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Research Article

Some Special Dual Direction Curves

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Abstract
In this paper, some special associated dual curves called dual involute-evolute-direction curves, dual Bertrand-direction curves, and dual Mannheim-direction curves are defined. Some relations between dual Frenet vectors and dual curvatures of these dual associated curves are given. Furthermore, useful methods to construct unit speed dual slant helices from unit speed dual helices by using dual involute-evolute-direction curves and dual Mannheim-direction curves are presented.

Keywords: Associated Curves, Direction Curves, Dual Bertrand Curves, Dual Helix, Dual Involute-Evolute Curves, Dual Mannheim Curves, Dual Slant Helix

Bazı Özel Dual Doğrultu Eğrileri

Öz
Bu çalışmada, dual involüt-evolüt-doğrultu eğrileri, dual Bertrand-doğrultu eğrileri ve dual Mannheim-doğrultu eğrileri denilen bazı özel bağlantılı eğriler tanımlanmıştır. Bu bağlantılı dual eğrilerin dual Frenet vektörleri ve dual eğrilikleri arasında bazı bağıntılar verilmiştir. Ayrıca, dual involüt-evolüt-doğrultu eğrileri ve dual Mannheim-doğrultu eğrileri kullanarak birim hızlı dual helislerden birim hızlı dual slant helisler üretmek için kullanışlı yöntemler sunulmuştur.

Anahtar Kelimeler: Bağlantili Eğriler, Doğrultu Eğrileri, Dual Bertrand Eğriler, Dual Helis, Dual Involüt-Evolüt Eğriler, Dual Mannheim Eğriler, Dual Slant Helis

1. Introduction

In the theory of curves, on the study of associated curves is important and interesting research area. One can find some properties such as Frenet vectors and curvatures of an original curve by using its associated curve. It can be said that a curve can be characterized by using its associated curve. In this research area, some well-known associated curves are the involute-evolute curve couples, Bertrand curves, and Mannheim partner curves. These curves have been studied in Minkowski space and dual space as well as in Euclidean space (Balgetir et al., 2004; Burke, 1960; Izumiya and Takeuchi, 2002; Liu and Wang, 2008; O'Neill, 2006; Turgut and Yılmaz, 2008).

Recently, a new type of associated curves, called direction curves, were introduced by Choi and Kim (2012). They defined direction curve of a given curve as the integral curve of a vector field generated by the Frenet vectors of the original curve. Then, they found some relations concerning Frenet vectors and curvatures between these curves. As applications of these curves, they gave a canonical method to construct general helices and slant helices which are widely used in science. This new type of associated curves has attracted attention of many authors. Choi et al. (2012) studied non-null direction curves, and Qian and Kim (2015) studied null direction curves in Minkowski space E_1^3. Körpınar et al. (2013) used Bishop frame instead of Frenet frame to examine direction curves. Macit and Düldül (2014) defined W-direction curve by using unit Darboux vector field W of a given curve and introduced V-

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direction curve of a curve lying on a surface. Moreover, they studied direction curves in Euclidean 4-space. Kızıltuğ and Önder (2015) investigated direction curves in a three dimensional compact Lie group. In the present paper, we first define dual $\tilde{X}$-direction curve of a given dual curve as an integral curve of dual unit vector field $\tilde{X}$ generated by dual Frenet vectors of the given curve. By considering this definition as an additional constraint, we define some special associated dual curves such as dual involute-evolute-direction curves, dual Bertrand-direction curves, and dual Mannheim-direction curves. For each one of all these dual direction curves, we find some relations between dual Frenet vectors and dual curvatures. Moreover, we investigate the cases that these dual direction curves are dual helices and dual slant helices. For dual involute-evolute-direction curves and dual Mannheim-direction curves, we give useful methods to construct a unit speed dual slant helix from a unit speed dual helix.

2. Preliminaries

In this section, we give a summary of basic concepts concerning dual space, dual curves, and dual special curves.

