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Asymptotics of the Solution of the Parabolic Problem with a Stationary Phase and an Additive Free Member

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Abstract: *In this paper we construct the asymptotics of the solution of the singularly perturbed parabolic problem with the stationary phase and the additive free term by using the regularization method for singularly perturbed problems. In this case, the asymptotic solution consists of regular and boundary layer terms. The boundary layer members are parabolic, power and rapidly oscillating boundary layer functions, and their products. These products are called angular boundary layer functions. Angular boundary layer functions have two components: the first is described by the product of a parabolic boundary layer function and a boundary layer function, which has a rapidly oscillating change.*

Keywords: *Asymptotics, singularly perturbed parabolic problem, stationary phase.*

1. INTRODUCTION

Singularly perturbed problems with rapidly oscillating free terms were studied in [1 - 3]. In [1], authors mostly focused on ordinary differential equations with a rapidly oscillating free term whose phase does not have stationary points. Using the regularization method for singularly perturbed problems [5], differential equations of parabolic type with a small parameter where fast-oscillating functions as a free member were studied in [2 - 3]. The asymptotic solutions constructed in [1 - 3] contain a boundary layer function having a rapidly oscillating character of change. In addition to such boundary-layer function, ordinary differential equations that contain an exponential boundary layer were studied in [1], parabolic equations with parabolic boundary layer were studied in [2], [3] and angular boundary layer were considered in [2], [6]. If the phase of the free term has stationary points, then boundary layers have an additional rise, which is due to a power character of a change. In this case, the asymptotic solution consists of regular and boundary layer terms. The boundary layer members are parabolic and power, rapidly oscillating boundary layer functions, and their products, which are called angular boundary layer functions [5]. In this paper we used the methods of [5], [6] to solve the parabolic problem.

2. ASYMPTOTIC CONSTRUCTION

2.1. Statement of the Problem

In this paper we study the following problem:

$$L_\varepsilon u\,(x,t,\varepsilon) \equiv \partial_t u - \varepsilon^2 a(x)\partial_x^2 u - b(x,t)u = \sum_{k=1}^{N} f_\kappa(x,t)\exp\left(\frac{i\theta_\kappa(t)}{\varepsilon}\right), (x,t)\in\Omega, \quad (1)$$

$$u(x,t,\varepsilon)|_{t=0} = u(x,t,\varepsilon)|_{x=0} = u(x,t,\varepsilon)|_{x=1} = 0$$

where $\varepsilon > 0$ – is a small parameter, $\Omega = \{ (x, t){:}\ x \in (0,1),\ t\in(0,T]\}$.
The problem is solved under the following assumptions:

1. $a(x) > 0,\ a(x) \in C^\infty[0,1],\ b(x,t),\ f(x,t) \in C^\infty(\overline{\Omega})$,
2. $\forall\, x \in [0,1]$ function $a(x) > 0$
3. $\theta_k'(t) = 0$ – is the phase function.

2.2. Regularization of the Problem

For the regularization of the problem (1) we introduce regularizing independent variables using methods described in [4] and [6]:

$$\eta = \frac{t}{\varepsilon^2},\ r_k = \frac{i\,[\theta_k(t)-\theta_k(0)]}{\varepsilon}, \xi_\nu = \frac{\varphi_\nu(x)}{\varepsilon}, i = \sqrt{-1}, \quad (2)$$

$$\zeta_\nu = \frac{\varphi_\nu(x)}{\varepsilon^2}, \varphi_\nu(x) = (-1)^{\nu-1}\int_{\nu-1}^{x}\frac{ds}{\sqrt{a(s)}}, \nu = 1,2\ ,$$

$$\sigma_k = \int_0^t exp\left(\frac{i\,[\theta_\kappa(s)-\theta_\kappa(0)]}{\varepsilon}\right)ds \equiv p_\kappa(t,\varepsilon), l = \overline{0,r}, j = \overline{0,k_l-1}$$

Instead of the desired function $u(x,t,\varepsilon)$ we study the extended function

