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On Bicomplex Pell and Pell-Lucas Numbers

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Abstract
In this paper, bicomplex Pell and bicomplex Pell-Lucas numbers are defined. Also, negabicomplex Pell and negabicomplex Pell-Lucas numbers are given. Some algebraic properties of bicomplex Pell and bicomplex Pell-Lucas numbers which are connected between bicomplex numbers and Pell and Pell-Lucas numbers are investigated. Furthermore, d'Ocagne's identity, Binet's formula, Cassini's identity and Catalan's identity for these numbers are given.

Keywords: Pell and Pell-Lucas numbers, Bicomplex number, Quaternion
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1. Introduction

Bicomplex numbers were introduced by Corrado Segre in 1892 [1]. G. Baley Price (1991), presented bicomplex numbers based on multi-complex spaces and functions in his book [2]. In recent years, fractal structures of these numbers have also been studied [3] . The set of bicomplex numbers can be expressed by the basis $\{1\,,i\,,j\,,i\,j\}$ as,

$$\mathbb{C}_2 = \{\, q = q_1 + iq_2 + jq_3 + i\,jq_4 \mid q_1, q_2, q_3, q_4 \in \mathbb{R}\} \tag{1.1}$$

or

$$\mathbb{C}_2 = \{\, q = (q_1 + iq_2) + j(q_3 + iq_4) \mid q_1, q_2, q_3, q_4 \in \mathbb{R}\} \tag{1.2}$$

where i,j and $i\,j$ satisfy the conditions

$$i^2 = -1,\;\; j^2 = -1,\;\; i\,j = j\,i.$$

Thus, any bicomplex number q is introduced as pairs of typical complex numbers with the additional structure of commutative multiplication (Table 1).

A set of bicomplex numbers $\mathbb{C}_2$ is a real vector space with addition and scalar multiplication operations. The vector space $\mathbb{C}_2$ equipped with bicomplex product is a real associative algebra. Also, the vector space together with the properties of multiplication and the product of the bicomplex numbers are a commutative algebra. Furthermore, three different conjugations can operate on bicomplex numbers [3], [4], [5] as follows:

Table 1. Multiplication scheme of bicomplex numbers

x	1	i	j	i j
1	1	i	j	i j
i	i	-1	i j	-j
j	j	i j	-1	-i
i j	i j	-j	-i	1

$$q = q_1 + i\,q_2 + j\,q_3 + i\,j\,q_4 = (q_1 + iq_2) + j\,(q_3 + iq_4),\; q \in \mathbb{C}_2$$
$$q_i{}^* = q_1 - iq_2 + jq_3 - ijq_4 = (q_1 - iq_2) + j\,(q_3 - iq_4),$$
$$q_j{}^* = q_1 + iq_2 - jq_3 - ijq_4 = (q_1 + iq_2) - j\,(q_3 + iq_4),$$
$$q_{ij}{}^* = q_1 - iq_2 - jq_3 + ijq_4 = (q_1 - iq_2) - j\,(q_3 - iq_4).$$

and properties of conjugation

1) $(q^*)^* = q$,
2) $(q_1\,q_2)^* = q_2{}^*\,q_1{}^*,\;\; q_1, q_2 \in \mathbb{C}_2$,
3) $(q_1 + q_2)^* = q_1{}^* + q_2{}^*$,
4) $(\lambda\, q)^* = \lambda\, q^*$,
5) $(\lambda\, q_1 \pm \mu\, q_2)^* = \lambda\, q_1{}^* \pm \mu\, q_2{}^*,\; \lambda, \mu \in \mathbb{R}$.

Therefore, the norm of the bicomplex numbers is defined as

$$N_{q_i} = \|q \times q_i{}^*\| = \sqrt{\left|q_1^2 + q_2^2 - q_3^2 - q_4^2 + 2\,j\,(q_1 q_3 + q_2 q_4)\right|},$$
$$N_{q_j} = \|q \times q_j{}^*\| = \sqrt{\left|q_1^2 - q_2^2 + q_3^2 - q_4^2 + 2\,i\,(q_1 q_2 + q_3 q_4)\right|},$$
$$N_{q_{ij}} = \|q \times q_{ij}{}^*\| = \sqrt{\left|q_1^2 + q_2^2 + q_3^2 + q_4^2 + 2\,i\,j\,(q_1 q_4 - q_2 q_3)\right|}.$$

Pell numbers were invented by John Pell but, these numbers are named after Edouard Lucas. Pell and Pell-Lucas numbers have important parts in mathematics. They have fundamental importance in the fields of combinatorics and number theory [6],[7],[8],[9].

The sequence of Pell numbers

$$1,\, 2,\, 5,\, 12,\, 29,\, 70,\, 169,\, 408,\, 985,\, 2378, \ldots, P_n, \ldots$$

is defined by the recurrence relation

$$P_n = 2\,P_{n-1} + P_{n-2}\,,\;\; (n \geq 2),$$

with $P_0 = 0, P_1 = 1$.

The sequence of Pell - Lucas numbers

$$2,\, 6,\, 14,\, 34,\, 82,\, 198,\, 478,\, 1154,\, 2786,\, 6726, \ldots, Q_n, \ldots$$

is defined by the recurrence relation

$$Q_n = 2\,Q_{n-1} + Q_{n-2}\,,\;\; (n \geq 2),$$

with $Q_0 = 2, Q_1 = 1$.
Also, the sequence of modified Pell numbers

$$1, 3, 7, 17, 41, 99, 329, 577, 1393, 3363, \ldots, q_n, \ldots$$

is defined by the recurrence relation

$$q_n = 2q_{n-1} + q_{n-2}, \quad (n \geq 2),$$

with $q_0 = 1, q_1 = 1$.
Furthermore, we can see the matrix representations of Pell and Pell-Lucas numbers in [1]-[3],[5], [8]. In 2018, Catarino defined bicomplex k-Pell quaternions in [10].

