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On Growth and Approximation of Generalized Biaxially Symmetric Potentials on Parabolic-Convex Sets

Devendra Kumar[1,2]*

Abstract
The regular, real-valued solutions of the second-order elliptic partial differential equation

$$\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}+\frac{2\alpha+1}{x}\frac{\partial F}{\partial y}+\frac{2\beta+1}{y}\frac{\partial F}{\partial x}=0,\alpha,\beta>\frac{-1}{2}$$

are known as generalized bi-axially symmetric potentials (GBSP's). McCoy [1] has showed that the rate at which approximation error $E_{2n}^{\frac{p}{2n}}(F;D),(p\geq 2,D$ is parabolic-convex set) tends to zero depends on the order of $GBSP$ F and obtained a formula for finite order. If $GBSP$ F is an entire function of infinite order then above formula fails to give satisfactory information about the rate of decrease of $E_{2n}^{\frac{p}{2n}}(F;D)$. The purpose of the present work is to refine above result by using the concept of index-q. Also, the formula corresponding to q-order does not always hold for lower q-order. Therefore we have proved a result for lower q-order also, which have not been studied so far.

Keywords: Parabolic-convex set, Index-q,q-order, Lower q-order, Generalized bi-axially symmetric potentials and elliptic partial differential equation
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[1] *Department of Mathematics, Faculty of Sciences Al-Baha University,P.O.Box-1988, Alaqiq, Al-Baha-65431, Saudi Arabia, K.S.A..*
[2] *Department of Mathematics, [Research and Post Graduate Studies], M.M.H.College, Model Town, Ghaziabad-201001, U.P., India.*
***Corresponding author**: d_kumar001@rediffmail.com
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1. Introduction

The linear second order elliptic partial differential equation is given in the form

$$\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}+\frac{2\alpha+1}{x}\frac{\partial F}{\partial y}+\frac{2\beta+1}{y}\frac{\partial F}{\partial x}=0,\alpha,\beta>\frac{-1}{2}, \tag{1.1}$$

which are in x and y cf. Gilbert [2]. A polynomial of degree n which is even in x and y is said to be a $GBSP$ polynomial of degree n if it satisfies (1.1). A $GBSP$ F that is regular about origin can be expanded as

$$F(x,y)=\sum_{n=0}^{\infty}a_n R_n^{(\alpha,\beta)}(x,y),$$

where

$$R_n^{(\alpha,\beta)}(x,y)=(x^2+y^2)^n P_n^{(\alpha,\beta)}((x^2-y^2)/(x^2+y^2))/P_n^{(\alpha,\beta)}(1),n=0,1,2,\ldots$$

and $P_n^{(\alpha,\beta)}(t)$ are Jacobi polynomials. Various authors such as Srivastava [3], McCoy [4], Kumar and Basu [5], Kumar and Bishnoi [6], Harfaoui [7], Kumar [8], Kadiri and Harfaoui [9], Kasana and Kumar [10]-[12] and Kapoor and Nautiyal [13] studied the growth and L_p-approximation of regular real-valued solutions of certain elliptic partial differential equations but our results are different from these authors.

There are so many applications of the solutions of (1.1) in several areas of mathematical physics, for example, its solutions arise in the Maxwell system for the modelling of electric or magnetic n-poles, potential scattering, in quasi-stationary (time independent)diffusion processes and as the initial data for parabolic partial differential equations.

Let D be a certain open set that is symmetric about the origin with Jordan boundary. We define the p-norm on D as:

$$\|.\|_p = (\frac{1}{A}\int\int_D |.|^p dxdy)^{\frac{1}{p}}, p \in [1,\infty), \|.\|_\infty = \sup_D |.|, \|1\|_p = 1.$$

The space $L^p(D)$ of real-valued $GBSP$ given by (1.2) is regular and even on D with finite p-norm and the space $l^p(D)$ of associated functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \tag{1.2}$$

where

$$R_n^{(\alpha,\beta)}(z,0) = z^{2n}, n = 0,1,2,\ldots$$

is analytic on D with finite p-norm. McCoy [1] developed a pair of integral transforms that are one to one maps between the space $L^p(D)$ of real-valued $GBSP$ F and the space $l^p(D)$ of associated f as:

$$F(x,y) = K_{\alpha,\beta}(f) = \int_0^\pi \int_0^1 f(\tau) k_{\alpha,\beta}(t,s) dt ds,$$