$\breve{u}(M,\varepsilon), M=(x,t,r,\eta,\sigma,\xi,\zeta), \sigma=(\sigma_1,\sigma_2 \dots \sigma_N),\ r=(r_1,r_2 \dots r_N),\ \xi=(\xi_1,\xi_2), \zeta=(\zeta_1,\zeta_2)$ such that its constriction by regularizing variables coincides with the desired solution:

$$\breve{u}(M,\varepsilon)|_{\gamma=p(x,t,\varepsilon)} \equiv u(x,t,\varepsilon) \qquad (3)$$
$$\gamma=(r,\sigma,\eta,\xi,\zeta)$$

Taking into account (2) and (3), we find the derivatives:

$$\partial_t u(x,t,\varepsilon) \equiv (\partial_t \breve{u}(M,\varepsilon)+\frac{1}{\varepsilon^2}\partial_\eta \breve{u}(M,\varepsilon)+\sum_{k=1}^{N}[\frac{i\theta_k{}'(t)}{\varepsilon}\partial_{r_k}\breve{u}(M,\varepsilon) + exp(r_k)\,\partial_{\sigma_k}\breve{u}(M,\varepsilon))|_{\gamma=p(x,t,\varepsilon)},$$

$$\partial_x u(x,t,\varepsilon) \equiv \left((\partial_x \breve{u}\,(M,\varepsilon)+\sum_{\nu=1}^{2}\left\{\frac{\varphi_\nu'(x)}{\varepsilon}\partial_{\xi_\nu}\breve{u}(M,\varepsilon)+\frac{\varphi_\nu'(x)}{\varepsilon^2}\partial_{\zeta_\nu}\breve{u}(M,\varepsilon)\right\}\right)|_{\gamma=p(x,t,\varepsilon)},$$

$$\partial_x^2 u(x,t,\varepsilon) \equiv \left((\partial_x^2 \breve{u}\,(M,\varepsilon) + \sum_{\nu=1}^{2}\left\{\left(\frac{\varphi_\nu'(x)}{\varepsilon}\right)^2\partial_{\xi_\nu}^2\breve{u}(M,\varepsilon)+\left(\frac{\varphi_\nu'(x)}{\varepsilon^2}\right)^2\partial_{\zeta_\nu}^2\breve{u}(M,\varepsilon)+\frac{1}{\varepsilon}D_{\xi,\nu}\breve{u}(M,\varepsilon) + \frac{1}{\varepsilon^2}D_{\zeta,\nu}\breve{u}(M,\varepsilon)\right\}\right)|_{\gamma=p(x,t,\varepsilon)} \qquad (4)$$

$$D_{\xi,\nu} \equiv 2\varphi_\nu'(x)\partial_{x,\xi_\nu}^2+\varphi_\nu''(x)\partial_{\xi_\nu},$$

$$D_{\zeta,\nu} \equiv 2\varphi_\nu'(x)\partial_{x,\zeta_\nu}^2+\varphi_\nu''(x)\partial_{\zeta_\nu}.$$

On the basis of (1), (2), (3), (4) for the extended function $\breve{u}(M,\varepsilon)$ we set the problem as:

$$\widetilde{L_\varepsilon}\breve{u}(M,\varepsilon) \equiv \frac{1}{\varepsilon^2}T_0\breve{u}(M,\varepsilon)+\sum_{k=1}^{N}\frac{i\theta'{}_k(t)}{\varepsilon}\partial_{r_k}\breve{u}(M,\varepsilon)+T_1\breve{u}(M,\varepsilon) = \sum_{k=1}^{N}f_\kappa(x,t)\,exp\left(r_k+\frac{i\theta_\kappa(0)}{\varepsilon}\right)+\ L_\zeta\breve{u}(M,\varepsilon)+\varepsilon L_\xi\breve{u}(M,\varepsilon)+\varepsilon^2 L_x\breve{u}(M,\varepsilon)$$

$$\breve{u}(M,\varepsilon)|_{t=r_k=\eta=0}=\breve{u}(M,\varepsilon)|_{x=0,\xi_1=\zeta_1=0}=\breve{u}(M,\varepsilon)|_{x=1,\xi_2=\zeta_2=0}=0, \qquad (5)$$

$$T_1 \equiv \partial_\eta - \sum_{\nu=1}^{2}\partial_{\zeta_\nu}^2,$$
$$T_2 \equiv \partial_t - \sum_{\nu=1}^{2}\partial_{\xi_\nu}^2 - b(x,t)+\sum_{k=1}^{N}\exp(r_k)\,\partial_{\sigma_k},$$

