



## EXISTENCE OF SOLUTION TO A NONLOCAL CONFORMABLE FRACTIONAL THERMISTOR PROBLEM

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**ABSTRACT.** We study a nonlocal thermistor problem for fractional derivatives in the conformable sense. Classical Schauder's fixed point theorem is used to derive the existence of a tube solution.

### 1. INTRODUCTION

The fractional calculus may be considered an old and yet novel topic. It has started from some speculations of Leibniz, in 1695 and 1697, followed later by Euler, in 1730, and has been strongly developed till present days [1, 2]. In recent years, considerable interest in fractional calculus has been stimulated by its many applications in several fields of science, including physics, chemistry, aerodynamics, electrodynamic of complex media, signal processing, and optimal control [3, 4, 5]. Most fractional derivatives are defined through fractional integrals [5, 6, 7, 8]. Due to this fact, those fractional derivatives inherit a nonlocal behavior, which leads to many interesting applications, including memory effects and future dependence [4, 9, 10, 11, 12, 13, 14].

In 2014, a new simpler and more efficient definition of fractional derivative, depending just on the basic limit definition of derivative, was introduced in [15]. See also [16, 17, 18, 19, 20] for further developments on conformable differentiation. The new notion is prominently compatible and conformable with the classical derivative. In contrast with other fractional definitions, this new concept satisfies more standard formulas for the derivative of the product and quotient of two functions and has a simpler chain rule. In addition to the conformable derivative, the conformable fractional integral has been also introduced, and Rolle and mean value theorems for conformable fractional differentiable functions obtained. The subject is nowadays under strong development [21, 22, 23, 24]. This is well explained by

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the fact that the new definition reflects a natural extension of the usual derivative to solve different types of fractional differential equations [25, 26, 27]. The main advantages of the conformable calculus, among others, are: (i) the simple nature of the conformable fractional derivative, which allows for many extensions of classical theorems in calculus (e.g., product and chain rules) that are indispensable in applications but not valid for classical fractional differential models; (ii) the conformable fractional derivative of a constant function is zero, which is not the case for some fractional derivatives like the Riemann–Liouville; (iii) conformable fractional derivatives, conformable chain rule, conformable integration by parts, conformable Gronwall’s inequality, conformable exponential function, conformable Laplace transform, all tend to the corresponding ones in usual calculus; (iv) while in the standard calculus there exist functions that do not have Taylor power series representations about certain points, in the theory of conformable calculus they do have; (v) a nondifferentiable function can be differentiable in the conformable sense.

The thermistor concept was first discovered in 1833 by Michael Faraday (1791–1867), who noticed that the silver sulfides resistance decreases as the temperature increases. This lead Samuel Ruben (1900–1988) to invent the first commercial thermistor in the 1930s. Roughly speaking, a thermistor is a circuit device that may be used either as a current limiting device or a temperature sensing device. Typically, it is a small cylinder made from a ceramic material whose electrical conductivity depends strongly on the temperature. The heat produced by an electrical current, passing through a conductor device, is governed by the so-called thermistor equations. Nowadays, thermistors can be found everywhere, in airplanes, air conditioners, cars, computers, medical equipment, hair dryers, portable heaters, incubators, electrical outlets, refrigerators, digital thermostats, ice sensors and aircraft wings, ovens, stove tops and in all kinds of appliances. Knowing it, it is not a surprise that a great deal of attention is currently paid, by many authors, to the study of thermistor problems [28, 29, 30]. In [31], existence and uniqueness of a positive solution to generalized nonlocal thermistor problems with fractional-order derivatives were discussed. Recently, Sidi Ammi et al. studied global existence of solutions for a fractional Caputo nonlocal thermistor problem [32]. Existence and uniqueness results for a fractional Riemann–Liouville nonlocal thermistor problem on arbitrary time scales are investigated in [33], while dynamics and stability results for Hilfer fractional-type thermistor problems are studied in [34]. The Hilfer fractional derivative has been used to interpolate both the Riemann–Liouville and the Caputo fractional derivative.