Also, for Pell, Pell-Lucas and modified Pell numbers the following properties hold:[6],[7],[8],[9]

$$P_m P_{n+1} + P_{m-1} P_n = P_{m+n},$$

$$P_m P_{n+1} - P_{m+1} P_n = (-1)^n P_{m-n},$$

$$P_m P_n - P_{m+r} P_{n-r} = (-1)^{n-r} P_{m+r-n} P_r,$$

$$Q_m Q_n - Q_{m+r} Q_{n-r} = 8(-1)^{n-r+1} P_{m+r-n} P_r,$$

$$P_{n-1} P_{n+1} - P_n^2 = (-1)^n,$$

$$P_n^2 + P_{n+1}^2 = P_{2n+1},$$

$$P_{n+1}^2 - P_{n-1}^2 = 2P_{2n},$$

$$2P_{n+1} P_n - 2P_n^2 = P_{2n},$$

$$P_n^2 + P_{n+3}^2 = 5P_{2n+3},$$

$$P_{2n+1} + P_{2n} = 2P_{n+1}^2 - 2P_n^2 - (-1)^n,$$

$$P_n^2 + P_{n-1} P_{n+1} = \frac{Q_n^2}{4},$$

$$P_{n+1} + P_{n-1} = Q_n,$$

$$P_n Q_n = P_{2n},$$

$$Q_n = 2q_n,$$

$$P_{n+1} - P_n = q_n,$$

$$P_{n+1} + P_n = q_{n+1},$$

and for nega Pell and pell-Lucas numbers the following properties hold,

$$P_{-n} = (-1)^{n+1} P_n,$$

$$Q_{-n} = (-1)^n Q_n.$$

In this paper, the bicomplex Pell and bicomplex Pell-Lucas numbers will be defined. The aim of this work is to present in a unified manner a variety of algebraic properties of both the bicomplex numbers as well as the bicomplex Pell and Pell-Lucas numbers and the negabicomplex Pell and Pell-Lucas numbers. In particular, using three types of conjugations, all the properties established for bicomplex numbers are also given for the bicomplex Pell and Pell-Lucas numbers. In addition, d'Ocagne's identity, Binet's formula, Cassini's identity and Catalan's identity for these numbers are given.

2. The bicomplex Pell and Pell-Lucas numbers

The bicomplex Pell and Pell-Lucas numbers BP_n and BPL_n are defined by the basis $\{1, i, j, ij\}$ as follows

$$\mathbb{C}_2^P = \{BP_n = P_n + iP_{n+1} + jP_{n+2} + ijP_{n+3} \,|\, P_n, \; n-th\,Pell\,number, n = 0,1,...\}. \tag{2.1}$$

and

$$\mathbb{C}_2^{PL} = \{BPL_n = Q_n + iQ_{n+1} + jQ_{n+2} + ijQ_{n+3} \,|\, Q_n, \; n-th\,Pell-Lucas\,number,, n = 0,1,...\} \tag{2.2}$$

where i, j and ij satisfy the conditions

$$i^2 = -1, \; j^2 = -1, \; ij = ji.$$

The bicomplex Pell and bicomplex Pell-Lucas numbers starting from $n = 0$, can be written respectively as;

$BP_0 = 0 + 1i + 2j + 5ij$, $BP_1 = 1 + 2i + 5j + 12ij$, $BP_2 = 2 + 5i + 12j + 29ij, \ldots$

$BPL_0 = 2 + 2i + 6j + 14ij$, $BPL_1 = 2 + 6i + 14j + 34ij$,
$BPL_2 = 6 + 14i + 34j + 82ij, \ldots$

Let BP_n and BP_m be two bicomplex Pell numbers such that

$$BP_n = P_n + iP_{n+1} + jP_{n+2} + ijP_{n+3}$$

and

$$BP_m = P_m + iP_{m+1} + jP_{m+2} + ijP_{m+3}.$$

Then, the addition and subtraction of these numbers are given by

$$\begin{aligned} BP_n \pm BP_m = \; & (P_n + iP_{n+1} + jP_{n+2} + ijP_{n+3}) \\ & \pm (P_m + iP_{m+1} + jP_{m+2} + ijP_{m+3}) \\ = \; & (P_n \pm P_m) + i(P_{n+1} \pm P_{m+1}) + j(P_{n+2} \pm P_{m+2}) \\ & + ij(P_{n+3} \pm P_{m+3}). \end{aligned}$$

The multiplication of a bicomplex Pell number by the real scalar λ is defined as

$$\lambda BP_n = \lambda P_n + i\lambda P_{n+1} + j\lambda P_{n+2} + ij\lambda P_{n+3}.$$

The multiplication of two bicomplex Pell numbers is defined by

$$\begin{aligned} BP_n \times BP_m = \; & (P_n + iP_{n+1} + jP_{n+2} + ijP_{n+3}) \\ & (P_m + iP_{m+1} + jP_{m+2} + ijP_{m+3}) \\ = \; & (P_nP_m - P_{n+1}P_{m+1} - P_{n+2}P_{m+2} + P_{n+3}P_{m+3}) \\ & + i(P_nP_{m+1} + P_{n+1}P_m - P_{n+2}P_{m+3} - P_{n+3}P_{m+2}) \\ & + j(P_nP_{m+2} + P_{n+2}P_m - P_{n+1}P_{m+3} - P_{n+3}P_{m+1}) \\ & + ij(P_nP_{m+3} + P_{n+3}P_m + P_{n+1}P_{m+2} + P_{n+2}P_{m+1}) \\ = \; & BP_m \times BP_n. \end{aligned}$$

The conjugation of the bicomplex Pell numbers is defined in three different ways as follows

$$(BP_n)_i^* = P_n - iP_{n+1} + jP_{n+2} - ijP_{n+3}, \tag{2.3}$$

$$(BP_n)_j^* = P_n + iP_{n+1} - jP_{n+2} - ijP_{n+3}, \tag{2.4}$$

$$(BP_n)^*_{ij} = P_n - iP_{n+1} - jP_{n+2} + ijP_{n+3}. \tag{2.5}$$

Theorem 2.1. *Let BP_n and BP_m be two bicomplex Pell numbers. In this case, we can give the following relations between the conjugates of these numbers:*

$$\begin{aligned}
(BP_n \times BP_m)^*_i = &\quad (BP_m)^*_i \times (BP_n)^*_i = (BP_n)^*_i \times (BP_m)^*_i,\\
(BP_n \times BP_m)^*_j = &\quad (BP_m)^*_j \times (BP_n)^*_j = (BP_n)^*_j \times (BP_m)^*_j,\\
(BP_n \times BP_m)^*_{ij} = &\quad (BP_m)^*_{ij} \times (BP_n)^*_{ij} = (BP_n)^*_{ij} \times (BP_m)^*_{ij}.
\end{aligned}$$

Proof. It can be proved easily by using (2.3)-(2.5). □

In the following theorem, some properties related to the conjugations of the bicomplex Pell numbers are given.

