reynolds's numbers, flows tend to be dominated by laminar flow, while at high Reynolds's numbers flows may sometimes intersect or

Email addresses: `Uatta517@gmail.com` (Atta Ullah), `kamalshah408@gmail.com` (Kamal Shah)

Received September 08, 2018, Accepted December 06, 2018, Online December 09, 2018.

move counter to the overall direction. The Reynolds's number has wide applications like it is used in fluid mechanics such as the movement of organisms swimming through water, the movement of air in atmosphere, for designing fountain heads, for water tubes and pumps etc.

LTADM [11] has been used as a powerful technique for solving non- linear ordinary and partial differential equations. In this method the unknown terms can be expressed in the form of infinite series like $u(x, y, t) = \sum_{n=0}^{\infty} u_n$ and the non- linear terms may be decomposed in terms of Adomian polynomials as $\mathbf{N}u(x, y, t) = \sum_{n=0}^{\infty} A_n$. Consider product of two functions uu_x, expand this non- linear term through Adomian polynomial as

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left(\sum_{i=0}^{\infty}(\lambda^i u_i)\right)\left(\sum_{i=0}^{\infty}(\lambda^i u_{i,x})\right)\right]\Big|_{\lambda=0} \tag{1.1}$$

for $n = 0$, we have, $A_0 = u_0 u_{0,x}$
for $n = 1$, we have, $A_1 = u_0 u_{1,x} + u_1 u_{0,x}$
for $n = 2$, we have, $A_2 = u_0 u_{2,x} + u_1 u_{1,x} + u_2 u_{0,x}$
generally we have, $A_n = u_0 u_{n,x} + u_1 u_{n-1,x} + u_2 u_{n-2,x} + \cdots + u_{n-1} u_{1,x} + u_n u_{0,x}$. This technique has been largely employed to solve large variety of problems such as Lane Emden Fowler equations [11], Falkner Skan equation [4], etc.

In present work, the proposed scheme LTADM is applied to two numerical problems of Burger's equations for the analytical exact/approximate solutions of these systems. Here we describe some preliminaries results as:

Definition 1.1. For the function $u(x, y, t)$ the Laplace transform of $u(x, y, t)$ is defined as

$$\mathcal{L}[u(x, y, t)] = \int_0^{\infty} \exp(-st)u(x, y, t)dt, s > 0,$$

for $x, y \in [a, b]$ and $t > 0$.

Theorem 1.2. *Suppose that X is Banach space and $T : X \to X$ is a contraction non linear mapping and if we consider the generated sequence of solutions via LTADM is written as*

$$u^{(k)} = T(u^{(k-1)}),\ k = 1, 2, 3, \cdots ,$$

then the following holds

1. $||u^{(k)} - u|| - X \le \mu^k ||Tu - u||_X,\ 0 \le \mu < 1$;
2. $u^{(k)}(x, t)$ *is always hold in the neighborhood* $u(x, t)$ *implies that*

$$u^{(k)} \in B(u, r) \subset X,\ \textit{where}\ B(u, r) = \{u^* \in \mathcal{B} :\ ||u^* - u||_X < r\};$$

3. $\lim_{k\to\infty} u^{(k)} = u$.

Proof. For proof see [10]. □

2. General idea for two dimensional system of Burger's equation

In this part we give basic idea for LTADM. Consider two dimensional system of Burger's equations [1] of the form

$$\begin{aligned} u_t + uu_x + vu_y &= \frac{1}{R}(u_{xx} + u_{yy}),\\ v_t + uv_x + vv_y &= \frac{1}{R}(v_{xx} + v_{yy}), \end{aligned} \tag{2.1}$$

with initial conditions

$$u(x, y, 0) = \alpha(x, y), v(x, y, 0) = \beta(x, y).$$

And $(x, y)\epsilon\Lambda$ and $t > 0$,
where $\Lambda = \{(x, y) : a \leq x \leq b, a \leq y \leq b\}$, R is the Reynolds numbers and $u(x, y, t), v(x, y, t)$ are velocity components.
Applying first Laplace transform, then inverse Laplace transform to equation (2.1) and using initial conditions we get