$$L_\xi \equiv a(x)\sum_{v=1}^{2} D_{\xi,v},$$

$$L_\zeta \equiv a(x)\sum_{v=1}^{2} D_{\zeta,v},$$

$$L_x \equiv a(x)\partial_x^2.$$

The problem (5) is regular in ε as ε → 0:

$$\left(\widetilde{L_\varepsilon}\breve{u}(M,\varepsilon)\right)|_{q=q(x,t,\varepsilon)} \equiv L_\varepsilon \breve{u}(x,t,\varepsilon). \tag{6}$$

2.3. Solution of Iterative Problems

The solution of the problem (5) is determined in the form of a series

$$\breve{u}(M,\varepsilon) = \sum_{v=0}^{\infty} \varepsilon^v u_v(M). \tag{7}$$

For the coefficients of this series we obtain the following iterative problems:

$$T_1 u_0(M) = 0, \qquad T_1 u_1(M) = -i\sum_{k=1}^{N} \theta_k'(t)\,\partial_{r_k} u_0(M),$$

$$T_1 u_2(M) = -i\sum_{k=1}^{N} \theta_k'(t)\partial_{r_k} u_1(M) - T_2 u_0(M) + \sum_{k=1}^{N} f_\kappa(x,t)\,exp\left(r_k + \frac{i\theta_\kappa(0)}{\varepsilon}\right), \tag{8}$$

$$+L_\zeta u_0(M)$$

$$T_1 u_v(M) = -i\sum_{k=1}^{N} \theta_k'(t)\partial_{r_k} u_{v-1}(M) - T_2 u_{v-2}(M) + L_\zeta u_{v-2} + L_\xi u_{v-3}(M)$$

$$+ L_x u_{v-4}(M),$$

The solution of this problem contains parabolic boundary layer functions, internal power boundary layer functions which are connected with a rapidly oscillating free term a phase which are vanishes at $t = t_l, l = 0, 1, \dots \dots, n$ in addition, the asymptotics also contain angular boundary layer functions. We introduce a class of functions in which the iterative problems are solved:

$$G_0 \cong C^\infty(\overline{\Omega}), G_1 = \left\{u(M): u(M) = \oplus_{l=1}^{2} G_0 \otimes erfc\left(\frac{\xi_l}{2\sqrt{t}}\right)\right\},$$

$$G_2 = \{u(M): u(M) = \oplus_{k=1}^{N} G_0 \otimes exp(r_k)\},$$

$$G_3 = \left\{u(M): u(M) = \oplus_{k=1}^{N} \oplus_{l=1}^{2} Y_k^l(N_l) \otimes exp(r_k), \left\|Y_k^l(N_l)\right\| < c\,exp\left(-\frac{\varsigma_l^2}{8\eta}\right)\right\},$$

$$G_4=\left\{u(M):u(M)=\oplus_{k=1}^{N}G_0\left(\oplus_{l=1}^{2}G_0\otimes erfc\left(\frac{\xi_l}{2\sqrt{t}}\right)\right)\sigma_k\right\},N_l=(x,t,\eta,\varsigma_1,\varsigma_2)$$

From these spaces we construct a new space:

$$G=\oplus_{l=0}^{4}G_l.$$

The element $u(M)\epsilon G$ has the form:

$$u(M)=v(x,t)+\sum_{l=1}^{2}w^l(x,t)erfc\left(\frac{\xi_l}{2\sqrt{t}}\right)+\sum_{k=1}^{N}\left[c_k(x,t)+\sum_{l=1}^{2}Y_k^l(N_l)\right]exp(r_k)+\sum_{k=1}^{N}\left[z_k(x,t)+\sum_{l=1}^{2}q_k^l(x,t)erfc\left(\frac{\xi_l^2}{2\sqrt{t}}\right)\right]\sigma_{k.} \quad (9)$$

2.4. Solvability of Intermediate Tasks

The iterative problems (9) in general form can be written as:

$$T_1u(M)=H(M). \quad (10)$$

Theorem 1. Suppose that the conditions 1)-3) and $H(M)\epsilon G_3$ are satisfied. Then equation (10) is solvable in G.