Theorem 2.2. *Let $(BP_n)^*_i$, $(BP_n)^*_j$ and $(BP_n)^*_{ij}$ be three kinds of conjugation of the bicomplex Pell numbers. The following relations hold:*

$$BP_n \times (BP_n)^*_i = \quad 2(-Q_{2n+3} + jP_{2n+3}), \tag{2.6}$$

$$\begin{aligned}BP_n \times (BP_n)^*_j = &\quad (P_n^2 - P_{n+1}^2 + P_{n+2}^2 - P_{n+3}^2)\\ &\quad +4i(P_{2n+3} + P_nP_{n+1}),\end{aligned} \tag{2.7}$$

$$BP_n \times (BP_n)^*_{ij} = \quad 6P_{2n+3} + 4ij(-1)^{n+1}, \tag{2.8}$$

$$BP_n \times (BP_n)^*_i + BP_{n-1} \times (BP_{n-1})^*_i = \quad -2(8P_{2n+2} + jQ_{2n+2}), \tag{2.9}$$

$$BP_n \times (BP_n)^*_j + BP_{n-1} \times (BP_{n-1})^*_j = \quad 12(-P_{2n+2} + iP_{2n+2}), \tag{2.10}$$

$$BP_n \times (BP_n)^*_{ij} + BP_{n-1} \times (BP_{n-1})^*_{ij} = \quad 6Q_{2n+2}. \tag{2.11}$$

Proof. (2.6): Using (2.1) and (2.3) we get,

$$\begin{aligned}
BP_n \times (BP_n)^*_i = &\quad (P_n^2 + P_{n+1}^2 - P_{n+2}^2 - P_{n+3}^2)\\
&\quad +2j(P_nP_{n+2} + P_{n+1}P_{n+3})\\
= &\quad P_{2n+1} - P_{2n+5} + 2jP_{2n+3}\\
= &\quad 2(-Q_{2n+3} + jP_{2n+3}).
\end{aligned}$$

(2.7): Using (2.1) and (2.4) we get,

$$\begin{aligned}
BP_n \times (BP_n)^*_j = &\quad (P_n^2 - P_{n+1}^2 + P_{n+2}^2 - P_{n+3}^2)\\
&\quad +2i(P_nP_{n+1} + P_{n+2}P_{n+3})\\
= &\quad (P_n^2 - P_{n+1}^2 + P_{n+2}^2 - P_{n+3}^2)\\
&\quad +4i(P_{2n+3} + P_nP_{n+1}).
\end{aligned}$$

(2.8): Using (2.1) and (2.5) we get,

$$\begin{aligned}
BP_n \times (BP_n)^*_{ij} = &\quad (P_n^2 + P_{n+1}^2 + P_{n+2}^2 + P_{n+3}^2)\\
&\quad +2ij(P_nP_{n+3} - P_{n+1}P_{n+2})\\
= &\quad (P_{2n+1} + P_{2n+5}) + 4ij(-1)^{n+1}\\
= &\quad 6P_{2n+3} + 4ij(-1)^{n+1}.
\end{aligned}$$

(2.9): Using (2.6) we get,

$$\begin{aligned} BP_n\times(BP_n)_i^*+BP_{n-1}\times(BP_{n-1})_i^* = & -2\,[\,(Q_{2n+3}+Q_{2n+1}) \\ & -j\,(P_{2n+3}+P_{2n+1})\,] \\ = & -2\,(\,8\,P_{2n+2}-j\,Q_{2n+2}\,). \end{aligned}$$

(2.10): Using (2.7) we get,

$$\begin{aligned} BP_n\times(BP_n)_j^*+BP_{n-1}\times(BP_{n-1})_j^* = & (P_{n-1}^2-P_{n+3}^2) \\ & +4\,i\,(P_n\,Q_n+Q_{2n+2}) \\ = & -12\,P_{2n+2}+4\,i\,(3\,P_{2n+2}) \\ = & -12\,(\,P_{2n+2}-i\,P_{2n+2}\,). \end{aligned}$$

(2.11): Using (2.8) we get,

$$\begin{aligned} BP_n\times(BP_n)_{ij}^*+BP_{n-1}\times(BP_{n-1})_{ij}^* = & \;6\,(\,P_{2n+3}+P_{2n+1}\,) \\ & +4\,i\,j\,[(-1)^{n+1}+(-1)^n] \\ = & \;6\,Q_{2n+2}. \end{aligned}$$

□

Therefore, the norm of the bicomplex Pell number BP_n is defined in three different ways as follows

$$N_{BP_n i} = \|BP_n\times BP_{ni}^*\| = \sqrt{2\,|-Q_{2n+3}+j\,P_{2n+3}\,|},$$

$$\begin{aligned} N_{BP_n j} = & \|BP_n\times BP_{nj}^*\| \\ = & \sqrt{|(P_n^2-P_{n+1}^2+P_{n+2}^2-P_{n+3}^2)+4\,i\,(P_{2n+3}+P_n\,P_{n+1})|}, \end{aligned} \tag{2.12}$$

$$N_{BP_n ij} = \|BP_n\times BP_{nij}^*\| = \sqrt{|\,6\,Q_{2n+3}+4\,i\,j\,(-1)^{n+1}\,|}. \tag{2.13}$$

