



MULTIPOINT SELFADJOINT QUASI-DIFFERENTIAL OPERATORS FOR FIRST ORDER

RUKIYE ÖZTÜRK MERT, BÜLENT YILMAZ, AND ZAMEDDIN I. ISMAILOV

ABSTRACT. In the present paper, the aim is to described all selfadjoint extensions of the minimal operator generated by first order linear symmetric multipoint quasi-differential operator expression in the direct sum of weighted Hilbert spaces of vector-functions defined at the semi-infinite intervals by using the Calkin-Gorbachuk method. We have also examine the structure of the spectrum of such extensions.

1. INTRODUCTION

The general theory of selfadjoint extensions of linear densely defined closed symmetric operator in any Hilbert space is mentioned for the first time in mathematical literature in famous works of J. von Neumann [9] and M.H. Stone [11]. In mathematical literature there are the Glazman-Krein-Naimark and Calkin-Gorbachuk methods (see[6,10]).

The scalar case of this theory has been studied by I.M. Glazman, M.A. Naimark, M.G. Krein, W.N. Everitt, L. Markus, A. Zettl, J. Sun, D. O'Regan, R. Agarwal [2-5, 8, 12] which is the motivation of this paper.

The main purpose of this paper is to generalized of mentioned above theory to infinite dimensional case of considered problems by using Calkin-Gorbachuk methods.

Here, the representation of all selfadjoint extension of the multipoint symmetric quasi-differential operator, generated by first order symmetric quasi-differential operator expression in the direct sum of weighted Hilbert spaces of vector-functions defined at the semi-infinite intervals, in terms of abstract boundary values are described. The structure of the spectrum of these selfadjoint extensions is also investigated.

Received by the editors: November 22, 2017; Accepted: June 01, 2018.

2010 *Mathematics Subject Classification.* 47A10.

Key words and phrases. Symmetric and selfadjoint differential operators, multipoint quasi-differential expression, deficiency indices, spectrum.

First of all, it is better to note the views of A. Zettl's and J. Sun's [12] on this area: A selfadjoint ordinary differential operator in Hilbert spaces is generated by two things:

- (1) A symmetric (formally selfadjoint) differential expression;
- (2) A boundary condition which determined selfadjoint differential operators;

Given such a selfadjoint differential operator, a basic question is: What is its spectrum?

2. STATEMENT OF THE PROBLEM

Let H be a separable Hilbert space and $a_1, a_2 \in \mathbb{R}$. Also assume that $\alpha_1 : (-\infty, a_1) \rightarrow (0, \infty)$, $\alpha_2 : (a_2, \infty) \rightarrow (0, \infty)$, $\int_{-\infty}^{a_1} \frac{dx}{\alpha_1(x)} = \infty$, $\int_{a_2}^{\infty} \frac{dx}{\alpha_2(x)} = \infty$, $\alpha_1 \in C(-\infty, a_1)$, $\alpha_2 \in C(a_2, \infty)$.

In the Hilbert space

$$\mathcal{H} = L^2_{\alpha_1}(H, (-\infty, a_1)) \oplus L^2_{\alpha_2}(H, (a_2, \infty))$$

of vector-functions on $(-\infty, a_1) \cup (a_2, \infty)$, consider the following linear multi-point differential operator expression for first order in the form

$$l(u) = (l_1(u_1), l_2(u_2)),$$

where $u = (u_1, u_2)$,

$$\begin{aligned} l_1(u_1) &= i(\alpha_1 u_1)' + A_1 u_1, \\ l_2(u_2) &= i(\alpha_2 u_2)' + A_2 u_2 \end{aligned}$$

and for simplicity, we assume that A_1 and A_2 are linear bounded selfadjoint operators in H .

The minimal L_{10} (L_{20}) and maximal L_1 (L_2) operators associated with differential expression l_1 (l_2) in $L^2_{\alpha_1}(H, (-\infty, a_1))$ ($L^2_{\alpha_2}(H, (a_2, \infty))$) can be constructed, by using the same technique in [7].