$$\tau^2 = \tau^2(x,y,t,s) = x^2 - y^2t^2 + 2ixyt\cos s,$$

$$f(z) = K_{\alpha,\beta}^{-1}(F) = \int_{-1}^{+1} F(r,\xi,r(1-\xi^2)^{\frac{1}{2}} j_{\alpha,\beta}(zr^{-1},\xi) d\xi,$$

where

$$k_{\alpha,\beta}(t,s) = \nu_{\alpha,\beta}(1-t^2)^{\alpha-\beta-1} t^{2\beta+1} (\sin s)^{2\alpha}$$

and

$$j_{\alpha,\beta}(\tau,\xi) = \eta_{\alpha,\beta} \frac{(1-\tau)}{(1+\tau)^{\alpha+\beta+2}} \times F[\frac{\alpha+\beta+2}{2}; \frac{\alpha+\beta+3}{2}; \beta+1; \frac{2\tau(1+\xi)}{(1+\tau)^2}].$$

Let us consider the set D which is parabolic-convex, that is,

$$(x+iy)^2 \in D \Leftrightarrow \{(\xi,\eta) : 4x^2(x^2-\eta^2) \le \xi \le x^2 - y^2\} \subsetneq D$$

or equivalently,

$$(x+iy)^2 \in D \Leftrightarrow \{(\xi,\eta) : \xi + i\eta = \tau^2(x,y,t,s), 0 \le t \le 1, 0 \le s \le \pi\} \subsetneq D.$$

For example:$D = \Delta : x^2 + y^2 < 1 or D = \{(\xi,\eta) : |\xi| < 1, |\eta| < (1+\xi^2)^{\frac{1}{2}}\}$.

Now we define optimal approximation errors as :

$$E_{2n}^p = E_{2n}^p(F;D) = \min\{\|F-H\|_p : H \in P_{2n}\},$$

$$e_{2n}^p = e_{2n}^p(f;D) = \min\{\|f-h\|_p : h \in p_{2n}\}, n = 0,1,2,\ldots,$$

where $P_{2n} = \{K_{\alpha,\beta}(h) : h \in p_{2n}\}$, and $p_{2n} = \{\sum_{k=0}^n a_k z^{2k} : a_k(\text{real}), 0 \le k \le n\}$.

McCoy [1, p.465] proved that

$$\lim_{n\to\infty} E_{2n}^{\frac{p}{2n}}(F;D) = 0 \tag{1.3}$$

if and only if, F is the restriction of an entire *GBSP* (analytic) function to D. McCoy [14] showed that a *GBSP* F is the restriction of an entire *GBSP* (analytic) function to D if and only if the $K_{\alpha,\beta}$ associate f is the restriction of an entire (analytic) function to D. And when the growth of an entire *GBSP* function with associate f is measured by order $\rho = \rho(F)$ and type $T = T(F)$ which are defined as in analytic function theory by

$$\rho = \limsup_{r\to\infty} \frac{\log\log M_r(F)}{\log r}, T = \limsup_{r\to\infty} \frac{\log M_r(F)}{r^\rho},$$

where

$$M_r(F) = \sup\{|F(x,y)| : x^2+y^2 < r^2\},$$

then $\rho(F) = \rho(f)$ and $T(F) = T(f)$.

For an entire F, (1.3) does not give any clue as to the rate at which $E_{2n}^{\frac{p}{2n}}(F;D)$ tends to zero. McCoy [1, p.467] has showed that this rate depends on the order of *GBSP* F. Moreover, he proved that

$$\limsup_{n\to\infty} \frac{2n\log n}{\log[\frac{1}{E_{2n}^p(F)}]} = \rho(F) \tag{1.4}$$

where $\rho(F)$ is the nonnegative real number if and only if, F is the restriction of an entire *GBSP* (analytic) function to D of order ρ.

However, if *GBSP* F is an entire function of infinite order, then (1.4) fails to give satisfactory information about the rate of decrease of $E_{2n}^{\frac{p}{2n}}(F;D)$. The purpose of the present work is to refine the result of McCoy [1, p.467] by using the concept of index of an entire function introduced by Sato [15, p.412] to the function of infinite order.