Theorem 2.3. *Let BP_n and BPL_n be the bicomplex Pell and bicomplex Pell-Lucas numbers, respectively.The following relations hold:*

$$\begin{aligned} BP_m\,BP_n+BP_{m+1}\,BP_{n+1} = & \;4\,(\,Q_{m+n+4}-i\,Q_{m+n+4} \\ & -j\,P_{m+n+4}+i\,j\,P_{m+n+4}\,), \end{aligned} \tag{2.14}$$

$$\begin{aligned} (BP_n)^2 = & \;4\,P_{2n+3}-4\,i\,P_{2n+3}+2\,j\,(P_{2n+1}-6\,P_{n+1}^2) \\ & +2\,i\,j\,(\,6\,P_n\,P_{n+1}+2\,P_{2n+1}\,), \end{aligned} \tag{2.15}$$

$$\begin{aligned} (BP_n)^2+(BP_{n+1})^2 = & \;4\,(\,Q_{2n+4}-i\,Q_{2n+4}-j\,P_{2n+4} \\ & +i\,j\,P_{2n+4}\,), \end{aligned} \tag{2.16}$$

$$\begin{aligned} (BP_{n+1})^2-(BP_{n-1})^2 = & -4\,(\,P_{2n+1}+2\,i\,Q_{2n+3}+2\,j\,P_{2n+3} \\ & +2\,i\,j\,P_{2n+3}\,) \end{aligned} \tag{2.17}$$

$$BP_n-i\,BP_{n+1}+j\,BP_{n+2}-i\,j\,BP_{n+3} = 4\,(-4\,P_{n+3}+j\,q_{n+3}\,), \tag{2.18}$$

$$\begin{aligned} BP_n-i\,BP_{n+1}-j\,BP_{n+2}-i\,j\,BP_{n+3} = & \;2\,(\,q_{n+1}-P_{n+5}+i\,P_{n+5} \\ & +j\,P_{n+4}-i\,j\,P_{n+3}\,). \end{aligned} \tag{2.19}$$

Proof. (2.14): By the equation (2.1) we get,

$$\begin{aligned} BP_m\,BP_n+BP_{m+1}\,BP_{n+1} = &\ (P_{m+n+1}-P_{m+n+3}-P_{m+n+5} \\ &\ +P_{m+n+7}) \\ &\ +2\,i\,(P_{m+n+2}-P_{m+n+6}) \\ &\ +2\,j\,(P_{m+n+3}-P_{m+n+5}) \\ &\ +2\,i\,j\,(2\,P_{m+n+4}) \\ = &\ 4\,(Q_{m+n+4}-i\,Q_{m+n+4}-j\,P_{m+n+4} \\ &\ +i\,j\,P_{m+n+4}). \end{aligned}$$

(2.15): By the equation (2.1) we get,

$$\begin{aligned} (BP_n)^2 = &\ (P_n^2-P_{n+1}^2-P_{n+2}^2+P_{n+3}^2)+2\,i\,(P_n\,P_{n+1}-P_{n+2}\,P_{n+3}) \\ &\ +2\,j\,(P_n\,P_{n+2}-P_{n+1}\,P_{n+3})+2\,i\,j\,(P_n\,P_{n+3}+P_{n+1}\,P_{n+2}) \\ = &\ 4\,P_{2n+3}-4\,i\,P_{2n+3}+2\,j\,(P_{2n+1}-6\,P_{n+1}^2) \\ &\ +2\,i\,j\,(6\,P_n\,P_{n+1}+2\,P_{2n+1}). \end{aligned}$$

(2.16): By the equations (2.1) and (2.14) we get,

$$\begin{aligned} (BP_n)^2+(BP_{n+1})^2 = &\ (P_n^2-P_{n+2}^2+P_{n+4}^2-P_{n+2}^2) \\ &\ +2\,i\,(P_{2n+2}-P_{2n+6})+2\,j\,(P_{2n+3}-P_{2n+5}) \\ &\ +2\,i\,j\,(2\,P_{2n+4}) \\ = &\ 4\,(Q_{2n+4}-i\,Q_{2n+4}-j\,P_{2n+4}+i\,j\,P_{2n+4}). \end{aligned}$$

(2.17) By the equations (2.1) and (2.14) we get,

$$\begin{aligned} (BP_{n+1})^2-(BP_{n-1})^2 = &\ (P_{n+1}^2-P_{n-1}^2+P_n^2-P_{n+2}^2) \\ &\ +2\,i\,[2\,(P_{2n+1}-P_{2n+5})] \\ &\ +2\,j\,(P_{2n+3}-5P_{2n+3}) \\ &\ +2\,i\,j\,[4\,(q_{2n+2}+P_{2n+2})] \\ = &\ 2\,(P_{2n}-P_{2n+2})+2\,i\,(-4\,Q_{2n+3}) \\ &\ +2\,j\,(-4\,P_{2n+3})+2\,i\,j\,(4\,P_{2n+3}) \\ = &\ -4\,(P_{2n+1}+2\,i\,Q_{2n+3}+2\,j\,P_{2n+3} \\ &\ +2\,i\,j\,P_{2n+3}) \end{aligned}$$

(2.18): By the equation (2.1) we get,

$$\begin{aligned} BP_n-i\,BP_{n+1}-j\,BP_{n+2}-i\,j\,BP_{n+3} = &\ (P_n+P_{n+2}+P_{n+4}-P_{n+6}) \\ &\ +2\,i\,(P_{n+5})+2\,j\,(P_{n+4}) \\ &\ -2\,i\,j\,(P_{n+3}) \\ = &\ -(4\,P_{n+1}+P_n)+2\,i\,P_{n+5} \\ &\ +2\,j\,P_{n+4}-2\,i\,j\,P_{n+3}. \end{aligned}$$

(2.19): By the equation (2.1) we get,

$$\begin{aligned} BP_n-i\,BP_{n+1}-j\,BP_{n+2}-i\,j\,BP_{n+3} = &\ (P_n+P_{n+2}+P_{n+4}-P_{n+6}) \\ &\ +2\,i\,(P_{n+5})+2\,j\,(P_{n+4}) \\ &\ -2\,i\,j\,(P_{n+3}) \\ = &\ -(4\,P_{n+1}+P_n)+2\,i\,P_{n+5} \\ &\ +2\,j\,P_{n+4}-2\,i\,j\,P_{n+3}. \end{aligned}$$

. □

Theorem 2.4. *(d'Ocagne's identity). For $n,m \geq 0$ d'Ocagne's identity for bicomplex Pell numbers BP_n and BP_m is given by*

$$BP_m\,BP_{n+1}-BP_{m+1}\,BP_n = 12\,(-1)^n\,P_{m-n}\,(j+i\,j). \tag{2.20}$$

Proof. (2.20): By the equation (2.1) we get,

$$\begin{aligned} BP_m BP_{n+1} - BP_{m+1} BP_n = & \ (-1)^n P_{m-n}(0) \\ & +i(-1)^n(P_{m-n-1}(0) \\ & +2j(-1)^n(P_{m-n-2}+P_{m-n+2}) \\ & +ij(-1)^n[(-P_{m-n-3}+P_{m-n+3} \\ & \quad +P_{m-n-1}-P_{m-n+1})] \\ = & \ 2j(-1)^n(6P_{m-n}) \\ & +ij(-1)^n 6(P_{m-n-1}-P_{m-n+1}) \\ = & \ 12(-1)^n P_{m-n}(j++ij). \end{aligned}$$