While previous works assume the electrical conductivity to be a smooth and bounded function from above and below, or a Lipschitz continuous function depending strongly in both time and temperature, in contrast, here we only use the hypothesis of continuity on the electrical conductivity. Motivated by the results of [35], we establish existence of a tube solution for a conformable fractional nonlocal

thermistor problem by means of Schauder’s fixed point theorem. More precisely, we are concerned with heat conduction in a thermistor used as a current surge regulator governed by the following equations:

$$u^{(\alpha)}(t) = \frac{\lambda f(t, u(t))}{\left(\int_a^T f(x, u(x)) dx\right)^2} = g(t, u(t)), \quad t \in [a, T],$$

$$u(a) = u_a,$$
(1)

where  $u$  describes the temperature of the conductor and  $u^{(\alpha)}(t)$  denotes the conformable fractional derivative of  $u$  at  $t$  of order  $\alpha$ ,  $\alpha \in (0, 1)$ . We assume that  $a$ ,  $T$  and  $\lambda$  are fixed positive reals. Moreover, as already mentioned, we assume the following hypothesis:

(H<sub>1</sub>)  $f : [a, T] \times \mathbb{R}^+ \rightarrow \mathbb{R}^{+*}$  is a continuous function.

The rest of the article is arranged as follows. In Section 2, we give preliminary definitions and set the basic concepts and necessary results from the simple and interesting conformable fractional calculus. Then, in Section 3, we prove existence of a tube solution via Schauder’s fixed point theorem.

## 2. PRELIMINARIES

We first recall the definition of conformable fractional derivative as given in [15].

**Definition 1** (Conformable fractional derivative [15]). *Let  $\alpha \in (0, 1)$  and  $f : [0, \infty) \rightarrow \mathbb{R}$ . The conformable fractional derivative of  $f$  of order  $\alpha$  is defined by*

$$T_\alpha(f)(t) := \lim_{\epsilon \rightarrow 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon}$$

for all  $t > 0$ . Often, we write  $f^{(\alpha)}$  instead of  $T_\alpha(f)$  to denote the conformable fractional derivative of  $f$  of order  $\alpha$ . In addition, if the conformable fractional derivative of  $f$  of order  $\alpha$  exists, then we simply say that  $f$  is  $\alpha$ -differentiable. If  $f$  is  $\alpha$ -differentiable in some  $t \in (0, a)$ ,  $a > 0$ , and  $\lim_{t \rightarrow 0^+} f^{(\alpha)}(t)$  exists, then we define  $f^{(\alpha)}(0) := \lim_{t \rightarrow 0^+} f^{(\alpha)}(t)$ .

If  $f$  is differentiable at a point  $t > 0$ , then  $T_\alpha(f)(t) = t^{1-\alpha} \frac{df}{dt}(t)$ .

**Remark 2.** *If  $f \in C^1$ , then one has*

$$\lim_{\alpha \rightarrow 1} T_\alpha(f)(t) = f'(t), \quad \lim_{\alpha \rightarrow 0} T_\alpha(f)(t) = t f'(t).$$

**Definition 3** (Conformable fractional integral [15]). *Let  $\alpha \in (0, 1)$ ,  $f : [a, \infty) \rightarrow \mathbb{R}$ . The conformable fractional integral of  $f$  of order  $\alpha$  from  $a$  to  $t$ , denoted by  $I_\alpha^\alpha(f)(t)$ , is defined by*

$$I_\alpha^\alpha(f)(t) := \int_a^t \frac{f(\tau)}{\tau^{1-\alpha}} d\tau,$$

where the above integral is the usual improper Riemann integral.