The operators $L_0 = L_{10} \oplus L_{20}$ and $L = L_1 \oplus L_2$ in the Hilbert space \mathcal{H} are called minimal and maximal operators associated with differential expression $l(\cdot)$ respectively. It is clear that the operator L_0 is symmetric and $L_0^* = L$ in \mathcal{H} . The minimal operator L_0 is not maximal. Indeed, differential expression $l(\cdot)$ with boundary condition $(\alpha_1 u_1)(a_1) = (\alpha_2 u_2)(a_2)$ generates a selfadjoint extension of L_0 .

The main goal in the present study is to describe all selfadjoint extensions of the minimal operator L_0 in \mathcal{H} in terms of boundary values and investigate the structure of the spectrum of these extensions.

3. DESCRIPTION OF SELFADJOINT EXTENSIONS

In this section, we investigate the abstract representation of all selfadjoint extensions of the minimal operator L_0 with the use of Calkin-Gorbachuk method in terms of boundary values.

We first prove the following lemma which we will need.

Lemma 3.1. *The deficiency indices of the operators L_{10} and L_{20} are in form*

$$(m(L_{10}), n(L_{10})) = (\dim H, 0),$$

$$(m(L_{20}), n(L_{20})) = (0, \dim H).$$

Proof. The general solutions of differential equations are as follows:

$$i(\alpha_1 u_1^\pm)'(t) \pm iu_1^\pm(t) = 0, \quad t < a_1,$$

$$i(\alpha_2 u_2^\pm)'(t) \pm iu_2^\pm(t) = 0, \quad t > a_2,$$

where

$$u_1^\pm(t) = \frac{1}{\alpha_1(t)} \exp\left(\mp \int_{-\infty}^t \frac{ds}{\alpha_1(s)}\right) f_1, \quad t < a_1, \quad f_1 \in H,$$

$$u_2^\pm(t) = \frac{1}{\alpha_2(t)} \exp\left(\pm \int_t^\infty \frac{ds}{\alpha_2(s)}\right) f_2, \quad t > a_2, \quad f_2 \in H$$

respectively.

Then it is obtained that

$$\begin{aligned} \|u_1^+\|_{L^2_{\alpha_1}(H, (-\infty, a_1))}^2 &= \int_{-\infty}^{a_1} \|u_1^+(t)\|_H^2 \alpha_1(t) dt = \int_{-\infty}^{a_1} \exp\left(-2 \int_{-\infty}^t \frac{ds}{\alpha_1(s)}\right) \frac{dt}{\alpha_1(t)} \|f_1\|_H^2 \\ &= \int_{-\infty}^{a_1} \exp\left(-2 \int_{-\infty}^t \frac{ds}{\alpha_1(s)}\right) d\left(\int_{-\infty}^t \frac{ds}{\alpha_1(s)}\right) \|f_1\|_H^2 = \frac{1}{2} \|f_1\|_H^2 < \infty. \end{aligned}$$

By simple calculations, we also have that

$$u_1^-(t) = \frac{1}{\alpha_1(t)} \exp\left(\int_{-\infty}^t \frac{ds}{\alpha_1(s)}\right) f_1 \notin L^2_{\alpha_1}(H, (-\infty, a_1)).$$

Consequently, the deficiency indices of the operator L_{10} can be expressed in the following form

$$(m(L_{10}), n(L_{10})) = (\dim H, 0).$$

In a similar way one can show that

$$(m(L_{20}), n(L_{20})) = (0, \dim H).$$

Therefore, this completes the proof of lemma. □

From last assertion one can easily see that

$$m(L_0) = m(L_{10}) + m(L_{20}) = \dim H$$

and

$$n(L_0) = n(L_{10}) + n(L_{20}) = \dim H.$$

Consequently, the symmetric minimal operator L_0 has a selfadjoint extension (see [6]).

In order to describe the all selfadjoint extensions of the minimal operator L_0 it is needed to construct a space if boundary values for it.