$$\begin{aligned} u(x,y,t) &= \alpha(x,y) - \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\Big(uu_x + vu_y\Big)\right] + \frac{1}{R}\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\Big(u_{xx} + u_{yy}\Big)\right], \\ v(x,y,t) &= \beta(x,y) - \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\Big(uv_x + vv_y\Big)\right] + \frac{1}{R}\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\Big(v_{xx} + v_{yy}\Big)\right]. \end{aligned} \tag{2.2}$$

In equation (2.2) involve four non- linear terms, by considered method we decompose these four non- linear terms through Adomian polynomials as

$$\begin{aligned} uu_x &= \sum_{n=0}^{\infty} A_n(x,y,t) \\ vu_y &= \sum_{n=0}^{\infty} B_n(x,y,t) \\ uv_x &= \sum_{n=0}^{\infty} C_n(x,y,t) \\ vv_x &= \sum_{n=0}^{\infty} D_n(x,y,t). \end{aligned} \tag{2.3}$$

The Adomian polynomials can be generated for all forms of non- linear terms. They are determined by the following relations

$$\begin{aligned} A_n(x,y,t) &= \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left(\sum_{i=0}^{\infty}(\lambda^i u_i)\right)\left(\sum_{i=0}^{\infty}(\lambda^i u_{i,x})\right)\right]\Bigg|_{\lambda=0}, \\ B_n(x,y,t) &= \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left(\sum_{i=0}^{\infty}(\lambda^i v_i)\right)\left(\sum_{i=0}^{\infty}(\lambda^i u_{i,y})\right)\right]\Bigg|_{\lambda=0}, \\ C_n(x,y,t) &= \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left(\sum_{i=0}^{\infty}(\lambda^i u_i)\right)\left(\sum_{i=0}^{\infty}(\lambda^i v_{i,x})\right)\right]\Bigg|_{\lambda=0}, \\ D_n(x,y,t) &= \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left(\sum_{i=0}^{\infty}(\lambda^i v_i)\right)\left(\sum_{i=0}^{\infty}(\lambda^i v_{i,x})\right)\right]\Bigg|_{\lambda=0}. \end{aligned} \tag{2.4}$$

and the unknown terms $u(x, y, t)$, and $v(x, y, t)$ decomposes by an infinite series of components as

$$u(x,y,t) = \sum_{n=0}^{\infty} u_n(x,y,t), \qquad v(x,y,t) = \sum_{n=0}^{\infty} v_n(x,y,t). \tag{2.5}$$

Using equations (2.3) and (2.5) in equation (2.2), we get

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x,y,t) &= \alpha(x,y) - \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\sum_{n=0}^{\infty}\Big(A_n + B_n\Big)\right] + \frac{1}{R}\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\sum_{n=0}^{\infty}\Big(u_{n,xx} + u_{n,yy}\Big)\right], \\ \sum_{n=0}^{\infty} v_n(x,y,t) &= \beta(x,y) - \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\sum_{n=0}^{\infty}\Big(C_n + D_n\Big)\right] + \frac{1}{R}\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\sum_{n=0}^{\infty}\Big(v_{n,xx} + v_{n,yy}\Big)\right]. \end{aligned} \tag{2.6}$$

Comparing both hand sides terms for $n = 0, 1, 2 \ldots$ we obtained the following terms

$$\begin{aligned} u_0(x,y,t) &= \alpha(x,y), \\ v_0(x,y,t) &= \beta(x,y). \end{aligned} \tag{2.7}$$

$$\begin{aligned} u_1(x,y,t) - \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\Big(A_0 + B_0\Big)\right] + \frac{1}{R}\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\Big(u_{0,xx} + u_{0,yy}\Big)\right], \\ v_1(x,y,t) = -\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\Big(C_0 + D_0\Big)\right] + \frac{1}{R}\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\Big(v_{0,xx} + v_{0,yy}\Big)\right]. \end{aligned} \tag{2.8}$$