**Theorem 4** (See [15]). *If  $f$  is a continuous function in the domain of  $I_\alpha^a$ , then  $T_\alpha(I_\alpha^a(f))(t) = f(t)$  for all  $t \geq a$ .*

**Notation 5.** *Let  $0 < a < b$ . We denote by  ${}_a\mathfrak{J}_a^b[f]$  the value of the integral  $\int_a^b \frac{f(t)}{t^{1-\alpha}} dt$ , that is,  ${}_a\mathfrak{J}_a^b[f] := I_\alpha^a(f)(b)$ . We also denote by  $C^{(\alpha)}([a, b], \mathbb{R})$ ,  $0 < a < b$ ,  $\alpha > 0$ , the set of all real-valued functions  $f : [a, b] \rightarrow \mathbb{R}$  that are  $\alpha$ -differentiable and for which the  $\alpha$ -derivative is continuous. We often abbreviate  $C^{(\alpha)}([a, b], \mathbb{R})$  by  $C^{(\alpha)}([a, b])$ .*

**Lemma 6** (See [35]). *Let  $r \in C^{(\alpha)}([a, b])$ ,  $0 < a < b$ , be such that  $r^{(\alpha)}(t) < 0$  on the set  $\{t \in [a, b] : r(t) > 0\}$ . If  $r(a) \leq 0$ , then  $r(t) \leq 0$  for every  $t \in [a, b]$ .*

**Theorem 7** (See [35]). *If  $g \in L^1([a, b])$ , then function  $x : [a, b] \rightarrow \mathbb{R}$  defined by*

$$x(t) := e^{-\frac{1}{\alpha}(\frac{t}{a})^\alpha} \left( e^{\frac{1}{\alpha}} x_0 + {}_a\mathfrak{J}_a^t \left[ \frac{g(s)}{e^{-\frac{1}{\alpha}(\frac{s}{a})^\alpha}} \right] \right)$$

*is solution to problem*

$$\begin{cases} x^{(\alpha)}(t) + \frac{1}{a^\alpha} x(t) = g(t), & t \in [a, b], \quad a > 0, \\ x(a) = x_0. \end{cases}$$

**Proposition 8** (See [35]). *If  $x : (0, \infty) \rightarrow \mathbb{R}$  is  $\alpha$ -differentiable at  $t \in [a, b]$ , then*

$$|x(t)|^{(\alpha)} = \frac{x(t) x^\alpha(t)}{|x(t)|}.$$

For proving our main results, we make use of the following auxiliary definition and lemmas.

**Definition 9** (See p. 112 of [36]). *Let  $X, Y$  be topological spaces. A map  $f : X \rightarrow Y$  is called compact if  $f(X)$  is contained in a compact subset of  $Y$ .*

**Lemma 10** (See [37]). *Let  $M$  be a subset of  $C([0, T])$ . Then  $M$  is precompact if and only if the following conditions hold:*

- (1)  $\{u(t) : u \in M\}$  is uniformly bounded,
- (2)  $\{u(t) : u \in M\}$  is equicontinuous on  $[0, T]$ .

**Lemma 11** (Schauder fixed point theorem [37]). *Let  $U$  be a closed bounded convex subset of a Banach space  $X$ . If  $T : U \rightarrow U$  is completely continuous, then  $T$  has a fixed point in  $U$ .*

### 3. MAIN RESULTS

We begin by introducing the notion of tube solution for problem (1).

**Definition 12.** *Let  $(v, M) \in C^{(\alpha)}([a, T], \mathbb{R}) \times C^{(\alpha)}([a, T], [0, \infty))$ . We say that  $(v, M)$  is a tube solution to problem (1) if*

- (i)  $(y - v(t)) (g(t, y) - v^{(\alpha)}) \leq M(t)M^{(\alpha)}(t)$  for every  $t \in [a, b]$  and every real number  $y$  such that  $|y - v(t)| = M(t)$ ;

- (ii)  $v^{(\alpha)}(t) = g(t, v(t))$  and  $M^{(\alpha)}(t) = 0$  for all  $t \in [a, b]$  such that  $M(t) = 0$ ;
- (iii)  $|u_a - v(a)| \leq M(a)$ .