As introduced by W. Clifford, a dual number can be defined as an ordered pair of real numbers as $\bar{a}=(a,a^*)$, where a and a^* are called real part and dual part of dual number, respectively. A dual number can also be expressed as $\bar{a}=a+\varepsilon a^*$, where ε is dual unit which satisfies the following rules (Veldkamp, 1976):

$$\varepsilon \neq 0,\ 0\varepsilon=\varepsilon 0=0,\ 1\varepsilon=\varepsilon 1=\varepsilon,\ \varepsilon^2=0.$$

The set of dual numbers is denoted by ID. Similarly to the dual numbers, a dual vector can also be defined as $\tilde{a}=\vec{a}+\varepsilon\vec{a}^*$, where $\vec{a},\vec{a}^*\in IR^3$ (Yang, 1963). The set of dual vectors is denoted by ID^3. ID^3 is a module over the ring ID and is called dual space or ID-module. The norm of a dual vector $\tilde{a}=\vec{a}+\varepsilon\vec{a}^*$ is given by

$$\|\tilde{a}\|=\|\vec{a}\|+\varepsilon\frac{\langle \vec{a},\vec{a}^*\rangle}{\|\vec{a}\|}.$$

If $\|\tilde{a}\|=1+\varepsilon 0$, then $\tilde{a}$ is called dual unit vector (Hacısalihoğlu, 1983).

$\tilde{\alpha}(t)=\alpha(t)+\varepsilon\alpha^*(t)$ is a curve in dual space ID^3 and is called dual space curve, where $\alpha(t)$ and $\alpha^*(t)$ are curves in the real space IR^3. The real part $\alpha(t)$ of the dual space curve $\tilde{\alpha}(t)$ is called indicatrix. The dual arc-length of the curve $\tilde{\alpha}(t)$ from t_1 to t is defined by

$$\bar{s}=\int_{t_1}^{t}\|\tilde{\alpha}'(t)\|dt=\int_{t_1}^{t}\|\alpha'(t)\|dt +\varepsilon\int_{t_1}^{t}\langle T,\alpha^{*\prime}(t)\rangle dt=s+\varepsilon s^*,$$

where T is unit tangent vector of the indicatrix $\alpha(t)$ (Önder and Uğurlu, 2013; Yücesan et al., 2007).

Let $\tilde{\alpha}(\bar{s})$ be a dual space curve with dual arc-length parameter $\bar{s}$. Dual unit tangent vector, dual unit principle normal vector, and dual unit binormal vector of $\tilde{\alpha}$ are defined, respectively, as

$$\tilde{T}=\frac{d\tilde{\alpha}}{d\bar{s}},\ \tilde{N}=(1/\bar{\kappa})\tilde{T}',\ \tilde{B}=\tilde{T}\times\tilde{N},$$

where $\bar{\kappa}=\bar{\kappa}(\bar{s})$ is called dual curvature and the prime indicates the derivative with respect to the dual arc-length parameter $\bar{s}$. It is assumed that $\bar{\kappa}(\bar{s})$ is never pure dual number, that is, the real part of the dual curvature is nonzero. The dual frame $\{\tilde{T}(\bar{s}),\tilde{N}(\bar{s}),\tilde{B}(\bar{s})\}$ is called dual Frenet frame along the dual space curve $\tilde{\alpha}(\bar{s})$. The dual Frenet derivative formulae can be given in matrix form as

$$\frac{d}{d\overline{s}}\begin{bmatrix}\tilde{T}\\ \tilde{N}\\ \tilde{B}\end{bmatrix}=\begin{bmatrix}0 & \overline{\kappa} & 0\\ -\overline{\kappa} & 0 & \overline{\tau}\\ 0 & -\overline{\tau} & 0\end{bmatrix}\begin{bmatrix}\tilde{T}\\ \tilde{N}\\ \tilde{B}\end{bmatrix} \tag{2.1}$$

where $\overline{\tau}=\overline{\tau}(s)$ is called dual torsion of $\tilde{\alpha}$ (Yücesan et al., 2007).

Now, we give some definitions and theorems concerning special dual curves and dual curve pairs.

Definition 2.1. (Lee et al., 2011; Özkaldı et al., 2009) A dual curve $\tilde{\alpha}$ with curvature $\overline{\kappa}\neq 0$ and torsion $\overline{\tau}\neq 0$ is called dual helix if its dual unit tangent vector $\tilde{T}$ makes a constant angle $\overline{\theta}$ with a constant dual unit vector $\tilde{d}$; that is, $\left\langle \tilde{T},\tilde{d}\right\rangle=\cos\overline{\theta}$ is constant along the curve.

Theorem 2.2. (Lee et al., 2011) Let $\tilde{\alpha}$ be a unit speed dual curve with curvature $\overline{\kappa}\neq 0$. $\tilde{\alpha}$ is a dual helix if and only if there exists a constant dual number $\overline{c}$ such that $\overline{\tau}=\overline{c}\,\overline{\kappa}$.