Definition 3.2. ([6]) Let \mathbb{H} be any Hilbert space and $S : D(S) \subset \mathbb{H} \rightarrow \mathbb{H}$ be a closed densely defined symmetric operator on the Hilbert space having equal finite or infinite deficiency indices. A triplet $(\mathfrak{H}, \gamma_1, \gamma_2)$, where \mathfrak{H} is a Hilbert space, γ_1 and γ_2 are linear mappings from $D(S^*)$ into \mathfrak{H} , is called a space of boundary values for the operator S , if for any $f, g \in D(S^*)$

$$(S^* f, g)_{\mathbb{H}} - (f, S^* g)_{\mathbb{H}} = (\gamma_1(f), \gamma_2(g))_{\mathfrak{H}} - (\gamma_2(f), \gamma_1(g))_{\mathfrak{H}}$$

while for any $F, G \in \mathfrak{H}$, there exists a function $f \in D(S^*)$ such that $\gamma_1(f) = F$ and $\gamma_2(f) = G$.

It is known that for any symmetric operator with equal deficiency indices, we have at least one space of boundary values (see [6]).

Theorem 3.3. *The triplet (H, γ_1, γ_2) , where*

$$\gamma_1 : D(L) \subset H \rightarrow H, \quad \gamma_1(u) = \frac{1}{\sqrt{2}}((\alpha_1 u_1)(a_1) - (\alpha_2 u_2)(a_2)), \quad u = (u_1, u_2) \in D(L),$$

$$\gamma_2 : D(L) \subset H \rightarrow H, \quad \gamma_2(u) = \frac{1}{i\sqrt{2}}((\alpha_1 u_1)(a_1) + (\alpha_2 u_2)(a_2)), \quad u = (u_1, u_2) \in D(L)$$

is a space of boundary values of the minimal operator L_0 in \mathcal{H} .

Proof. In this case, the following holds for any $u = (u_1, u_2)$ and $v = (v_1, v_2)$ from $D(L)$:

$$\begin{aligned} & (Lu, v)_{\mathcal{H}} - (u, Lv)_{\mathcal{H}} \\ &= (i(\alpha_1 u_1)' + A_1 u_1, v_1)_{L^2_{\alpha_1}(H, (-\infty, a_1))} + (i(\alpha_2 u_2)' + A_2 u_2, v_2)_{L^2_{\alpha_2}(H, (a_2, \infty))} \\ &\quad - (u_1, i(\alpha_1 v_1)' + A_1 v_1)_{L^2_{\alpha_1}(H, (-\infty, a_1))} - (u_2, i(\alpha_2 v_2)' + A_2 v_2)_{L^2_{\alpha_2}(H, (a_2, \infty))} \\ &= i[(\alpha_1 u_1)', v_1]_{L^2_{\alpha_1}(H, (-\infty, a_1))} + (u_1, (\alpha_1 v_1)')_{L^2_{\alpha_1}(H, (-\infty, a_1))} \\ &\quad + i[(\alpha_2 u_2)', v_2]_{L^2_{\alpha_2}(H, (a_2, \infty))} + (u_2, (\alpha_2 v_2)')_{L^2_{\alpha_2}(H, (a_2, \infty))} \\ &= i[(\alpha_1 u_1)', \alpha_1 v_1]_{L^2_{\alpha_1}(H, (-\infty, a_1))} + (\alpha_1 u_1, (\alpha_1 v_1)')_{L^2_{\alpha_1}(H, (-\infty, a_1))} \\ &\quad + i[(\alpha_2 u_2)', \alpha_2 v_2]_{L^2_{\alpha_2}(H, (a_2, \infty))} + (\alpha_2 u_2, (\alpha_2 v_2)')_{L^2_{\alpha_2}(H, (a_2, \infty))} \\ &= i[(\alpha_1 u_1, \alpha_1 v_1)'_{L^2_{\alpha_1}(H, (-\infty, a_1))} + (\alpha_2 u_2, \alpha_2 v_2)'_{L^2_{\alpha_2}(H, (a_2, \infty))}] \\ &= i[(\alpha_1 u_1)(a_1), (\alpha_1 v_1)(a_1)]_H - ((\alpha_2 u_2)(a_2), (\alpha_2 v_2)(a_2))_H \\ &= (\gamma_1(u), \gamma_2(v))_H - (\gamma_2(u), \gamma_1(v))_H. \end{aligned}$$