Generally we write

$$\begin{aligned} u_{n+1}(x,y,t) = -\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\Big(A_n + B_n\Big)\right] + \frac{1}{R}\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\Big(u_{n,xx} + u_{n,yy}\Big)\right], \\ v_{n+1}(x,y,t) = -\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\Big(C_n + D_n\Big)\right] + \frac{1}{R}\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\Big(v_{n,xx} + v_{n,yy}\Big)\right]. \end{aligned} \tag{2.9}$$

The obtained terms written in the form of infinite series as

$$\begin{aligned} u(x,y,t) &= u_0(x,y,t) + u_1(x,y,t) + u(x,y,t) + \ldots, \\ v(x,y,t) &= v_0(x,y,t) + v_1(x,y,t) + v_2(x,y,t) \ldots \end{aligned} \tag{2.10}$$

which constitute the solution.

3. Applications

In this section, we illustrate the above presented method, by two problems.

Example 3.1. Consider the two dimensional system of Burger's equations [1] with initial conditions

$$u(x,y,0) = x + y, \quad v_0(x,y,0) = x - y.$$

And the domain has been taken as $\Lambda = \{(x,y) : 0 \le x \le 0.7, 0 \le y \le 0.7\}$.
Taking same procedure of generalized form, then using the given initial conditions, we lead

$$\begin{aligned} u(x,y,t) = x + y - \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\Big(uu_x + vu_y\Big)\right] + \frac{1}{R}\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\Big(u_{xx} + u_{yy}\Big)\right], \\ v(x,y,t) = x - y - \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\Big(uv_x + vv_y\Big)\right] + \frac{1}{R}\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\Big(v_{xx} + v_{yy}\Big)\right]. \end{aligned} \tag{3.1}$$

Putting equations (2.3) and (2.5) in equation (3.9), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x,y,t) = x + y - \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\sum_{n=0}^{\infty}\Big(A_n + B_n\Big)\right] + \frac{1}{R}\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\sum_{n=0}^{\infty}\Big(u_{n,xx} + u_{n,yy}\Big)\right], \\ \sum_{n=0}^{\infty} v_n(x,y,t) = x - y - \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\sum_{n=0}^{\infty}\Big(C_n + D_n\Big)\right] + \frac{1}{R}\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\sum_{n=0}^{\infty}\Big(v_{n,xx} + v_{n,yy}\Big)\right]. \end{aligned} \tag{3.2}$$

Comparing both hand sides terms for $n = 0, 1, 2 \ldots$ we obtained the following terms

$$\begin{aligned} u_0(x,y,t) &= x + y, \\ v_0(x,y,t) &= x - y. \end{aligned} \tag{3.3}$$

$$\begin{aligned} u_1(x,y,t) &= -2xt, \\ v_1(x,y,t) &= -2yt. \end{aligned} \tag{3.4}$$

$$\begin{aligned} u_2(x,y,t) &= -4(x+y)\frac{t^2}{2!}, \\ v_2(x,y,t) &= -4(x-y)\frac{t^2}{2!}. \end{aligned} \tag{3.5}$$

$$\begin{aligned} u_3(x,y,t) &= -12x\frac{t^3}{3!}, \\ v_3(x,y,t) &= -12y\frac{t^3}{3!}. \end{aligned} \tag{3.6}$$

$$\begin{aligned} u_4(x,y,t) &= 12(5x+4y)\frac{t^4}{4!}, \\ v_4(x,y,t) &= 12(7x-6y)\frac{t^4}{4!}. \end{aligned} \tag{3.7}$$