**Notation 13.** We introduce the following notation:

$$\mathbf{T}(v, M) := \left\{ u \in C^{(\alpha)}([a, T], \mathbb{R}) : |u(t) - v(t)| \leq M(t), t \in [a, T] \right\}.$$

Consider the following problem:

$$\begin{cases} u^{(\alpha)} + \frac{1}{a^\alpha}u(t) = g(t, \tilde{u}(t)) + \frac{1}{a^\alpha}\tilde{u}(t), & t \in [a, T], \quad a > 0, \\ u(a) = u_a, \end{cases} \tag{2}$$

where

$$\tilde{u}(t) := \begin{cases} \frac{M(t)}{|u-v(t)|} (u(t) - v(t)) + v(t) & \text{if } |u - v(t)| > M(t), \\ u(t) & \text{otherwise.} \end{cases} \tag{3}$$

In order to apply Schauder’s fixed point theorem, let us define the operator  $\mathbf{K} : C([a, T]) \rightarrow C([a, T])$  by

$$\mathbf{K}(u)(t) := e^{-\frac{1}{\alpha}(\frac{t}{a})^\alpha} \left( e^{\frac{1}{\alpha}}u_a + {}_\alpha\mathfrak{J}_a^t \left[ \frac{g(s, \tilde{u}(s)) + \frac{1}{a^\alpha}\tilde{u}(s)}{e^{-\frac{1}{\alpha}(\frac{s}{a})^\alpha}} \right] \right).$$

**Proposition 14.** If  $(v, M) \in C^{(\alpha)}([a, T], \mathbb{R}) \times C^{(\alpha)}([a, T], [0, \infty))$  is a tube solution to (1), then  $\mathbf{K} : C([a, T]) \rightarrow C([a, T])$  is compact and problem (2)–(3) has a solution.

*Proof.* Let  $\epsilon > 0$  and  $\{u_n\}_{n \in \mathbb{N}}$  be a sequence of  $C([a, T], \mathbb{R})$  that converges to  $u \in C([a, T], \mathbb{R})$ . Remark that  $L(t) = e^{-\frac{1}{\alpha}(\frac{t}{a})^\alpha}$  is a decreasing function on  $[a, T]$ . Then,  $L(T) \leq L(t) \leq L(a)$  for all  $t \in [a, T]$ . It results that

$$\begin{aligned} |\mathbf{K}(u_n(t)) - \mathbf{K}(u(t))| &= \left| e^{-\frac{1}{\alpha}(\frac{t}{a})^\alpha} \left( e^{\frac{1}{\alpha}}u_a + {}_\alpha\mathfrak{J}_a^t \left[ \frac{g(s, \tilde{u}_n(s)) + \frac{1}{a^\alpha}\tilde{u}_n(s)}{e^{-\frac{1}{\alpha}(\frac{s}{a})^\alpha}} \right] \right) \right. \\ &\quad \left. - e^{-\frac{1}{\alpha}(\frac{t}{a})^\alpha} \left( e^{\frac{1}{\alpha}}u_a + {}_\alpha\mathfrak{J}_a^t \left[ \frac{g(s, \tilde{u}(s)) + \frac{1}{a^\alpha}\tilde{u}(s)}{e^{-\frac{1}{\alpha}(\frac{s}{a})^\alpha}} \right] \right) \right| \\ &\leq \frac{L(a)}{L(T)} {}_\alpha\mathfrak{J}_a^t \left[ \left| \left( g(s, \tilde{u}_n(s)) + \frac{1}{a^\alpha}\tilde{u}_n(s) \right) - \left( g(s, \tilde{u}(s)) + \frac{1}{a^\alpha}\tilde{u}(s) \right) \right| \right] \\ &\leq \frac{L(a)}{L(T)} \left( {}_\alpha\mathfrak{J}_a^t [|g(s, \tilde{u}_n(s)) - g(s, \tilde{u}(s))|] + \frac{1}{a^\alpha} {}_\alpha\mathfrak{J}_a^t [|\tilde{u}_n(s) - \tilde{u}(s)|] \right) \end{aligned}$$