Thus, if

$$\rho(q) = \limsup_{r\to\infty} \frac{\log^{[q]} M_r(F)}{\log r}, q \geq 2$$

where $\log^{[0]} M_r(F) = M_r(F)$ and $\log^{[q]} M_r(F) = \log(\log^{[q-1]} M_r(F))$, then *GBSP* F is said to be of index-q if $\rho(q-1) = \infty$ while $\rho(q) < \infty$. If *GBSP* F is of index-q we shall call $\rho(q)$ the q-order of F. Analogous to lower order, the concept of lower q-order can be introduced. Thus *GBSP* F, that is an entire function of index-q, is said to be lower q-order $\lambda(q)$ if

$$\lambda(q) = \liminf_{r\to\infty} \frac{\log^{[q]} M_r(F)}{\log r}, q \geq 2.$$

2. Auxiliary results

In this section we shall prove some lemmas which will be useful in the sequel.

Lemma 2.1. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of index-q(≥ 2) and lower q-order $\lambda(q)$ and let $\nu(r)$ denote the rank of the maximum term $\mu(r)$ for $|z| = r$,i.e. $\mu(r) = \max_{n\geq 0}\{|a_n|r^n\}$ and $\nu(r) = \max\{n : \mu(r) = |a_n|r^n\}$.*

Then

$$\lambda(q) = \liminf_{r\to\infty} \frac{\log^{[q-1]} \nu(r)}{\log r} = \liminf_{r\to\infty} \frac{\log^{[q]} \mu(r)}{\log r}.$$

Proof. The proof follows on the lines of Whittaker [16, Thm. 1] for $q = 2$, so we omit the proof. □

Lemma 2.2. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of index-q(≥ 2) and lower q-order $\lambda(q)$ and let $\{n_k\}$ denote the range of the step function $\nu(r)$, then*

$$\lambda(q) = \liminf_{r\to\infty} \frac{\log^{[q-1]} n_k}{\log \xi(n_{k+1})}$$

where the $\xi(n_k)$ denote the jump points of $\nu(r)$.

Proof. For $q = 2$, the proof is due to Gray and Shah [17, Lemma 1]. □

Lemma 2.3. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}$ be an entire function of index-$q(\geq 2)$ and lower q-order $\lambda(q)$ such that $\varphi(k) = |\frac{a_k}{a_{k+1}}|^{\frac{1}{(n_{k+1}-n_k)}}$ forms an increasing function of k for $k > k_o$; then*

$$\lambda(q) = \liminf_{k\to\infty} \frac{(n_{k+1} - n_k) \log^{[q-1]} n_k}{\log|\frac{a_k}{a_{k+1}}|}.$$

Proof. For $q = 2$, the proof is due to Juneja and Kapoor [18]. So we omit the proof. □

Lemma 2.4. *Let $\{n_k\}$ be an increasing sequence of positive integers and let $\{a_n\}$ be a sequence of complex numbers such that $|a_{n_k}| < 1$ for $k > k_o$; then for $q \geq 2$*

$$\liminf_{k\to\infty} \frac{n_k \log^{[q-1]} n_k}{\log|a_{n_k}|^{-1}} \geq \liminf_{k\to\infty} \frac{(n_k - n_{k-1}) \log^{[q-1]} n_{k-1}}{\log|\frac{a_{n_{k-1}}}{a_{n_k}}|}.$$

Proof. The proof follows on the lines of Juneja [21, Lemma 2] for $q = 2$, so we omit the proof. □

3. Main results

Theorem 3.1. *For fixed $p \geq 2$, let the $F \in L^p(D)$ be the restriction of an entire GBSP (analytic) function to D of index-$q(\geq 2)$. Then F is of q-order $\rho(q)$ if and only if*

$$\rho(q) = \limsup_{n\to\infty} \frac{2n \log n}{\log[\frac{1}{E^p_{2n}(F)}]}. \tag{3.1}$$

Proof. The proof follows on the lines of [1, Thm. 2(i)], so we omit the details. □

However, the result corresponding to (3.1) does not always hold for the lower q-order. The following theorem is corresponding to (3.1) for the lower q-order of a *GBSP* F.

Theorem 3.2. *For fixed $p \geq 2$, let the $F \in L^p(D)$ be the restriction of an entire GBSP (analytic) function to D of index $q(\geq 2)$. Then F is of lower q-order $\lambda(q)$ if and only if*

$$\lambda(q) = \max_{\{n_k\}} \liminf_{k\to\infty} \frac{2n_k \log^{[q-1]} n_{k-1}}{-\log E^p_{2n_k}(F)},$$

where maximum is taken over all increasing sequence $\{n_k\}$ of natural numbers.