Infinite series form of terms as

$$\begin{aligned} u(x,y,t) &= x+y-2xt-4(x+y)\frac{t^2}{2!}-12x\frac{t^3}{3!}+12(5x+4y)\frac{t^4}{4!}, \\ v(x,y,t) &= x-y-2yt-4(x-y)\frac{t^2}{2!}-12y\frac{t^3}{3!}+12(7x-6y)\frac{t^4}{4!} \end{aligned} \tag{3.8}$$

Applying the convergence theorem to above infinite series solution, suppose

$$\Phi_n = T(\Phi_{n-1}), \qquad \Phi_{n-1} = (U_{n-1}, V_{n-1})$$

where

$$U_0 = u_0 + u_1, \qquad V_0 = v_0 + v_1$$

and

$$U_n = \sum_{i=0}^{2n+1} u_i, \qquad V_n = \sum_{i=0}^{2n+1} v_i \quad \text{where} \quad t \le \sqrt{\frac{\mu}{2}}, \qquad 0 \le \mu \le 1$$

By convergence theorem for $n = 0$, we get

$$\|U_0 - \phi\|_X = \|(u_0+u_1) - \phi\|_X = \mu^0\| - \tfrac{2t^2}{1-2t^2}(x+y-2xt)\|_X.$$

For $n = 1$

$$\|U_1 - \phi\|_X = \|(u_0+u_1+u_2) - \phi\|_X = \| - \tfrac{4t^4}{1-2t^2}(x+y-2xt)\|_X.$$

$$\|U_1 - \phi\|_X = \|(u_0+u_1+u_2) - \phi\|_X \le \mu^1\| - \tfrac{2t^2}{1-2t^2}(x+y-2xt)\|_X.$$

This implies that

$$\|U_1 - \phi\|_X = \mu^1 \ \|U_0 - \phi\|_X.$$

For $n = 2$

$$\|U_2 - \phi\|_X = \|(u_0+u_1+u_2+u_3) - \phi\|_X = \| - \tfrac{8t^6}{1-2t^2}(x+y-2xt)\|_X.$$

$$\|U_2 - \phi\|_X = \|(u_0+u_1+u_2+u_3) - \phi\|_X \le \mu^1\| - \tfrac{2t^2}{1-2t^2}(x+y-2xt)\|_X.$$

This implies that

$$\|U_2 - \phi\|_X = \mu^2 \ \|U_0 - \phi\|_X.$$

For $k = n$, we have

$$\|U_n - \phi\|_X = \mu^n \ \|U_0 - \phi\|_X.$$

Taking limit as $n \to \infty$

$$\lim_{n\to\infty} \|U_n - \phi\|_X = \lim_{n\to\infty} \mu^n \ \|U_0 - \phi\|_X = 0.$$

Therefore

$$u(x, y, t) = \lim_{n\to\infty} U_n = \phi = \frac{x + y - 2xt}{1 - 2t^2}.$$

Similarly

$$v(x, y, t) = \lim_{n\to\infty} V_n = \phi = \frac{x + y - 2yt}{1 - 2t^2}.$$

Hence the approximate solutions tends to exact solutions by taking $n \to \infty$ which demonstrates the efficiency of the proposed method for a nonlinear coupled system of Burger's equations.

Example 3.2. Consider the two dimensional system of Burger's equations [5] with initial conditions

$$u(x, y, 0) = sin(\pi x) + cos(\pi y), \qquad v_0(x, y, 0) = x + y.$$

And the domain has been taken as $\Lambda = \{(x, y \leq x \leq 0.7, 0 \leq y \leq 0.7\}$.
Taking same procedure of generalized form, then using the given initial conditions, we lead

$$\begin{aligned} u(x, y, t) &= \sin(\pi x) + \cos(\pi y) - \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\Big(uu_x + vu_y\Big)\right] + \frac{1}{R}\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\Big(u_{xx} + u_{yy}\Big)\right], \\ v(x, y, t) &= x + y - \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\Big(uv_x + vv_y\Big)\right] + \frac{1}{R}\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\Big(v_{xx} + v_{yy}\Big)\right]. \end{aligned} \tag{3.9}$$