Now let $f_1, f_2 \in H$. Let us find the function $u = (u_1, u_2) \in D(L)$ such that

$$\gamma_1(u) = \frac{1}{\sqrt{2}}((\alpha_1 u_1)(a_1) - (\alpha_2 u_2)(a_2)) = f_1$$

and

$$\gamma_2(u) = \frac{1}{i\sqrt{2}}((\alpha_1 u_1)(a_1) + (\alpha_2 u_2)(a_2)) = f_2.$$

From this we can obtain that

$$(\alpha_1 u_1)(a_1) = \frac{(if_2 + f_1)}{\sqrt{2}}, \quad (\alpha_2 u_2)(a_2) = \frac{(if_2 - f_1)}{\sqrt{2}}$$

If we choose the functions $u_1(\cdot)$ and $u_2(\cdot)$ as

$$u_1(t) = \frac{1}{\alpha_1(t)} \exp\left(\int_t^{a_1} \frac{ds}{\alpha_1(s)}\right) \frac{(if_2 + f_1)}{\sqrt{2}}, \quad t < a_1,$$

$$u_2(t) = \frac{1}{\alpha_2(t)} \exp\left(-\int_{a_2}^t \frac{ds}{\alpha_2(s)}\right) \frac{(if_2 - f_1)}{\sqrt{2}}, \quad t > a_2$$

it is clear that $(u_1, u_2) \in D(L)$ and $\gamma_1(u_1) = f_1, \gamma_2(u_2) = f_2$ which complete the proof. □

Using the Calkin-Gorbachuk method [6], we immediately obtain the following

Theorem 3.4. *If \tilde{L} is a selfadjoint extension of the minimal operator L_0 in \mathcal{H} , then it is generated by the differential operator expression $l = (l_1, l_2)$ and boundary condition*

$$(\alpha_2 u_2)(a_2) = W(\alpha_1 u_1)(a_1),$$

where $W : H \rightarrow H$ is a unitary operator. Moreover, the unitary operator W in H is determined uniquely by the extension \tilde{L} , i.e. $\tilde{L} = L_W$ and vice versa.

Proof. It is known that all selfadjoint extensions of the minimal operator \tilde{L}_0 are described by the differential-operator expression $l = (l_1, l_2)$ with boundary condition

$$(V - E)\gamma_1(u) + i(V + E)\gamma_2(u) = 0, \quad u = (u_1, u_2) \in D(L),$$

where $V : H \rightarrow H$ is a unitary operator. Therefore from Lemma 3.3, we obtain

$$\begin{aligned} & (V - E) \frac{1}{\sqrt{2}} ((\alpha_1 u_1)(a_1) - (\alpha_2 u_2)(a_2)) \\ & + i(V + E) \frac{1}{i\sqrt{2}} ((\alpha_1 u_1)(a_1) + (\alpha_2 u_2)(a_2)) = 0. \end{aligned}$$

Hence it is obtained that

$$(\alpha_2 u_2)(a_2) = -V(\alpha_1 u_1)(a_1).$$

Choosing $W = -V$ in last boundary condition we have

$$(\alpha_2 u_2)(a_2) = W(\alpha_1 u_1)(a_1).$$

□

4. SPECTRUM OF SELFADJOINT EXTENSIONS

In this section, we will investigate the structure of the spectrum of the selfadjoint extension L_W of the minimal operator L_0 in \mathcal{H} .

Now we can give the following result which deals with the point spectrum of the operator L_W .

Theorem 4.1. *The point spectrum $\sigma_p(L_W)$ of the selfadjoint extension L_W is empty.*

Proof. Consider the following eigenvalue problem

$$l(u) = \lambda u, \quad u = (u_1, u_2) \in \mathcal{H}, \quad \lambda \in \mathbb{R},$$

with boundary condition

$$(\alpha_2 u_2)(a_2) = W(\alpha_1 u_1)(a_1).$$

Then we have

$$\begin{aligned} i(\alpha_1 u_1)'(t) + A_1 u_1(t) &= \lambda u_1(t), \quad t < a_1, \\ i(\alpha_2 u_2)'(t) + A_2 u_2(t) &= \lambda u_2(t), \quad t > a_2, \\ (\alpha_2 u_2)(a_2) &= W(\alpha_1 u_1)(a_1). \end{aligned}$$