or

$$\begin{aligned} g(s, \tilde{u}_n(s)) - g(s, \tilde{u}(s)) &= \frac{\lambda f(s, \tilde{u}_n(s))}{\left(\int_a^T f(x, \tilde{u}_n(x)) dx\right)^2} - \frac{\lambda f(s, \tilde{u}(s))}{\left(\int_a^T f(x, \tilde{u}(x)) dx\right)^2} \\ &= \frac{\lambda}{\left(\int_a^T f(x, \tilde{u}_n(x)) dx\right)^2} (f(s, \tilde{u}_n(s)) - f(s, \tilde{u}(s))) \\ &\quad + \lambda f(s, \tilde{u}(s)) \left( \frac{1}{\left(\int_a^T f(x, \tilde{u}_n(x)) dx\right)^2} - \frac{1}{\left(\int_a^T f(x, \tilde{u}(x)) dx\right)^2} \right) \\ &= I_1 + I_2. \end{aligned}$$

Since there is a constant  $R > 0$  such that  $\|\tilde{u}\|_{C([a, T], \mathbb{R})} < R$ , there exists an index  $N$  such that  $\|\tilde{u}_n\|_{C([a, T], \mathbb{R})} \leq R$  for all  $n > N$ . Thus,  $f$  is uniformly continuous and, consequently, uniformly bounded on  $[a, T] \times B_R(0)$ . Then, there exist positive constants  $A$  and  $B$  such that  $A \leq f(s, v) \leq B$  for all  $(s, v) \in [a, T] \times B_R(0)$ . Thus, for a well chosen  $D$ , which will be given below, one has

$$\exists \eta > 0, \quad |\tilde{u}_n - \tilde{u}| < \eta, \quad \forall x \in [a, T], \quad |f(x, \tilde{u}_n) - f(x, \tilde{u})| < D$$

and

$$\begin{aligned} |I_1| &\leq \frac{\lambda}{A^2(T-a)^2} |f(s, \tilde{u}_n(s)) - f(s, \tilde{u}(s))| \\ &\leq \frac{\lambda D}{A^2(T-a)^2}. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} |I_2| &\leq \frac{\lambda B \left| \left(\int_a^T f(x, \tilde{u}_n(x)) dx\right)^2 - \left(\int_a^T f(x, \tilde{u}(x)) dx\right)^2 \right|}{\left(\int_a^T f(x, \tilde{u}_n(x)) dx\right)^2 \left(\int_a^T f(x, \tilde{u}(x)) dx\right)^2} \\ &\leq \frac{\lambda B}{A^4(T-a)^4} \left| \left(\int_a^T f(x, \tilde{u}_n) - f(x, \tilde{u}) d\tau\right) \left(\int_a^T f(x, \tilde{u}_n) + f(x, \tilde{u}) d\tau\right) \right| \\ &\leq \frac{2\lambda B^2}{A^4(T-a)^3} \left(\int_a^T |f(x, \tilde{u}_n) - f(x, \tilde{u})| d\tau\right) \\ &\leq \frac{2\lambda B^2 D}{A^4(T-a)^2}. \end{aligned}$$

Then,

$$|I_1 + I_2| \leq \lambda D \left( \frac{1}{A^2(T-a)^2} + \frac{2B^2}{A^4(T-a)^2} \right) := E$$

and it follows that

$$|\mathbf{K}(u_n(t)) - \mathbf{K}(u(t))| \leq \frac{L(a)}{L(T)} \left( {}_\alpha \mathfrak{J}_a^T(E) + \frac{1}{a^\alpha} {}_\alpha \mathfrak{J}_a^T(\eta) \right).$$