Proof. Let the free term $H(M)\epsilon G_3$ be representable in the form:

$$H(M)=\sum_{k=1}^{N}\sum_{l=1}^{2}H_k^l(N_l),\ \ \|H_k^l(N_l)\|<c\,exp\left(\frac{\varsigma_l^2}{8\eta}\right),$$

then, by directly substituting function $u(M)\epsilon\ G$ from (9) in (10) we see that this function is a solution of a given problem if and only if the function $Y_k^l(N_l)$ is a solution of equation:

$$\partial_\eta Y_k^l(N_l)=\partial_{\varsigma_l}^2Y_k^l(N_l)+H_k^l(N_l),\ l=1,2,\ k=1,2,\dots,N. \quad (11)$$

With the corresponding boundary conditions, this equation has a solution which has the estimate:

$$\|Y_k^l(N_l)\|<c\,exp\left(\frac{\varsigma_l^2}{8\eta}\right).$$

The theorem is proved.

Theorem 2. Suppose that the conditions of Theorem 1 are satisfied. Then, under the following additional conditions

1. $u(M)|_{t=\eta=0}=0,u(M)|_{x=l-1,\xi_l=0,\varsigma_l=0}=0,l=1,2;$
2. $L_\varsigma u(M)=0,L_\xi u(M)=0;$
3. $i\sum_{k=1}^{N}\theta_k'(t)\partial_{r_k}u_v(M)+T_2u_{v-1}(M)+h(M)\in G_3.$

equation (10) is uniquely solvable.

Proof. By Theorem 1 equation (10) has a solution that can be written in the form (9). If condition 1) is satisfied, we obtain:

$$v(x,t)|_{t=0} = -\sum_{k=1}^{N} c_k(x,0), w^l(x,t)|_{t=0} = \bar{w}^l(x),$$
$$Y_k^l(N_l)|_{t=\eta=0} = 0, q_k^l(x,t)|_{t=0} = \bar{q}_k^l(x), d_k^l(x,t)|_{t=0} = \bar{d}_k^l(x), \quad (12)$$
$$w^l(x,t)|_{x=l-1} = -c_k(l-1,t), q_k^l(x,t)|_{x=l-1} = -z_k(l-1,t), l = 1,2.$$

Due to the fact that the function $erfc\left(\frac{\theta}{2\sqrt{t}}\right)$ is zero at $\theta = 0$ the values for $w^l(x,t)|_{t=0}, q_k^l(x,t)|_{t=0}$ are chosen arbitrarily.

We calculate:

$$\begin{aligned}
i\sum_{k=1}^{N}\theta_k'(t)\partial_{r_k}u_v(M) + T_2u_{v-1}(M) + h(M) \\
= i\sum_{k=1}^{N}\theta_k'(t)\left[c_{k,v}(x,t) + \sum_{l=1}^{2}Y_{k,v}^l(N_l)\right]\exp(r_k) \\
+ [\partial_t v_{v-1}(x,t) - b(x,t)v_{v-1}(x,t)] \\
+ \sum_{l=1}^{2}[\partial_t w_{v-1}^l(x,t) - b(x,t)w_{v-1}^l(x,t)]erfc\left(\frac{\xi_l}{2\sqrt{t}}\right) \\
+ \sum_{k=1}^{N}\Bigg[\partial_t c_{k,v-1}(x,t) - b(x,t)c_{k,v-1}(x,t) \\
+ \sum_{l=1}^{2}(\partial_t Y_{k,v-1}^l(N_l) - b(x,t)Y_{k,v-1}^l(N_l))\Bigg]\exp(\tau_k) \\
+ \sum_{k=1}^{N}\Bigg\{\partial_t z_{k,v-1}(x,t) - (x,t)z_{k,v-1}(x,t) \\
+ \sum_{l=1}^{2}[\partial_t q_{k,v-1}^l(x,t) - b(x,t)q_{k,v-1}^l(x,t)]erfc\left(\frac{\xi_l}{2\sqrt{t}}\right)\Bigg\}\sigma_k \\
+ \sum_{k=1}^{N}\left[z_{k,v-1}(x,t) + \sum_{l=1}^{2}q_{k,v-1}^l(x,t)erfc\left(\frac{\xi_l}{2\sqrt{t}}\right)\right]exp(\tau_k) + h_0(x,t) \\
+ \sum_{l=1}^{2}h_1^l(x,t)\,erfc\left(\frac{\xi_l}{2\sqrt{t}}\right) \\
+ \sum_{k=1}^{N}\left[h_2^k(x,t) + \sum_{l=1}^{2}h_2^{l,k}(x,t)\right]exp(\tau_k) \\
+ \sum_{k=1}^{N}\left[h_3^k(x,t) + \sum_{l=1}^{2}h_3^{l,k}(x,t)erfc\left(\frac{\xi_l}{2\sqrt{t}}\right)\right]\sigma_k.
\end{aligned} \quad (13)$$