□

Theorem 2.5. *Let BP_n and BPL_n be the bicomplex Pell number and the bicomplex Pell-Lucas numbers respectively. The following relations are satisfied*

$$BP_{n+1} + BP_{n-1} = BPL_n, \tag{2.21}$$

$$BP_{n+1} - BP_{n-1} = 2BP_n, \tag{2.22}$$

$$BP_{n+2} + BP_{n-2} = 6BP_n. \tag{2.23}$$

$$BP_{n+2} - BP_{n-2} = 2BPL_n, \tag{2.24}$$

$$BP_{n+1} + BP_n = \frac{1}{2} BPL_{n+1}, \tag{2.25}$$

$$BP_{n+1} - BP_n = \frac{1}{2} BPL_n, \tag{2.26}$$

$$BPL_{n+1} + BPL_{n-1} = 4BP_n, \tag{2.27}$$

$$BPL_{n+1} - BPL_{n-1} = 2BPL_n, \tag{2.28}$$

$$BPL_{n+2} + BPL_{n-2} = 6BPL_n, \tag{2.29}$$

$$BPL_{n+2} - BPL_{n-2} = 8BP_n, \tag{2.30}$$

$$BPL_{n+1} + BPL_n = 4BP_{n+1}, \tag{2.31}$$

$$BPL_{n+1} - BPL_n = 4BP_n. \tag{2.32}$$

Proof. (2.21):By the equation (2.1) we get,

$$\begin{aligned} BP_{n+1}+BP_{n-1} = & \ (P_{n+1}+P_{n-1})+i\,(P_{n+2}+P_n) \\ & +j\,(P_{n+3}+P_{n+1})+i\,j\,(P_{n+4}+P_{n+2}) \\ = & \ (Q_n+i\,Q_{n+1}+j\,Q_{n+2}+i\,j\,Q_{n+3}) \\ = & \ BPL_n\,, \end{aligned}$$

(2.22): By the equation (2.1) we get,

$$\begin{aligned} BP_{n+1}-BP_{n-1} = & \ (P_{n+1}-P_{n-1})+i\,(P_{n+2}-P_n) \\ & +j\,(P_{n+3}-P_{n+1})+i\,j\,(P_{n+4}-P_{n+2}) \\ = & \ 2\,(P_n+i\,P_{n+1}+j\,P_{n+2}+i\,j\,P_{n+3}) \\ = & \ 2\,BP_n. \end{aligned}$$

(2.23): By the equation (2.1) we get,

$$\begin{aligned} BP_{n+2}+BP_{n-2} = & \ (P_{n+2}+P_{n-2})+i\,(P_{n+3}+P_{n-1}) \\ & +j\,(P_{n+4}+P_n)+i\,j\,(P_{n+5}+P_{n+1}) \\ = & \ 6\,(P_n+i\,P_{n+1}+j\,P_{n+2}+i\,j\,P_{n+3}) \\ = & \ 6\,BP_n. \end{aligned}$$

(2.24): By the equation (2.1) we get,

$$\begin{aligned} BP_{n+2}-BP_{n-2} = & \ (P_{n+2}-P_{n-2})+i\,(P_{n+3}-P_{n-1}) \\ & +j\,(P_{n+4}-P_n)+i\,j\,(P_{n+5}-P_{n+1}) \\ = & \ 2\,(Q_n+i\,Q_{n+1}+j\,Q_{n+2}+i\,j\,Q_{n+3}) \\ = & \ 2\,BPL_n\,. \end{aligned}$$

(2.25): By the equation (2.1) we get,

$$\begin{aligned} BP_{n+1}+BP_n = & \ (P_{n+1}+P_n)+i\,(P_{n+2}+P_{n+1}) \\ & +j\,(P_{n+3}+P_{n+2})+i\,j\,(P_{n+4}+P_{n+3}) \\ = & \ (q_{n+1}+i\,q_{n+2}+j\,q_{n+3}+i\,j\,q_{n+4}) \\ = & \ \tfrac{1}{2}\,(Q_{n+1}+i\,Q_{n+2}+j\,Q_{n+3}+i\,j\,Q_{n+4}) \\ = & \ \tfrac{1}{2}\,BPL_{n+1} \end{aligned}$$

where the property (1.17) of the modified Pell number is used.
(2.26): By the equation (2.1) we get,

$$\begin{aligned} BP_{n+1}-BP_n = & \ (P_{n+1}-P_n)+i\,(P_{n+2}-P_{n+1}) \\ & +j\,(P_{n+3}-P_{n+2})+i\,j\,(P_{n+4}-P_{n+3}) \\ = & \ (q_n+i\,q_{n+1}+j\,q_{n+2}+i\,j\,q_{n+3}) \\ = & \ \tfrac{1}{2}\,(Q_n+i\,Q_{n+1}+j\,Q_{n+2}+i\,j\,Q_{n+3}) \\ = & \ \tfrac{1}{2}\,BPL_n \end{aligned}$$

where the property (1.17) of the modified Pell number is used.
(2.27): By the equation (2.2) we get,

$$\begin{aligned} BPL_{n+1}+BPL_{n-1} = & \ (Q_{n+1}+Q_{n-1})+i\,(Q_{n+2}+Q_n) \\ & +j\,(Q_{n+3}+Q_{n+1})+i\,j\,(Q_{n+4}+Q_{n+2}) \\ = & \ 4\,(P_n+iP_{n+1}+j\,P_{n+2}+i\,j\,P_{n+3}) \\ = & \ 4\,BP_n. \end{aligned}$$

(2.28): By the equation (2.2) we get,

$$\begin{aligned} BPL_{n+1}-BPL_{n-1} = & \ (Q_{n+1}-Q_{n-1})+i\,(Q_{n+2}-Q_n) \\ & +j\,(Q_{n+3}-Q_{n+1})+i\,j\,(Q_{n+4}-Q_{n+2}) \\ = & \ 2\,(Q_n+i\,Q_{n+1}+j\,Q_{n+2}+i\,j\,Q_{n+3}) \\ = & \ 2\,BPL_n \end{aligned}$$