On the other hand, we can estimate the right hand side of the above inequality by

$$\begin{aligned} \frac{L(a)}{L(T)} {}_\alpha \mathfrak{J}_a^T(E) &\leq \frac{L(a)}{L(T)} E {}_\alpha \mathfrak{J}_a^T(1) \\ &\leq \frac{L(a)}{L(T)} E \frac{T^\alpha - a^\alpha}{\alpha} = \frac{\epsilon}{2} \end{aligned}$$

and

$$\begin{aligned} \frac{L(a)}{L(T)} \frac{1}{a^\alpha} {}_\alpha \mathfrak{J}_a^T(\eta) &\leq \frac{L(a)}{L(T)} \frac{\eta}{a^\alpha} {}_\alpha \mathfrak{J}_a^T(1) \\ &\leq \frac{L(a)}{L(T)} \frac{\eta}{a^\alpha} \frac{T^\alpha - a^\alpha}{\alpha} = \frac{\epsilon}{2}. \end{aligned}$$

If we set

$$E = \frac{\epsilon}{2} \frac{L(T)}{L(a)} \frac{\alpha}{T^\alpha - a^\alpha} = \frac{\alpha \epsilon L(T)}{2L(a)T^\alpha - a^\alpha}$$

and choose

$$D = \frac{E}{\lambda} \left( \frac{1}{A^2(T-a)^2} + \frac{2B^2}{A^4(T-a)^2} \right)^{-1}$$

and

$$\eta = \frac{\alpha \epsilon a^\alpha L(T)}{2L(a)(T^\alpha - a^\alpha)},$$

then

$$|\mathbf{K}(u_n(t)) - \mathbf{K}(u(t))| \leq \epsilon.$$

This proves the continuity of  $\mathbf{K}$ . To finish the proof of Proposition 14, we prove three technical lemmas.

**Lemma 15.** *If  $f$  is locally Lipschitzian, then the operator  $\mathbf{K}$  is continuous.*

*Proof.* It is a direct consequence of the inequality

$$|\mathbf{K}(u_n(t)) - \mathbf{K}(u(t))| \leq c \|u_n(t) - u(t)\|,$$

which tends to zero as  $n$  goes to  $+\infty$ . □

**Lemma 16.** *The set  $\mathbf{K}(C([a, T]))$  is uniformly bounded.*

*Proof.* Let  $u_n \in C([a, b])$ . We have

$$\begin{aligned} |\mathbf{K}(u_n)(t)| &= \left| e^{-\frac{1}{\alpha}(\frac{t}{a})^\alpha} \left( e^{\frac{1}{\alpha}u_a} + \alpha \mathfrak{J}_a^t \left[ \frac{g(s, \tilde{u}_n(s)) + \frac{1}{a^\alpha} \tilde{u}_n(s)}{e^{-\frac{1}{\alpha}(\frac{s}{a})^\alpha}} \right] \right) \right| \\ &\leq L(a) \left( e^{\frac{1}{\alpha}|u_a|} + \frac{1}{K(T)} \alpha \mathfrak{J}_a^T \left[ \left| g(s, \tilde{u}_n(s)) + \frac{1}{a^\alpha} \tilde{u}_n(s) \right| \right] \right) \\ &\leq L(a) \left( e^{\frac{1}{\alpha}|u_a|} + \frac{1}{L(T)} \alpha \mathfrak{J}_a^T [|g(s, \tilde{u}_n(s))|] + \frac{1}{L(T)a^\alpha} \alpha \mathfrak{J}_a^T [|\tilde{u}_n(s)|] \right). \end{aligned}$$

Similarly to above, there is an  $R > 0$  such that  $|\tilde{u}_n(s)| \leq R$  for all  $s \in [a, T]$  and all  $n \in \mathbb{N}$ . Since function  $f$  is compact on  $[a, T] \times B_R(0)$ , it is uniformly bounded and, as a consequence,  $g$  is also uniformly bounded. We deduce that

$$\begin{aligned} |g(s, \tilde{u}_n)| &\leq \frac{\lambda |f(s, \tilde{u}_n(s))|}{\left( \int_a^T f(s, \tilde{u}_n(s)) ds \right)^2} \\ &\leq \frac{\lambda B}{A^2(T - a)^2} = G. \end{aligned}$$