The condition (3) of the theorem will be ensured, if we choose arbitrarily in (9) as the solutions of the following equations:

$$\begin{gathered}
\partial_t v_{v-1}(x,t) - b(x,t)v_{v-1}(x,t) = -h_0(x,t),\\
\partial_t w_{v-1}^l(x,t) - b(x,t)w_{v-1}^l(x,t) = -h_1^l(x,t),\\
\partial_t Y_{k,v-1}^l(N_l) - b(x,t)Y_{k,v-1}^l(N_l) = -\left(h_2^{l,k}(x,t) + q_{k,v-1}^l(x,t)erfc\left(\frac{\varsigma_l}{2\sqrt{\eta}}\right)\right),\\
\partial_t z_{k,v-1}(x,t) - b(x,t)z_{k,v-1}(x,t) = -h_3^k(x,t),\\
\partial_t q_{k,v-1}^l(x,t) - b(x,t)q_{k,v-1}^l(x,t) = -h_3^{l,k}(x,t),\\
i\theta_k'(t)c_{k,v}(x,t) = -z_{k,v-1}(x,t) - \left[\partial_t c_{k,v-1}(x,t) - b(x,t)c_{k,v-1}(x,t)\right] - h_2^k(x,t).
\end{gathered} \tag{14}$$

After the choice of arbitrariness, expression (13) is rewritten as

$$i\sum_{k=1}^{N}\theta_k'(t)\partial_{r_k}u_v(M) + T_2u_{v-1}(M) + h(M) = \sum_{k=1}^{N}\sum_{l=1}^{2}\left[i\theta_k'(t)Y_{k,v}^l(N_l)\right]\exp(\tau_k) \in G_3$$

In (14), transition is made from $\xi_l/2\sqrt{t}$ to variable $\varsigma_l/2\sqrt{\eta}$. The function $Y_k^l(N_l)$ is defined as the solution of equation (11) under the boundary conditions from (12) in the form:

$$\begin{aligned}
Y_k^l(N_l) = d_k^l(x,t)erfc\left(\frac{\varsigma_l}{2\sqrt{\eta}}\right)\\
+\frac{1}{2\sqrt{\pi}}\int_0^\eta\int_0^\infty \frac{H_k^l(\cdot)}{\sqrt{\eta-\tau}}\left[exp\left(-\frac{(\varsigma_l - y)^2}{4(\eta-\tau)}\right) - exp\left(-\frac{(\varsigma_l + y)^2}{4(\eta-\tau)}\right)\right]dyd\tau.
\end{aligned} \tag{15}$$

We substitute this function in the corresponding equation from (14), then with respect to $d_k^l(x,t)$ we obtain a differential equation. By solving it under the initial condition $d_k^l(x,t)|_{t=0} = \bar{d}_k^l(x)$, we find

$$d_k^l(x,t) = \bar{d}_k^l(x,t)B(x,t) + P_k^l(x,t), B(x,t) = \exp\left(\int_0^t b(x,s)ds\right),$$

where $P_k^l(x,t)$-is known function.

By substituting the obtained function into condition for $d_k^l(x,t)|_{x=l-1}$ from (12) we define the value of $\bar{d}_k^l(x)|_{x=l-1}$. The obtained value is used as an initial condition for a differential equation with respect to $\bar{d}_k^l(x)$, which is obtained after substitution $d_k^l(x,t)$ into the first condition of 2). With that we ensure fulfillment of this condition and uniqueness of the function $Y_k^l(N_l)$. Due to the fact that $\theta_k'(t_k) = 0$, the last equation from (14) is solvable if

$$z_{k,v-1}^l(x,0) = -h_2^k(x,0) - \left[\partial_t c_{k,v-1}(x,t) - b(x,t)c_{k,v-1}(x,t)\right]|_{t=0}.$$

The obtained ratio is used as the initial condition for the differential equation with respect to $z_{k,v-1}^l(x,t)$ from (14).