Definition 2.3. (Lee et al., 2011) A dual curve $\tilde{\alpha}$ is called dual slant helix if its dual unit principle normal vector $\tilde{N}$ makes a constant dual angle $\overline{\theta}$ with a constant dual unit vector $\tilde{d}$; that is, $\left\langle \tilde{N},\tilde{d}\right\rangle=\cos\overline{\theta}$ is constant along the curve.

Theorem 2.4. (Lee et al., 2011) Let $\tilde{\alpha}: ID\rightarrow ID^3$ be a unit speed dual curve with curvature $\overline{\kappa}\neq 0$. $\tilde{\alpha}$ is a dual slant helix if and only if

$$\frac{\overline{\kappa}^2}{(\overline{\kappa}^2+\overline{\tau}^2)^{3/2}}\left(\frac{\overline{\tau}}{\overline{\kappa}}\right)'$$

is constant.

Definition 2.5. (Şenyurt et al., 2015) Let $\tilde{\alpha}: ID\rightarrow ID^3$ and $\tilde{\alpha}^*: ID\rightarrow ID^3$ be dual curves. If the dual unit tangent vector of $\tilde{\alpha}$ is orthogonal to the dual unit tangent vector of $\tilde{\alpha}^*$, then the dual curve $\tilde{\alpha}$ is called dual involute of the dual curve $\tilde{\alpha}^*$, and $\tilde{\alpha}^*$ is also called dual evolute of the dual curve $\tilde{\alpha}$.

Theorem 2.6. (Şenyurt et al., 2015) Let $\tilde{\alpha}: ID\rightarrow ID^3$ and $\tilde{\alpha}^*: ID\rightarrow ID^3$ be dual curves with dual Frenet frame $\{\tilde{T},\tilde{N},\tilde{B}\}$ and $\{\tilde{T}^*,\tilde{N}^*,\tilde{B}^*\}$, respectively. If the dual curve $\tilde{\alpha}$ is dual involute of $\tilde{\alpha}^*$, then $\tilde{T}=\tilde{N}^*$.

Definition 2.7. (Arslan Güven and Ağaoğlu, 2014) Let $\tilde{\alpha}: ID\rightarrow ID^3$ and $\tilde{\alpha}^*: ID\rightarrow ID^3$ be dual curves. If these exists a corresponding relationship between the dual space curves $\tilde{\alpha}$ and $\tilde{\alpha}^*$ so that the dual unit principle normal vectors of $\tilde{\alpha}$ and $\tilde{\alpha}^*$ are linear dependent to each other at the corresponding points of the dual curves, that is, $\tilde{N}=\tilde{N}^*$, then $\tilde{\alpha}$ and $\tilde{\alpha}^*$ are called dual Bertrand curves.

Definition 2.8. (Güngör and Tosun, 2010; Özkaldı et al., 2009) Let $\tilde{\alpha}: ID\rightarrow ID^3$ and $\tilde{\alpha}^*: ID\rightarrow ID^3$ be dual curves. If these exists a corresponding relationship between the dual space curves $\tilde{\alpha}$ and $\tilde{\alpha}^*$ so that the dual unit principle normal vector of $\tilde{\alpha}$ coincides with the dual unit binormal vector of $\tilde{\alpha}^*$ at the corresponding points of the dual curves, that is, $\tilde{B}=\tilde{N}^*$, then $\tilde{\alpha}$ is called a dual Mannheim curve and $\tilde{\alpha}^*$ is also called a dual Mannheim partner curve of $\tilde{\alpha}$.

3. Dual Direction Curves

Let $\tilde{\alpha}: ID\rightarrow ID^3$ be a unit speed dual curve with dual Frenet frame $\left\{\tilde{T}(\overline{s}),\tilde{N}(\overline{s}),\tilde{B}(\overline{s})\right\}$ and dual curvatures $\{\overline{\kappa}(\overline{s}),\overline{\tau}(\overline{s})\}$. Consider a dual vector field $\tilde{X}$ expressed by

$$\begin{aligned}\tilde{X}(\overline{s})=\overline{x}(\overline{s})\tilde{T}(\overline{s})+\overline{y}(\overline{s})\tilde{N}(\overline{s})\\ +\overline{z}(\overline{s})\tilde{B}(\overline{s})\end{aligned} \tag{3.1}$$

where $\overline{x}$, $\overline{y}$ and $\overline{z}$ are dual functions of $\overline{s}$ which is dual arc-length parameter of the dual curve $\tilde{\alpha}$. It can be assumed that $\tilde{X}$ is unit. Thus, the following equality holds

$$\bar{x}^2(\bar{s})+\bar{y}^2(\bar{s})+\bar{z}^2(\bar{s})=1. \qquad (3.2)$$

By differentiating equation (3.2), we get

$$\bar{x}(\bar{s})\bar{x}'(\bar{s})+\bar{y}(\bar{s})\bar{y}'(\bar{s})+\bar{z}(\bar{s})\bar{z}'(\bar{s})=0. \quad (3.3)$$

Now we can give the definitions of dual $\tilde{X}$-direction curve and dual $\tilde{X}$-donor curve as follows.