Proof. Let $F \in L^p(D)$ be the restriction of an entire *GBSP* (analytic) function to D of index-$q(\geq 2)$ and lower q-order $\lambda(q)$. Following Bernstein's [19, p.176] and A.Giroux [20, p.52], it follows that for

$$e^p_{2n}(f) \leq e^{\infty}_{2n}(f) \leq \frac{2B(r)}{r^{2n}(r-1)} \tag{3.2}$$

for any $r > 1$, where $B(r) = \max_{z\in\Im_r} |f(z)|$ and $\Im_r$ with $r > 1$ denotes the closed interior of the ellipse with foci ± 1, with half-major axis $(r^2+1)/2r$ and half-minor axis $(r^2-1)/2r$. The closed disks $D_1(r)$ and $D_2(r)$ bound the ellipse $\Im_r$ in the sense that

$$D_1(r) = \{z : |z| \leq \frac{r^2-1}{2r}\} \subsetneq \Im_r \subsetneq D_2(r) = \{z : |z| \leq \frac{r^2+1}{2r}\}.$$

From above it follows that

$$M(\frac{r^2-1}{2r}) \leq B(r) \leq M(\frac{r^2+1}{2r}) \text{for all} \quad r > 1. \tag{3.3}$$

Consequently, (3.2) and (3.3) give for any sequence $\{n_k\}$ of positive integers that

$$M(\frac{r^2+1}{2r}) \geq e^p_{2n_k}(f) r^{2n_k} \tag{3.4}$$

for any $r > 3$ and $k = 1, 2, \ldots$. Now using the optimal approximates [1, eq.12]

$$E_{2n}^{\frac{p}{2n}}(F) \leq w^{\frac{1}{2np}} e_{2n}^{\frac{p}{2n}}(f), w = w(\alpha, \beta, p : D)$$

in (3.4) we obtain

$$M(\frac{r^2+1}{2r}) \geq w^{\frac{-1}{p}} E_{2n_k}^{p} r^{2n_k}. \tag{3.5}$$

Now let

$$\liminf_{k\to\infty} \frac{2n_k \log^{[q-1]} n_{k-1}}{-\log E_{2n_k}^{p}(F)} = \eta^*(\{2n_k\}) \equiv \eta^*. \tag{3.6}$$

Since *GBSP* F is an entire function, (3.6) gives $0 \leq \eta^* \leq \infty$. First, let $0 < \eta^* < \infty$, then for

$$E_{2n_k}^{p}(F) > [\log^{[q-1]} n_{k-1}]^{-\frac{2n_k}{(\eta^*-\varepsilon)}}$$

for $k > k_o = k_o(\varepsilon)$. Let $r_k = e(\log^{[q-2]} n_{k-1})^{\frac{1}{(\eta^*-\varepsilon)}}$ for $k = 1, 2, 3, \ldots$. If $r_k \leq r \leq r_{k+1}, k > k_o$ then (3.5) gives

$$\begin{aligned}
\log M(\frac{r^2+1}{2r}) &\geq \left\{\log E_{2n_k}^{p}(F) + 2n_k \log r - \frac{1}{p}\log w\right\} \\
&\geq \log E_{2n_k}^{p}(F) + 2n_k \log r_k - \frac{1}{p}\log w \\
&> 2n_k \\
&= 2\exp^{[q-2]}(\frac{r_{k+1}}{e})^{(\eta^*-\varepsilon)}.
\end{aligned}$$

So

$$\begin{aligned}
\log^{[q]} M(\frac{r^2+1}{2r}) &> (\eta^*-\varepsilon)\log r_{k+1} - (\eta^*-\varepsilon) \\
&\geq (\eta^*-\varepsilon)\log r - (\eta^*-\varepsilon)
\end{aligned}$$

or

$$\lambda(q) = \liminf_{r\to\infty} \frac{\log^{[q]} M_r(F)}{\log r} \geq \eta^*$$

which obviously holds for every increasing sequence $\{n_k\}$ of positive integers, we have

$$\lambda(q) \geq \max_{\{n_k\}} \eta^*(\{2n_k\}) = \eta^{**}. \tag{3.7}$$

Now for each $n \geq 0$ there exists a unique $h \in p_{2n}$ such that

$$\|f - p_{2n}\|_p = e_{2n}^{p}(f), n = 0, 1, \ldots.$$

Further, since $\|p_{2n+1} - p_{2n}\|_p$ is bounded above by $2e_{2n}^{p}(f)$, we have by [20, p.42];