(2.29): By the equation (2.2) we get,

$$\begin{aligned} BPL_{n+2}+BPL_{n-2} = &\ (Q_{n+2}+Q_{n-2})+i\,(Q_{n+3}+Q_{n-1}) \\ &+j\,(Q_{n+4}+Q_n)+i\,j\,(Q_{n+5}+Q_{n+1}) \\ = &\ 6\,(Q_n+i\,Q_{n+1}+j\,Q_{n+2}+i\,j\,Q_{n+3}) \\ = &\ 6\,BPL_n. \end{aligned}$$

(2.30): By the equation (2.2) we get,

$$\begin{aligned} BPL_{n+2}-BPL_{n-2} = &\ (Q_{n+2}-Q_{n-2})+i\,(Q_{n+3}-Q_{n-1}) \\ &+j\,(Q_{n+4}-Q_n)+i\,j\,(Q_{n+5}-Q_{n+1}) \\ = &\ 8\,(P_n+i\,P_{n+1}+j\,P_{n+2}+i\,j\,P_{n+3}) \\ = &\ 8\,BP_n. \end{aligned}$$

(2.31): By the equation (2.2) we get,

$$\begin{aligned} BPL_{n+1}+BPL_n = &\ (Q_{n+1}+Q_n)+i\,(Q_{n+2}+Q_{n+1}) \\ &+j\,(Q_{n+3}+Q_{n+2})+i\,j\,(Q_{n+4}+Q_{n+3}) \\ = &\ 4\,P_{n+1}+i\,P_{n+2}+j\,P_{n+3}+i\,j\,P_{n+4} \\ = &\ 4\,BP_{n+1}. \end{aligned}$$

(2.32): By the equation (2.2) we get,

$$\begin{aligned} BPL_{n+1}-BPL_n = &\ (Q_{n+1}-Q_n)+i\,(Q_{n+2}-Q_{n+1}) \\ &+j\,(Q_{n+3}-Q_{n+2})+i\,j\,(Q_{n+4}-Q_{n+3}) \\ = &\ 4\,P_n+i\,P_{n+1}+j\,P_{n+2}+i\,j\,P_{n+3} \\ = &\ 4\,BP_n. \end{aligned}$$

□

Theorem 2.6. *If BP_n and BPL_n are bicomplex Pell and bicomplex Pell-Lucas numbers, respectively. For $n \geq 0$, the identities of negabicomplex Pell and negabicomplex Pell-Lucas numbers are*

$$BP_{-n} = \ (-1)^{n+1}\,BP_n+(-1)^n\,Q_n\,(\,i+2\,j+5\,i\,j\,). \tag{2.33}$$

and

$$BPL_{-n} = \ (-1)^n\,BPL_n+8\,(-1)^{n+1}\,P_n\,(\,i+2\,j+5\,i\,j\,). \tag{2.34}$$

Proof. (2.33): Using the identity of negapell numbers $P_{-n}=(-1)^{n+1}\,P_n$ we get

$$\begin{aligned} BP_{-n} = &\ P_{-n}+i\,P_{-n+1}+j\,P_{-n+2}+i\,j\,P_{-n+3} \\ = &\ P_{-n}+i\,P_{-(n-1)}+j\,P_{-(n-2)}+i\,j\,P_{-(n-3)} \\ = &\ (-1)^{n+1}\,P_n+i\,(-1)^n\,P_{n-1}+j\,(-1)^{n-1}\,P_{n-2} \\ &+i\,j\,(-1)^{n-2}\,P_{n-3} \\ = &\ (-1)^{n+1}\,(P_n+i\,P_{n+1}+j\,P_{n+2}+i\,j\,P_{n+3}) \\ &-i\,(-1)^{n+1}\,P_{n+1}-j\,(-1)^{n+1}\,P_{n+2}-i\,j\,(-1)^{n+1}\,P_{n+3} \\ &+i\,(-1)^n\,P_{n-1}+j\,(-1)^{n+1}\,P_{n-2}+i\,j\,(-1)^n\,P_{n-3} \\ = &\ (-1)^{n+1}\,BP_n+(-1)^n\,(P_{n+1}+P_{n-1})\,i \\ &+(-1)^n\,(P_{n+2}-P_{n-2})\,j+(-1)^n\,(P_{n+3}+P_{n-3})\,i\,j \\ = &\ (-1)^{n+1}\,BP_n+(-1)^n\,Q_n\,(\,i+2\,j+5\,i\,j\,) \end{aligned}$$

(2.34): Using the identity of negapell-Lucas numbers $Q_{-n}=(-1)^n\,Q_n$ we get

$$
\begin{aligned}
BPL_{-n} = & \quad Q_{-n} + i\,Q_{-n+1} + j\,Q_{-n+2} + i\,j\,Q_{-n+3} \\
= & \quad Q_{-n} + i\,Q_{-(n-1)} + j\,Q_{-(n-2)} + i\,j\,Q_{-(n-3)} \\
= & \quad (-1)^n\,Q_n + i\,(-1)^{n-1}\,Q_{n-1} + j\,(-1)^{n-2}\,Q_{n-2} \\
& \qquad + i\,j\,(-1)^{n-3}\,Q_{n-3} \\
= & \quad (-1)^{n+1}\,(Q_n + i\,Q_{n+1} + j\,Q_{n+2} + i\,j\,Q_{n+3}) \\
& \qquad - i\,(-1)^n\,Q_{n+1} - j\,(-1)^n\,Q_{n+2} \\
& \qquad - i\,j\,(-1)^n\,Q_{n+3} \\
& \qquad + i\,(-1)^{n-1}\,Q_{n-1} + j\,(-1)^n\,Q_{n-2} \\
& \qquad + i\,j\,(-1)^{n-1}\,Q_{n-3} \\
= & \quad (-1)^{n+1}\,BPL_n + (-1)^{n+1}\,(Q_{n+1} + Q_{n-1})\,i \\
& \qquad + (-1)^{n+1}\,(Q_{n+2} - Q_{n-2})\,j \\
& \qquad + (-1)^{n+1}\,(Q_{n+3} + Q_{n-3})\,i\,j \\
= & \quad (-1)^n\,BPL_n + 8\,(-1)^{n+1}\,P_n\,(i + 2\,j + 5\,i\,j)
\end{aligned}
$$

□

Theorem 2.7. *Binet's Formula. Let BP_n and BPL_n be the bicomplex Pell and bicomplex Pell-Lucas numbers respectively. For $n \geq 1$, Binet's formula for these numbers are as follows:*

$$BP_n = \frac{1}{\alpha - \beta}(\hat{\alpha}\,\alpha^n - \hat{\beta}\,\beta^n) \tag{2.35}$$

and

$$BPL_n = \hat{\alpha}\,\alpha^n + \hat{\beta}\,\beta^n \tag{2.36}$$

where $\hat{\alpha} = 1 + i\,\alpha + j\,\alpha^2 + i\,j\,\alpha^3$, $\alpha = 1 + \sqrt{2}$ *and* $\hat{\beta} = 1 + i\,\beta + j\,\beta^2 + i\,j\,\beta^3$, $\beta = 1 - \sqrt{2}$.