Putting equations (2.3) and (2.5) in equation (3.9), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x, y, t) &= \sin(\pi x) + \cos(\pi y) - \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\sum_{n=0}^{\infty}\Big(A_n + B_n\Big)\right] + \\ &\quad + \frac{1}{R}\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\sum_{n=0}^{\infty}\Big(u_{n,xx} + u_{n,yy}\Big)\right], \\ \sum_{n=0}^{\infty} v_n(x, y, t) &= x + y - \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\sum_{n=0}^{\infty}\Big(C_n + D_n\Big)\right] \\ &\quad + \frac{1}{R}\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\sum_{n=0}^{\infty}\Big(v_{n,xx} + v_{n,yy}\Big)\right]. \end{aligned} \tag{3.10}$$

Comparing both hand sides terms for $n = 0, 1, 2 \ldots$ we obtained the following terms

$$\begin{aligned} u_0(x, y, t) &= \sin(\pi x) + \cos(\pi y), \\ v_0(x, y, t) &= x + y. \end{aligned} \tag{3.11}$$

Taking Reynolds's number $R = 50$, and for the sack of simplicity we put value of u_0 (initial condition) in u_1, u_2 terms

$$\begin{aligned} u_1(x, y, t) &= t\Big[-u_0\pi\cos(\pi x) + (x + y)\pi\sin(\pi y) - \frac{1}{50}\pi^2 u_0\Big], \\ v_1(x, y, t) &= -t\Big[u_0 + x + y)\Big]. \end{aligned} \tag{3.12}$$

$$\begin{aligned}u_2(x,y,t) = \frac{t^2}{2!}\Bigg[&\left(u_0\pi\cos(\pi x) + \frac{\pi^2}{50}u_0 - (x+y)\pi\sin(\pi y)\right)\pi\cos(\pi x) + \frac{1}{2500}\pi^4\sin(\pi x)\\ &+ u_0\left(\pi^2\cos^2(\pi x) - \sin(\pi x) - \pi^2\sin(\pi x)\cos(\pi y) - \pi\sin(\pi y) + \frac{\pi^3}{50}\cos(\pi x)\right)\\ &- \left(u_0 + (x+y) + \frac{\pi^2}{50} + \frac{\pi^2}{50}(x+y) + 1\right)\pi\sin(\pi y) + \frac{1}{50}\Bigg(- \pi^2\sin(\pi y)\\ &+ 3\pi^3\sin(\pi x) + \cos(\pi y) + \pi^3 u_0\Bigg)\cos(\pi x) + \left((x+y) + \frac{1}{25} + \frac{1}{2500}\pi^2\right)\pi^2\cos(\pi y)\Bigg],\\ v_2(x,y,t) = \frac{t^2}{2!}\Bigg[&\left(2\pi\cos(\pi x) + \frac{1}{25}\pi^2 + 2\right)u_0 - (x+y)\Big(2\pi\sin(\pi y) + 1\Big)\Bigg].\end{aligned} \tag{3.13}$$

The other terms can be similarly computed. Infinite series which form the solutions as provided as

$$\begin{aligned}u(x,y,t) &= u_0(x,y,t) + u_1(x,y,t) + u_2(x,y,t) + \dots,\\ v(x,y,t) &= v_0(x,y,t) + v_1(x,y,t) + v_2(x,y,t) + \dots.\end{aligned} \tag{3.14}$$

In the following Figures 1 to 4, we have plotted the given solutions for three terms and four terms respectively.