This ends the proof of Lemma 16. □

**Lemma 17.** *The set  $\mathbf{K}(C([a, T]))$  is equicontinuous.*

*Proof.* For  $t_1, t_2 \in [a, T]$ , we have

$$\begin{aligned} &|\mathbf{K}(u_n)(t_2) - \mathbf{K}(u_n)(t_1)| \\ &= \left| e^{-\frac{1}{\alpha}(\frac{t_2}{a})^\alpha} \left( e^{\frac{1}{\alpha}u_a} + \alpha \mathfrak{J}_a^{t_2} \left[ \frac{g(s, \tilde{u}_n(s)) + \frac{1}{a^\alpha} \tilde{u}_n(s)}{e^{-\frac{1}{\alpha}(\frac{s}{a})^\alpha}} \right] \right) \right. \\ &\quad \left. - e^{-\frac{1}{\alpha}(\frac{t_1}{a})^\alpha} \left( e^{\frac{1}{\alpha}u_a} + \alpha \mathfrak{J}_a^{t_1} \left[ \frac{g(s, \tilde{u}_n(s)) + \frac{1}{a^\alpha} \tilde{u}_n(s)}{e^{-\frac{1}{\alpha}(\frac{s}{a})^\alpha}} \right] \right) \right| \\ &\leq e^{\frac{1}{\alpha}|u_a|} \left| e^{-\frac{1}{\alpha}(\frac{t_1}{a})^\alpha} - e^{-\frac{1}{\alpha}(\frac{t_2}{a})^\alpha} \right| + \left| \alpha \mathfrak{J}_{t_1}^{t_2} \left[ \frac{g(s, \tilde{u}_n(s)) + \frac{1}{a^\alpha} \tilde{u}_n(s)}{e^{-\frac{1}{\alpha}(\frac{s}{a})^\alpha}} \right] \right| \\ &\leq e^{\frac{1}{\alpha}|u_a|} \left| e^{-\frac{1}{\alpha}(\frac{t_1}{a})^\alpha} - e^{-\frac{1}{\alpha}(\frac{t_2}{a})^\alpha} \right| + \frac{1}{L(T)} \left| \alpha \mathfrak{J}_{t_1}^{t_2} \left( G + \frac{R}{a^\alpha} \right) \right| \\ &\leq e^{\frac{1}{\alpha}|u_a|} \left| e^{-\frac{1}{\alpha}(\frac{t_1}{a})^\alpha} - e^{-\frac{1}{\alpha}(\frac{t_2}{a})^\alpha} \right| + \frac{1}{L(T)} \left( G + \frac{R}{a^\alpha} \right) |\alpha \mathfrak{J}_{t_1}^{t_2}(1)| \\ &\leq e^{\frac{1}{\alpha}|u_a|} \left| e^{-\frac{1}{\alpha}(\frac{t_1}{a})^\alpha} - e^{-\frac{1}{\alpha}(\frac{t_2}{a})^\alpha} \right| + \frac{1}{L(T)} \left( G + \frac{R}{a^\alpha} \right) |t_1^\alpha - t_2^\alpha|. \end{aligned}$$

The right hand of the above inequality does not depend on  $u$  and tends to zero as  $t_2 \rightarrow t_1$ . This proves that the sequence  $(\mathbf{K}(u_n))_{n \in \mathbb{N}}$  is equicontinuous. □

By the Arzelà–Ascoli theorem, which asserts that a subset is relatively compact if and only if it is bounded and equicontinuous [36, p. 607],  $\mathbf{K}(C([a, b]))$  is relatively compact and therefore  $\mathbf{K}$  is compact. Consequently, by the Schauder fixed point



theorem, it has a fixed point (see [36]), which is a solution to problem (2)–(3). We have just proved Proposition 14.  $\square$

We are now ready to state the main result of the paper.