The general solution of these differential equations are as follows:

$$\begin{aligned} u_1(t; \lambda) &= \frac{1}{\alpha_1(t)} \exp\left(i(A_1 - \lambda) \int_t^{a_1} \frac{ds}{\alpha_1(s)}\right) f_\lambda^{(1)}, \quad f_\lambda^{(1)} \in H, \quad t < a_1, \\ u_2(t; \lambda) &= \frac{1}{\alpha_2(t)} \exp\left(i(A_2 - \lambda) \int_{a_2}^t \frac{ds}{\alpha_2(s)}\right) f_\lambda^{(2)}, \quad f_\lambda^{(2)} \in H, \quad t > a_2. \end{aligned}$$

It is clear that $f_\lambda^{(1)} \neq 0$ and $f_\lambda^{(2)} \neq 0$ the functions $u_1(\cdot, \lambda) \notin L^2_{\alpha_1}(H, (-\infty, a_1))$ and $u_2(\cdot, \lambda) \notin L^2_{\alpha_2}(H, (a_2, \infty))$.

Consequently, for every unitary operator W in H , we obtain that $\sigma_p(L_W) = \emptyset$. □

Later on, since the residual spectrum of any selfadjoint operator in any Hilbert space is empty, then we study the continuous spectrum of the selfadjoint extensions L_W of the minimal operator L_0 in \mathcal{H} . From the general theory of linear selfadjoint operators in Hilbert spaces it is well-known that

$$\sigma(L_W) \subset \mathbb{R}.$$

One can immediately obtain the following.

Theorem 4.2. *The continuous spectrum $\sigma_c(L_W)$ of the selfadjoint extension L_W in \mathcal{H} coincides with \mathbb{R} , i.e. $\sigma_c(L_W) = \mathbb{R}$.*

Proof. For $\lambda \in \mathbb{C}$, $\lambda_i = \text{Im}\lambda > 0$ and $f = (f_1, f_2) \in \mathcal{H}$ one can see that

$$\begin{aligned} \|R_\lambda(L_W)f(t)\|_{\mathcal{H}}^2 &= \left\| \frac{1}{\alpha_1(t)} \exp \left(i(A_1 - \lambda E) \int_{a_1}^t \frac{ds}{\alpha_1(s)} \right) f_\lambda \right. \\ &\quad + \frac{i}{\alpha_1(t)} \int_t^{a_1} \exp \left(i(A_1 - \lambda E) \int_s^t \frac{d\tau}{\alpha_1(\tau)} \right) f_1(s) ds \left. \right\|_{L_{\alpha_1}^2(H, (-\infty, a_1))}^2 \\ &\quad + \left\| \frac{i}{\alpha_2(t)} \int_t^\infty \exp \left(i(A_2 - \lambda E) \int_s^t \frac{d\tau}{\alpha_2(\tau)} \right) f_2(s) ds \right\|_{L_{\alpha_2}^2(H, (a_2, \infty))}^2 \\ &\geq \left\| \frac{1}{\alpha_2(t)} \int_t^\infty \exp \left(i(A_2 - \lambda E) \int_s^t \frac{d\tau}{\alpha_2(\tau)} \right) f_2(s) ds \right\|_{L_{\alpha_2}^2(H, (a_2, \infty))}^2. \end{aligned}$$

The vector functions $f^*(t; \lambda)$ have the form

$$f^*(t; \lambda) = \left(0, \frac{1}{\alpha_2(t)} \exp \left(i(A_2 - \bar{\lambda}) \int_{a_2}^t \frac{ds}{\alpha_2(s)} \right) f \right), \lambda \in \mathbb{C},$$