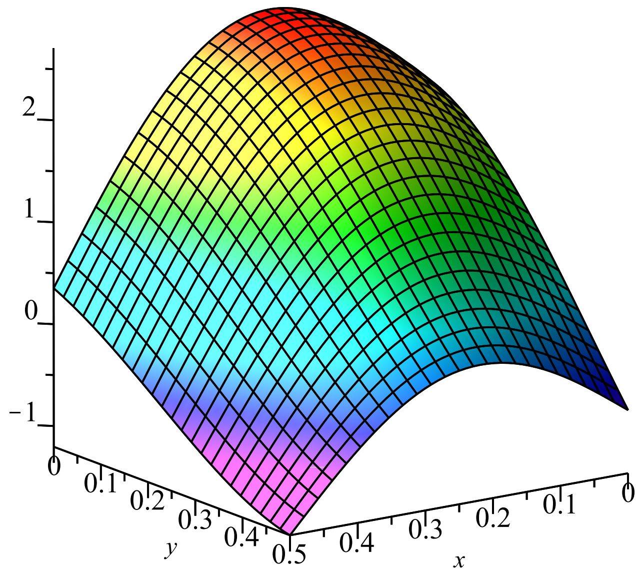

Figure 1. The plot of first three terms approximate solution of v at time level $t = 0.8$ of Example 3.2.

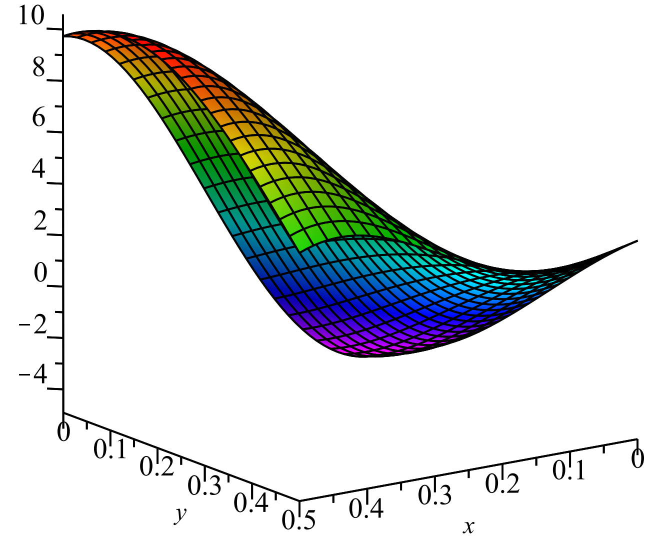

Figure 2. The plot of first four terms approximate solution of v at time level $t = 0.8$ of Example 3.2.

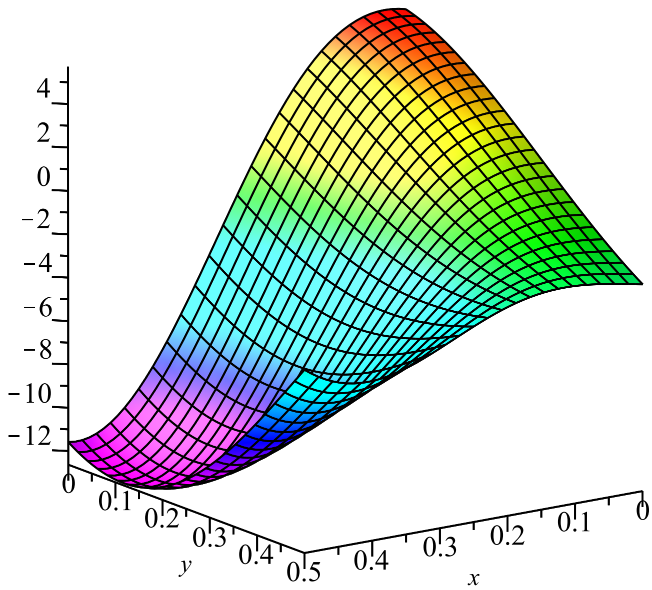

Figure 3. The plot of first three terms approximate solution of u at time level $t = 0.8$ of Example 3.2.