**Theorem 18.** *If  $(v, M) \in C^{(\alpha)}([a, T], \mathbb{R}) \times C^{(\alpha)}([a, T], [0, \infty))$  is a tube solution to (1), then problem (1) has a solution  $u \in C^{(\alpha)}([a, T], \mathbb{R}) \cap \mathbf{T}(v, M)$ .*

*Proof.* It remains to show that for every solution  $u$  to problem (2)–(3),  $u \in \mathbf{T}(v, M)$ . We argue as in [35]. Consider the set  $A := \{t \in [a, T] : |u(t) - v(t)| > M(t)\}$ . If  $t \in A$ , then by Proposition 8 one has

$$(|u(t) - v(t)| - M(t))^{(\alpha)} = \frac{(u(t) - v(t))(u^{(\alpha)}(t) - v^{(\alpha)}(t))}{|u(t) - v(t)|} - M^{(\alpha)}(t).$$

Thus, since  $(v, M)$  is a tube solution to problem (1), we have on  $\{t \in A : M(t) > 0\}$  that

$$\begin{aligned} & (|u(t) - v(t)| - M(t))^{(\alpha)} \\ &= \frac{(u(t) - v(t))(u^{(\alpha)}(t) - v^{(\alpha)}(t))}{|u(t) - v(t)|} - M^{(\alpha)}(t) \\ &= \frac{(u(t) - v(t))(g(t, \tilde{u}(t)) + (\frac{1}{a^\alpha}\tilde{u}(t) - \frac{1}{a^\alpha}u(t)) - v^{(\alpha)}(t))}{|u(t) - v(t)|} - M^{(\alpha)}(t) \\ &= \frac{(\tilde{u}(t) - v(t))(g(t, \tilde{u}(t)) - v^{(\alpha)}(t))}{M(t)} + \frac{(\tilde{u}(t) - v(t))(\tilde{u}(t) - u(t))}{a^\alpha M(t)} - M^{(\alpha)}(t) \\ &= \frac{(\tilde{u}(t) - v(t))(g(t, \tilde{u}(t)) - v^{(\alpha)}(t))}{M(t)} \\ &\quad + \left[ \frac{M(t)}{|u(t) - v(t)|} - 1 \right] \frac{|u(t) - v(t)|^2}{a^\alpha |u(t) - v(t)|} - M^{(\alpha)}(t) \\ &= \frac{(\tilde{u}(t) - v(t))(g(t, \tilde{x}(t)) - v^{(\alpha)}(t))}{M(t)} + \left[ \frac{M(t)}{a^\alpha} - \frac{|u(t) - v(t)|}{a^\alpha} \right] - M^{(\alpha)}(t) \\ &\leq \frac{M(t)M^\alpha(t)}{M(t)} + \frac{1}{a^\alpha} [M(t) - |u(t) - v(t)|] - M^{(\alpha)}(t) \\ &< 0. \end{aligned}$$

On the other hand, by Definition 12, we have on  $t \in \{\tau \in A : M(\tau) = 0\}$  that

$$\begin{aligned} & (|u(t)-v(t)| - M(t))^{(\alpha)} \\ &= \frac{(u(t) - v(t)) \left( g(t, \tilde{u}(t)) + \left( \frac{1}{a^\alpha} \tilde{u}(t) - \frac{1}{a^\alpha} u(t) \right) - v^{(\alpha)}(t) \right)}{|u(t) - v(t)|} - M^{(\alpha)}(t) \\ &= \frac{(u(t) - v(t)) \left( g(t, \tilde{u}(t)) - v^{(\alpha)}(t) \right)}{|u(t) - v(t)|} - \frac{1}{a^\alpha} |u(t) - v(t)| - M^{(\alpha)}(t) \\ &< -M^{(\alpha)}(t) \\ &= 0. \end{aligned}$$

If we set  $r(t) := |u(t) - v(t)| - M(t)$ , then  $r^{(\alpha)} < 0$  on  $A := \{t \in [a, T] : r(t) > 0\}$ . Moreover,  $r(a) \leq 0$  since  $u$  satisfies  $|u_a - v(a)| \leq M(a)$ . It follows from Lemma 6 that  $A = \emptyset$ . Therefore,  $u \in T(v, M)$  and the proof of the theorem is complete.  $\square$

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