$\lambda_i = \text{Im}\lambda > 0$, $f \in H$ belong to \mathcal{H} . Indeed,

$$\begin{aligned} \|f^*(t; \lambda)\|_{\mathcal{H}}^2 &= \int_{a_2}^\infty \frac{1}{\alpha_2(t)} \left\| \exp \left(i(A_2 - \bar{\lambda}) \int_{a_2}^t \frac{ds}{\alpha_2(s)} \right) f \right\|_H^2 dt \\ &= \int_{a_2}^\infty \frac{1}{\alpha_2(t)} \exp \left(-2\lambda_i \int_{a_2}^t \frac{ds}{\alpha_2(s)} \right) dt \|f\|_H^2 \\ &= \frac{1}{2\lambda_i} \|f\|_H^2 < \infty. \end{aligned}$$

For such functions $f^*(\lambda; \cdot)$, we have

$$\begin{aligned} &\|R_\lambda(L_W)f^*(\lambda; \cdot)\|_{\mathcal{H}}^2 \\ &\geq \left\| \frac{i}{\alpha_2(t)} \int_t^\infty \frac{1}{\alpha_2(s)} \exp \left(i(A_2 - \lambda E) \int_s^t \frac{d\tau}{\alpha_2(\tau)} + i(A_2 - \bar{\lambda} E) \int_{a_2}^t \frac{ds}{\alpha_2(s)} \right) f ds \right\|_{L_{\alpha_2}^2(H, (a_2, \infty))}^2 \end{aligned}$$

$$\begin{aligned}
 &= \left\| \frac{1}{\alpha_2(t)} \exp \left(-i\lambda \int_{a_2}^t \frac{d\tau}{\alpha_2(\tau)} + iA_2 \int_{a_2}^t \frac{d\tau}{\alpha_2(\tau)} \right) \right. \\
 &\quad \left. \times \int_t^\infty \frac{1}{\alpha_2(s)} \exp \left(-2\lambda_i \int_{a_2}^s \frac{d\tau}{\alpha_2(\tau)} \right) f(s) ds \right\|_{L_{\alpha_2}^2(H, (a_2, \infty))}^2 \\
 &= \left\| \frac{1}{\alpha_2(t)} \exp \left(\lambda_i \int_{a_2}^t \frac{d\tau}{\alpha_2(\tau)} \right) \int_t^\infty \frac{1}{\alpha_2(s)} \exp \left(-2\lambda_i \int_{a_2}^s \frac{d\tau}{\alpha_2(\tau)} \right) ds \right\|_{L_{\alpha_2}^2(H, (a_2, \infty))}^2 \|f\|_H^2 \\
 &= \left\| \frac{1}{2\lambda_i \alpha_2(t)} \exp \left(-\lambda_i \int_{a_2}^t \frac{d\tau}{\alpha_2(\tau)} \right) \right\|_{L_{\alpha_2}^2(H, (a_2, \infty))}^2 \|f\|_H^2 \\
 &= \frac{1}{4\lambda_i^2} \int_{a_2}^\infty \frac{1}{\alpha_2(t)} \exp \left(-2\lambda_i \int_{a_2}^t \frac{d\tau}{\alpha_2(\tau)} \right) dt \|f\|_H^2 \\
 &= \frac{1}{8\lambda_i^3} \|f\|_H^2.
 \end{aligned}$$

Using the above inequality one can obtain the following

$$\|R_\lambda(L_W)f^*(\lambda; \cdot)\|_{\mathcal{H}} \geq \frac{\|f\|_H^2}{2\sqrt{2}\lambda_i\sqrt{\lambda_i}} = \frac{1}{2\lambda_i} \|f^*(\lambda; t)\|_{\mathcal{H}},$$

i.e., for $\lambda_i = \text{Im}\lambda > 0$ and $f \neq 0$

$$\frac{\|R_\lambda(L_W)f^*(\lambda; \cdot)\|_{\mathcal{H}}}{\|f^*(\lambda; t)\|_{\mathcal{H}}} \geq \frac{1}{2\lambda_i}.$$

On the other hand it is clear that

$$\|R_\lambda(L_W)\| \geq \frac{\|R_\lambda(L_W)f^*(\lambda; \cdot)\|_{\mathcal{H}}}{\|f^*(\lambda; t)\|_{\mathcal{H}}}, \quad f \neq 0.$$

Consequently, we have

$$\|R_\lambda(L_W)\| \geq \frac{1}{2\lambda_i}, \quad \text{for } \lambda \in \mathbb{C} \text{ and } \lambda_i = \text{Im}\lambda > 0$$

which shows that every $\lambda_r \in \mathbb{R}$ belong to continuous spectrum of the extension L_W . This completes the proof. \square

Note: In case when $\alpha_1 = \alpha_2 = 1$ the similar results have been obtained in [1].