$$|p_{2n+1} - p_{2n}| \leq 2e_{2n}^{p}(f) r^{2n+1} \tag{3.8}$$

for all $z \in \Im_r$ for any $r > 1$. Thus we can write

$$f(z) = p_0(z) + \sum_{i=0}^{\infty} (p_{2i+1}(z) - p_{2i}(z))$$

and this series converges uniformly in any bounded domain of the complex plane. So, (3.8) gives

$$|f(z)| \leq |p_0(z)| + 2\sum_{i=0}^{\infty} e_{2i}^{p}(f) r^{2i+1}$$

for any $z \in \Im_r$ and from the definition of $B(r)$

$$B(r) \leq A_o + 2\sum_{i=0}^{\infty} e_{2i}^{p}(f) r^{2i+1}.$$

So (3.3) gives

$$M(\frac{r^2-1}{2r}) \leq A_o + 2\sum_{i=0}^{\infty} e_{2i}^{p}(f) r^{2i+1}. \tag{3.9}$$

Using the optimal approximate [1, eq.(13)]

$$e_{2n}^{\frac{p}{2n}}(f) \leq \delta^{\frac{1}{2np}} E_{2n}^{\frac{p}{2n}}(F), \delta = \delta(\alpha, \beta, p : D)$$

in (3.9) we get

$$M(\frac{r^2-1}{2r}) \leq A_o + 2\sum_{i=0}^{\infty} \delta^{\frac{1}{p}} E_{2i}^{p}(F) r^{2i+1}. \tag{3.10}$$

Obviously, the function $g(z) = \sum_{n=0}^{\infty} E_{2n}^{p}(F)\delta^{\frac{1}{p}} z^{2n+1}$ is an entire function. Let $\{n_k\}$ denote the range of $\nu(r)$ for this function. Consider the function $\tilde{g}(z) = \sum_{k=0}^{\infty} E_{2n_k}^{p}(F)\delta^{\frac{1}{p}} z^{2n_k+1}$. It is easily seen that $\tilde{g}(z)$ is also an entire function and that $g(z)$ and $\tilde{g}(z)$ have the same maximum term for every z. It follows that both have same lower q-order. If we denote this by $\lambda_o(q)$ then since $\tilde{g}(z)$ satisfies the hypothesis of Lemma 2.3, we have

$$\begin{aligned}
\lambda_0(q) &= \liminf_{k\to\infty} \frac{2(n_k - n_{k-1}) \log^{[q-1]} n_{k-1}}{\log(\frac{E_{2n_{k-1}}^{p}(F)}{E_{2n_k}^{p}(F)})} \\
&\leq \liminf_{k\to\infty} \frac{2n_k \log^{[q-1]} n_{k-1}}{-\log E_{2n_k(F)}^{p}} \\
&\leq \max \liminf_{k\to\infty} \frac{2n_k \log^{[q-1]} n_{k-1}}{-\log E_{2n_k(F)}^{p}} = \eta^{**}
\end{aligned}. \tag{3.11}$$

Thus (3.10) and (3.11) give

$$\begin{aligned}
M(\frac{r^2-1}{2r}) &\leq A_o + 2g(r) \\
&\leq O(1) + 2\exp^{[q-1]}(r^{\eta^{**}+\varepsilon})
\end{aligned}$$

for a sequence $r_1, r_2, \ldots \to \infty$. Hence, it gives that

$$\lambda(q) \leq \eta^{**}$$

which shows that the lower q-order of *GBSP* F does not exceed η^{**}. Thus, if *GBSP* F is of lower q-order $\lambda(q)$, then (3.7) shows that $\eta^{**} < \lambda(q)$. If $\eta^{**} < \lambda(q)$, then the above arguments show that *GBSP* F would be of lower q-order less than η^{**}, a contradiction. Thus, we must have $\eta^{**} = \lambda(q)$. □

The following theorem depicts the influence of $\lambda(q)$ on the rate of decrease of $E_{2n}^{p}(F)$.

Theorem 3.3. *For fixed $p \geq 2$, let the $F \in L^p(D)$ be the restriction of an entire GBSP (analytic) function to D of index q. Then, F is of lower q-order $\lambda(q)$ if and only if*

$$\lambda(q) \quad = \max_{\{n_k\}} \liminf_{k\to\infty} \frac{2(n_k - n_{k-1}) \log^{[q-1]} n_{k-1}}{\log(\frac{E_{2n_{k-1}}^{p}(F)}{E_{2n_k}^{p}(F)})},$$

where maximum is taken over all increasing sequences $\{n_k\}$ of natural numbers.

Proof. In view of Lemma 2.3 and Lemma 2.4 with above arguments the proof is immediate. □

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