Proof. (2.35):

$$
\begin{aligned}
BP_n = & \quad P_n + i\,P_{n+1} + j\,P_{n+2} + i\,j\,P_{n+3} \\
= & \quad \tfrac{\alpha^n - \beta^n}{\alpha - \beta} + i\,\tfrac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} + j\,\tfrac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta} + i\,j\,\tfrac{\alpha^{n+3} - \beta^{n+3}}{\alpha - \beta} \\
= & \quad \tfrac{\alpha^n\,(1 + i\,\alpha + j\,\alpha^2 + i\,j\,\alpha^3) - \beta^n\,(1 + i\,\beta + j\,\beta^2 + i\,j\,\beta^3)}{\alpha - \beta} \\
= & \quad \tfrac{\hat{\alpha}\,\alpha^n - \hat{\beta}\,\beta^n}{\alpha - \beta}
\end{aligned}
$$

and (2.36):

$$
\begin{aligned}
BPL_n = & \quad Q_n + i\,Q_{n+1} + j\,Q_{n+2} + i\,j\,Q_{n+3} \\
= & \quad \alpha^n + \beta^n + i\,(\alpha^{n+1} + \beta^{n+1}) + j\,(\alpha^{n+2} + \beta^{n+2}) + i\,j\,(\alpha^{n+3} + \beta^{n+3}) \\
= & \quad \alpha^n\,(1 + i\,\alpha + j\,\alpha^2 + i\,j\,\alpha^3) + \beta^n\,(1 + i\,\beta + j\,\beta^2 + i\,j\,\beta^3) \\
= & \quad \hat{\alpha}\,\alpha^n + \hat{\beta}\,\beta^n.
\end{aligned}
$$

Binet's formula of the bicomplex Pell number is the same as Binet's formula of the Pell number [7]. □

Theorem 2.8. *Cassini's Identity Let BP_n and BPL_n be the bicomplex Pell and bicomplex Pell-Lucas numbers, respectively. For $n \geq 1$, Cassini's identities for BP_n and BPL_n are as follows:*

$$BP_{n-1}\,BP_{n+1} - BP_n^2 = 12\,(-1)^n\,(j + i\,j) \tag{2.37}$$

and

$$BPL_{n-1}\,BPL_{n+1} - BPL_n^2 = 8.12\,(-1)^{n+1}\,(j + i\,j). \tag{2.38}$$

Proof. (2.37): Using (2.1) we get

$$\begin{aligned}
BP_{n-1}\,BP_{n+1}-(BP_n)^2 = \;& (P_{n-1}+i\,P_n+j\,P_{n+1}+i\,j\,P_{n+2}) \\
& (P_{n+1}+i\,P_{n+2}+j\,P_{n+3}+i\,j\,P_{n+4}) \\
& -[P_n+i\,P_{n+1}+j\,P_{n+2}+i\,j\,P_{n+3}]^2 \\
= \;& [\,(P_{n-1}P_{n+1}-P_n^2) \\
& \quad -(P_nP_{n+2}+P_{n+1}^2) \\
& \quad -(P_{n+1}P_{n+3}-P_{n+2}^2) \\
& \quad +(P_{n+2}P_{n+4}-P_{n+3}^2)\,] \\
& +i\,[\,(P_{n+2}P_{n-1}-P_{n+1}P_n) \\
& \quad -(P_{n+4}P_{n+1}-P_{n+3}P_{n+2})\,] \\
& +j\,[\,(P_{n+1}P_{n+1}-P_nP_{n+2}) \\
& \quad -(P_{n+2}P_{n+2}-P_{n+1}P_{n+3}) \\
& \quad +(P_{n+3}P_{n-1}-P_{n+2}P_n) \\
& \quad -(P_{n+4}P_n-P_{n+3}P_{n+1})\,] \\
& +i\,j\,(P_{n+4}P_{n-1}-P_{n+3}P_n) \\
= \;& 12\,(-1)^n\,(\,j+i\,j).
\end{aligned}$$

(2.38): Using (2.2) we get

$$\begin{aligned}
BPL_{n-1}\,BPL_{n+1}-(BPL_n)^2 = \;& (Q_{n-1}+i\,Q_n+j\,Q_{n+1}+i\,j\,Q_{n+2}) \\
& (Q_{n+1}+i\,Q_{n+2}+j\,Q_{n+3}+i\,j\,Q_{n+4}) \\
& -[Q_n+i\,Q_{n+1}+j\,Q_{n+2}+i\,j\,Q_{n+3}]^2 \\
= \;& [\,(Q_{n-1}Q_{n+1}-Q_n^2) \\
& \quad +(Q_{n+1}^2-Q_{n+2}Q_n) \\
& \quad +(Q_{n+2}^2-Q_{n+3}Q_{n+1}) \\
& \quad +(Q_{n+4}Q_{n+2}-Q_{n+3}^2)\,] \\
& +i\,[(Q_{n+2}Q_{n-1}-Q_{n+1}Q_n) \\
& \quad +(Q_{n+3}Q_{n+2}-Q_{n+4}Q_{n+1})\,] \\
& +j\,[\,(Q_{n+1}Q_{n+1}-Q_nQ_{n+2}) \\
& \quad +(Q_{n+1}Q_{n+3}-(Q_{n+2}Q_{n+2}) \\
& \quad +(Q_{n+3}Q_{n-1}-Q_{n+2}Q_n) \\
& \quad +(Q_{n+3}(Q_{n+1}-Q_{n+4}Q_n)\,] \\
& +i\,j\,(Q_{n+4}Q_{n-1}-Q_{n+3}Q_n) \\
= \;& 8.12\,(-1)^{n+1}\,(\,j+i\,j).
\end{aligned}$$

where the identities of the Pell and Pell-Lucas numbers $P_m\,P_{n+1}-P_{m+1}\,P_n=(-1)^n P_{m-n}$ and $Q_m\,Q_{n+1}-Q_{m+1}Q_n=8\,(-1)^{n+1}\,P_{m-n}$ are used. ☐