The equation with respect to $v_{v-1}(x,t)$ under the initial condition from (12) determines this function uniquely. Equations with respect to $w_{k,v-1}^l(x,t), q_{k,v-1}^l(x,t)$ under the corresponding condition from (12) have solutions in the form:

$$w_{k,v-1}^{l}(x,t)=\overline{w}_{k,v-1}^{l}(x)B(x,t)+H_{1,v-1}^{l}(x,t),$$
$$q_{k,v-1}^{l}(x,t)=\overline{q}_{k,v-1}^{l}(x)B(x,t)+H_{2,v-1}^{l}(x,t) \tag{16}$$

where $H_{1,v-1}^{l}(x,t), H_{2,v-1}^{l}(x,t)$- are known functions.

By substituting (16) into the conditions under $x=l-1$ from (12), we define values of $\overline{w}_{k,v-1}^{l}(x)|_{x=l-1}, \overline{q}_{k,v-1}^{l}(x)|_{x=l-1}$. These conditions are used for solving differential equations which are obtained from second condition of (2):

$$L_{\xi}\left(w_{k,v-1}^{l}(x,t)erfc\left(\frac{\xi_l}{2\sqrt{t}}\right)\right)=0, L_{\xi}\left(q_{k,v-1}^{l}(x,t)erfc\left(\frac{\xi_l}{2\sqrt{t}}\right)\right)=0.$$

Thus function $u(M)$ is determined uniquely. The theorem is proved.

2.5. Solution of iteration problems

Equation (8) is homogeneous for $k=0$, therefore by Theorem 1, it has a solution in G in the form:

$$\begin{aligned} u_0(M)=v_0(x,t)+\sum_{l=1}^{2} w^{l}(x,t)erfc\left(\frac{\xi_l}{2\sqrt{t}}\right) \\ +\sum_{k=1}^{N}\left\{\left(c_{k,0}(x,t)+\sum_{l=1}^{2} Y_{k,0}^{l}(N_l)\right)e^{r_k}\right. \\ \left.+\left[z_{k,0}(x,t)+\sum_{l=1}^{2} q_{k,0}^{l}(x,t)erfc\left(\frac{\xi_l}{2\sqrt{t}}\right)\right]\sigma_k\right\} \end{aligned} \tag{17}$$

The function $Y_{k,0}^{l}(N_l)$ is solution of the equation $\partial_{\eta}Y_{k,0}^{l}(N_l)=\partial_{\varsigma_l}^{2}Y_{k,0}^{l}(N_l)$ which satisfies

$$Y_{k,0}^{l}(N_l)|_{t=\eta=0}=0, Y_{k,0}^{l}(N_l)|_{x=l-1,\ \varsigma_l=0}=-c_{k,0}(l-1,t).$$

From the last problem we define

$$Y_{k,0}^{l}(N_l)=d_{k,0}^{l}(x,t)\ erfc\left(\frac{\varsigma_l}{2\sqrt{\eta}}\right), d_{k,0}^{l}(x,t)|_{x=l-1}=-c_{k,0}(l-1,t), \text{where } d_{k,0}^{l}(x,t)|_{t=0}=\bar{d}_{k,0}^{l}(x).$$

$\bar{d}_{k,0}^{l}(x)$ is arbitrary function. In the next step the equation (8) for $k = 1$ takes the form:

$$T_1u_1(M)=-i\sum_{k=1}^{N}\theta_k'(t)\left[c_{k,0}(x,t)+\sum_{l=1}^{2}Y_{k,0}^{l}(N_l)\right]e^{r_k}.$$