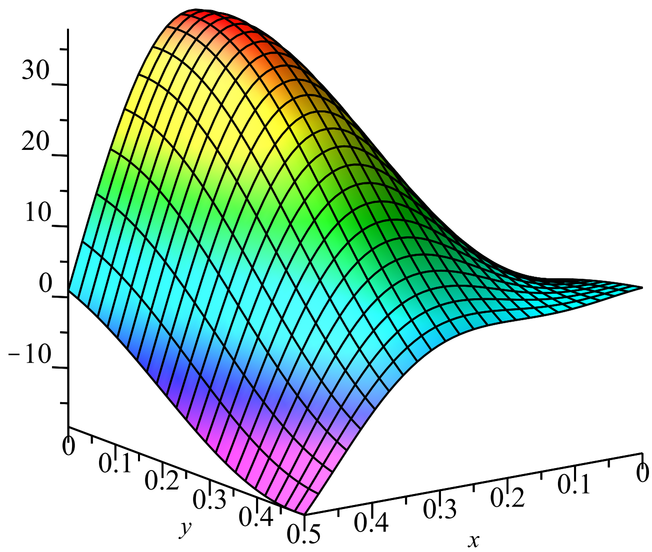

Figure 4. The plot of first four terms approximate solution of u at time level $t = 0.8$ of Example 3.2.

4. Conclusion

We have been successfully applied the LTADM for obtaing analytical approximate solutions to system of Burger's equations. By doing so we obtained approximate solutions which is closed to the exact solutions. A results which described the convergency of the proposed method has been given. We observed that LTADM is easy and sufficient techniques for analytical solution of such like problems.

Acknowledgment We are really thank full to the reviewers for their useful suggestions and corrections.

References

[1] J.Biazar and H.Aminikhah, Study of convergence of homotopy perturbation method for system of partial differential equations (2009).

[2] J.M.Burger, A Mathematical model illustrating the theory of turbulence. *Academic press, New York,* (1948).

[3] M.Dehghan, J.Manafian and A.Saadatmandi, Solving nonlinear fractional partial differential equations using the homotopy analysis method. *Num. Meth. Partial Diff. Equns*, **26**(2), 448 –479 (2010).

[4] N.S.Elgazery, Numerical solution for the falkner-skan equations. *Chaos, Solitons and Fractals,* 35 (**4**), 738–746 (2008).

[5] Y.Chen and H.L.An, Numerical solutions of coupled Burgers equations with time-and spacefractional derivatives. *Applied Mathematics and Computation,* **200**(1), 87 – 95 (2008).

[6] D.D.Ganji, H.Tari, M. Bakhshi Joobari, Variational iteration method and homotopy perturbation method for nonlinear evolution equations. *Int. J. Comput. Math. Appl,* 54, 1018–1027 (2007).

[7] Y.Hu, Y.Luo and Z.Lu, Analytical solution of the linear fractional differential equation by Adomian decomposition method. *J.Comput.Appl.Math,* 215, 220–229 (2008).

[8] H.Naher, F.A.Abdullah and M.A.Akbar, The exp-function method for new exact solutions of the nonlinear partial differential equations. *International Journal of Physical Sciences,* (**29**)6, 6706–6716 (2011).

[9] C.Rogers and F.William, Shadwick Backlund transformations and their applications, Academic press (1982).

[10] K.Shah, H.Khalil, R.A.Khan, Analytical Solutions of Fractional Order Diffusion Equations by Natural Transform Method, *Iran Journal of Science and Technology (Trans Sci:A)*, 1–12 (2016).

[11] A.Ullah and K.Shah, Numerical analysis of Lane EmdenFowler equations. *Journal of Taibah University for Science,* 1658–1663 (2018).

[12] A.M.Yang, Y.Z.Zhang and Y.Long Te Yang, Fourier transforms to heat-conduction in a semi-infnite fractal bar. *Termal Science,* 37 (**3**), 707–713 (2013).