Theorem 2.9. *Catalan's Identity. Let BP_n and BPL_n be the bicomplex Pell and bicomplex Pell-Lucas numbers, respectively. For $n \geq 1$, Catalan's identities for BP_n and BPL_n are as follows*

$$(BP_n)^2-BP_{n+r}\,BP_{n-r}=12\,(-1)^{n-r}\,{P_r}^2\,(\,j+i\,j\,), \tag{2.39}$$

and

$$(BPL_n)^2-BPL_{n+r}\,BPL_{n-r}=8\,.\,12\,(-1)^{n-r}\,{P_r}^2\,(\,j+i\,j\,). \tag{2.40}$$

respectively.

Proof. (2.39): Using (2.1) we get

$$\begin{aligned}
BP_n^2 - BP_{n+r}\,BP_{n-r} = \; & [(P_n^2 - P_{n+r}\,P_{n-r}) \\
& -(P_{n+1}^2 - P_{n+r+1}\,P_{n-r+1}) \\
& -(P_{n+2}^2 - P_{n+r+2}\,P_{n-r+2}) \\
& +(P_{n+3}^2 - P_{n+r+3}\,P_{n-r+3})] \\
& +i[(P_n\,P_{n+1} - P_{n+r}\,P_{n-r+1}) \\
& -(P_{n+2}\,P_{n+3} - P_{n+r+2}\,P_{n-r+3}) \\
& +(P_{n+1}\,P_n - P_{n+r+1}\,P_{n-r}) \\
& -(P_{n+3}\,P_{n+2} - P_{n+r+3}\,P_{n-r+2})] \\
& +j[(P_n\,P_{n+2} - P_{n+r}\,P_{n-r+2}) \\
& -(P_{n+1}\,P_{n+3} - P_{n+r+1}\,P_{n-r+3}) \\
& +(P_{n+2}\,P_n - P_{n+r+2}\,P_{n-r}) \\
& -(P_{n+3}\,P_{n+1} - P_{n+r+3}\,P_{n-r+1})] \\
& +i\,j[(P_n\,P_{n+3} - P_{n+r}\,P_{n-r+3}) \\
& +(P_{n+1}\,P_{n+2} - P_{n+r+1}\,P_{n-r+2}) \\
& +(P_{n+3}\,P_n - P_{n+r+3}\,P_{n-r}) \\
& +(P_{n+2}\,P_{n+1} - P_{n+r+2}\,P_{n-r+1})] \\
= \; & (-1)^{n-r}\,P_r^2\,(0+0i+12\,j+12\,i\,j) \\
= \; & 12\,(-1)^{n-r}\,P_r^2\,(j+i\,j).
\end{aligned}$$

(2.40): Using (2.2) we get

$$\begin{aligned}
(BPL_n)^2 - BPL_{n+r}\,BPL_{n-r} = \; & [(Q_n^2 - Q_{n+r}\,Q_{n-r}) \\
& -(Q_{n+1}^2 - Q_{n+r+1}\,Q_{n-r+1}) \\
& -(Q_{n+2}^2 - Q_{n+r+2}\,Q_{n-r+2}) \\
& +(Q_{n+3}^2 - Q_{n+r+3}\,Q_{n-r+3})] \\
& +i[(Q_n\,Q_{n+1} - Q_{n+r}\,Q_{n-r+1}) \\
& -(Q_{n+2}\,Q_{n+3} - Q_{n+r+2}\,Q_{n-r+3}) \\
& +(Q_{n+1}\,Q_n - Q_{n+r+1}\,Q_{n-r}) \\
& -(Q_{n+3}\,Q_{n+2} - Q_{n+r+3}\,Q_{n-r+2})] \\
& +j[(Q_n\,Q_{n+2} - Q_{n+r}\,Q_{n-r+2}) \\
& -(Q_{n+1}\,Q_{n+3} - Q_{n+r+1}\,Q_{n-r+3}) \\
& +(Q_{n+2}\,Q_n - Q_{n+r+2}\,Q_{n-r}) \\
& -(Q_{n+3}\,Q_{n+1} - Q_{n+r+3}\,Q_{n-r+1})] \\
& +i\,j[(Q_n\,Q_{n+3} - Q_{n+r}\,Q_{n-r+3}) \\
& +(Q_{n+1}\,Q_{n+2} - Q_{n+r+1}\,Q_{n-r+2}) \\
& +(Q_{n+3}\,Q_n - Q_{n+r+3}\,Q_{n-r}) \\
& +(Q_{n+2}\,Q_{n+1} - Q_{n+r+2}\,Q_{n-r+1})] \\
= \; & 8\,(-1)^{n-r}\,P_r^2\,(0+0i+12\,j+12\,i\,j) \\
= \; & 8.12\,(-1)^{n-r}\,P_r^2\,(j+i\,j).
\end{aligned}$$

where the identities of the Pell and Pell-Lucas numbers are used as follows,

$$\begin{aligned}
P_mP_n - P_{m+r}P_{n-r} = \; & (-1)^{n-r}P_{m+r-n}\,P_r, \\
P_n\,P_n - P_{n-r}\,P_{n+r} = \; & (-1)^{n-r}\,P_r^2, \\
Q_mQ_n - Q_{m+r}Q_{n-r} = \; & (-1)^{n-r+1}P_{m+r-n}\,P_r, \\
Q_n\,Q_n - Q_{n-r}\,Q_{n+r} = \; & (-1)^{n-r+1}\,P_r^2.
\end{aligned}$$

□

3. Conclusion

In this study, a number of new algebraic results on bicomplex Pell and bicomplex Pell-Lucas numbers are derived. Also, negabicomplex Pell and negabicomplex Pell-Lucas numbers are given. Furthermore, d'Ocagne's identity, Binet's formula, Cassini's identity and Catalan's identity for these numbers are generated.

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