According to the Theorem 1, this equation is solvable in U, if $c_{k,0}(x,t)$=0 and the function $Y_{k,0}^{l}(N_l)$ is solution of the differential equation $\partial_{\eta}Y_{k,0}^{l}(N_l)=\partial_{\varsigma_l}^{2}Y_{k,0}^{l}(N_l)+H_{k,0}^{l}(N_l)$ and its solution can be written in the form (14), where $H_k^{l}(0)=i\theta_k'(t)Y_{k,0}^{l}(N_l)$. By satisfying conditions 1) -3) of Theorem 1 we obtain (see (14)):

$$\begin{gathered}\partial_t v_0-b(x,t)v_0(x,t)=0, \partial_t w_0^{l}(x,t)-b(x,t)w_0^{l}(x,t)=0,\\ \partial_t d_{k,0}^{l}(x,t)-b(x,t)d_{k,0}^{l}(x,t)=-q_{k,0}^{l}(x,t),\end{gathered} \tag{18}$$

$$\partial_t z_{k,0}(x,t) - b(x,t) z_{k,0}(x,t) = 0,$$

$$\partial_t q_{k,0}^l(x,t) - b(x,t) q_{k,0}^l(x,t) = 0,$$

$$i\theta_k'(t) c_{k,1}(x,t) = -z_{k,0}(x,t) + f_k(x,t) \exp\left(\frac{i\theta_\text{к}(0)}{\varepsilon}\right),$$

$$L_\varsigma\left(d_{k,0}^l(x,t) erfc\left(\frac{\varsigma_l}{2\sqrt{\eta}}\right)\right) = 0$$

When the equation is solved with respect to $d_{k,0}^l(x,t)$, in the $q_{k,0}^l(x,t) erfc\left(\frac{\xi_l}{2\sqrt{t}}\right)$ occurs a transition:

$$\frac{\xi_l}{2\sqrt{t}} = \frac{\varsigma_l}{2\sqrt{\eta}}.$$

The initial conditions for equations (18) are determined from (12). Functions $w_0^l(x,t), d_{k,0}^l(x,t), q_{k,0}^l(x,t)$ are expressed through arbitrary functions $\bar{w}_0^l(x), \bar{d}_{k,0}^l(x), \bar{q}_{k,0}^l(x)$. These arbitrary functions provide the condition

$$L_\xi u_k(m) = 0, L_\varsigma u_k(m) = 0$$

ensuring the solvability of the equation with respect to $c_{k,1}^l(x,t)$. Suppose that

$$Z_{k,0}(x,t)|_{t=0} = f_k(x,t) exp\left(\frac{i\theta_k(0)}{\varepsilon}\right).$$

This relation is used by the initial condition for determining $Z_{k,0}(x,t)$ by putting in to the equation (18).

Repeating this process, we can determine all the coefficients of $u_k(m)$ of the partial sum.

$$u_{\varepsilon n}(m) = \sum_{i=0}^{n} \varepsilon^i u_i(m).$$

In each iteration with respect to $v_i(x,t), w_i^l(x,t), d_{k,i}^l(x,t), z_{k,i}(x,t), q_{k,i}^l(x,t)$, we obtain inhomogeneous equations.

2.6. Assessment of the Remainder Term.

For the remainder term

$$R_{\varepsilon n}(x,t,\varepsilon) \equiv R_{\varepsilon n}(m,\varepsilon)|_{\gamma=\rho(x,t,\varepsilon)} = u(x,t,\varepsilon) - \sum_{i=0}^{n} \varepsilon^i u_i(m)|_{\gamma=\rho(x,t,\varepsilon)},$$

taking into account (3), (6), we obtain the equation

$$L_\varepsilon R_{\varepsilon n}(x,t,\varepsilon) = \varepsilon^{n+1} g_n(x,t,\varepsilon)$$

with homogeneous boundary conditions. Using the maximum principle, as in [7], we get the estimate

$$|R_{\varepsilon n}(x,t,\varepsilon)| < c\varepsilon^{n+1}. \tag{19}$$

Theorem 3. Suppose that conditions 1) -3) are satisfied. Then the constructed solution is an asymptotic solution of problem (1), i.e. $\forall n = 0,1,2,\dots$ the estimate is fair (18).

3. CONCLUSION

It is shown that the asymptotic solution of the problem contains parabolic, power, rapidly oscillating, and angular boundary layer functions. Angular boundary layer functions have two components: the first is described by the product of a parabolic boundary layer function and a boundary layer function, which has a rapidly oscillating change.

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