Finally, we can provide an example for Theorem 4.2.

Example 4.3. All selfadjoint extensions L_φ of the minimal operator L_0 generated by multipoint differential expression

$$l(u) = (l_1(u_1), l_2(u_2)) = (i(tu_1)'(t, x) + xu_1(t, x), i(t^{-1}u_2)(t, x) + xu_2(t, x))$$

in the direct sum $L_t^2((-\infty, -1) \times (0, 1)) \oplus L_t^2((1, \infty) \times (0, 1))$ written in terms of boundary values are described the following boundary conditions are described

$$u_2(1) = e^{i\varphi} u_1(-1), \quad \varphi \in [0, 2\pi).$$

Moreover, spectrum of such extension $\sigma(L_\varphi) = \sigma_c(L_\varphi) = \mathbb{R}$.

REFERENCES

- [1] Bairamov E., Öztürk Mert, R, Ismailov, Z., Selfadjoint Extensions of a Singular Differential Operator, *Journal of Mathematical Chemistry*, 50: (2012), 1100-1110.
- [2] El-Gebeily, M.A., O'Regan, D., Agarwal, R., Characterization of Self-adjoint Ordinary Differential Operators, *Mathematical and Computer Modelling*, 54 (2011), 659-672.
- [3] Everitt, WN, Markus L., The Glazman-Krein-Naimark Theorem for Ordinary Differential Operators, *Operator Theory, Advances and Applications*, 98 (1997), 118-130.
- [4] Everitt, W.N., Poulkou A., Some Observations and Remarks on Differential Operators Generated by First-Order Boundary Value Problems, *Journal of Computational and Applied Mathematics*, 153 (2003), 201-211.
- [5] Glazman, IM., On the Theory of Singular Differential Operators, *Uspehi Math. Nauk.*, 40 (1950), 102-135 (English translation in Amer. Math. Soc. Translations 1962; (1), 4: 331-372).
- [6] Gorbachuk, VI, Gorbachuk, ML., Boundary Value Problems for Operator-Differential Equations, First ed., Kluwer Academic Publisher: Dordrecht, 1991.
- [7] Hörmander, L., On the Theory of General Partial Differential Operators, *Acta Mathematica*, 94 (1955), 161-248.
- [8] Naimark, MA., Linear Differential Operators II. Ungar, New York, 1968.
- [9] von Neumann, J., Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren, *Math. Ann.* 102 (1929-1930), 49-131.
- [10] Rofe-Beketov, FS, Kholkin, AM., Spectral Analysis of Differential Operators, World Scientific Monograph Series in Mathematics v.7. 2005.
- [11] Stone, MH., Linear Transformations in Hilbert Spaces and Their Applications in Analysis, *American Math. Soc. Coloq.*, (1932), 15.
- [12] Zettl, A, Sun, J., Survey Article: Self-adjoint Ordinary Differential Operators and Their Spectrum, *Rocky Mountain Journal of Mathematics*, 45, 1 (2015), 763-886.

Current address: Rukiye ÖZTÜRK MERT: Department of Mathematics, Art and Science Faculty, Hitit University, 19030, Çorum, Turkey

E-mail address: rukiyeozturkmert@hitit.edu.tr

ORCID Address: <http://orcid.org/0000-0001-8083-5304>

Current address: Bülent YILMAZ: Department of Mathematics, Marmara University, 34722, Kadıköy, Istanbul, Turkey

E-mail address: bulentyilmaz@marmara.edu.tr

ORCID Address: <http://orcid.org/0000-0002-1394-230X>

Current address: Zameddin I. Ismailov: Department of Mathematics, Faculty of Sciences, Karadeniz Technical University, 61080, Trabzon, Turkey

E-mail address: zameddin.ismailov@gmail.com

ORCID Address: <http://orcid.org/0000-0001-5193-5349>