Definition 3.1. Let $\tilde{\alpha}$ be a dual curve and $\tilde{X}$ be a dual unit vector field satisfying the equations (3.1) and (3.2). The dual integral curve $\tilde{\alpha}^*: ID \to ID^3$ of $\tilde{X}$ is called a dual $\tilde{X}$-direction curve of $\tilde{\alpha}$. The curve $\tilde{\alpha}$ whose dual $\tilde{X}$-direction curve is $\tilde{\alpha}^*$ is also called the dual $\tilde{X}$-donor curve of $\tilde{\alpha}^*$.

Remark 3.2. Since $\tilde{X}$ is unit, it is clear that the dual arc-length parameter $\bar{s}^*$ of the $\tilde{X}$-direction curve $\tilde{\alpha}^*$ of $\tilde{\alpha}$ is equal to $\bar{s}+\bar{c}$, where $\bar{c}$ is a constant dual number. Without loss of generality, we assume that $\bar{s}^*=\bar{s}$.

Let $\tilde{\alpha}$ be a unit speed dual curve with the dual Frenet frame $\{\tilde{T},\tilde{N},\tilde{B}\}$ and the curvatures $\{\bar{\kappa},\bar{\tau}\}$, $\tilde{X}$ be a dual unit vector field satisfying the equations (3.1) and (3.2), and $\tilde{\alpha}^*$ be dual $\tilde{X}$-direction curve of $\tilde{\alpha}$ with the dual Frenet frame $\{\tilde{T}^*,\tilde{N}^*,\tilde{B}^*\}$ and the curvatures $\{\bar{\kappa}^*,\bar{\tau}^*\}$. By differentiating (3.1) with respect to $\bar{s}$, using dual Frenet formulae, and considering the fact that $\dfrac{d\tilde{\alpha}^*}{d\bar{s}}=\tilde{T}^*=\tilde{X}$, we have

$$\begin{aligned}\bar{\kappa}^*\tilde{N}^* &= (\bar{x}'-\bar{y}\bar{\kappa})\tilde{T}+(\bar{y}'+\bar{x}\bar{\kappa}-\bar{z}\bar{\tau})\tilde{N}\\ &\quad+(\bar{z}'+\bar{y}\bar{\tau})\tilde{B}\end{aligned}. \quad (3.4)$$

By using equality (3.4), we can give definitions of dual involute-evolute-direction curves, dual Bertrand-direction curves, and dual Mannheim curves, and study some properties of these curves.

4. Dual Involute-Evolute-Direction Curves

In this section, we define dual involute-evolute-direction curves and obtain some relations between these curves.

Definition 4.1. Let $\tilde{\alpha}: ID \to ID^3$ be a dual curve in dual space and $\tilde{\alpha}^*: ID \to ID^3$ be dual $\tilde{X}$-direction curve of $\tilde{\alpha}$. If $\tilde{\alpha}^*$ is a dual evolute of $\tilde{\alpha}$, then $\tilde{\alpha}^*$ is called dual evolute-direction curve of $\tilde{\alpha}$. Then, $\tilde{\alpha}$ is also called dual involute-donor curve of $\tilde{\alpha}^*$.

Proposition 4.2. Let $\tilde{\alpha}: ID \to ID^3$ be a unit speed dual curve in dual space. $\tilde{\alpha}^*$ is dual evolute-direction curve of $\tilde{\alpha}$ if and only if the dual functions in (3.1) are as follows

$$\bar{x}(\bar{s})=0,\ \bar{y}(\bar{s})=\sin\left(\int\bar{\tau}\,d\bar{s}\right),$$
$$\bar{z}(\bar{s})=\cos\left(\int\bar{\tau}\,d\bar{s}\right).$$

Proof. From Theorem 2.6, we know that $\tilde{N}^*=\tilde{T}$. By using equality (3.4), we have the following system of differential equations

$$\left.\begin{aligned}&\bar{x}'-\bar{y}\bar{\kappa}=\bar{\kappa}^*\\ &\bar{y}'+\bar{x}\bar{\kappa}-\bar{z}\bar{\tau}=0\\ &\bar{z}'+\bar{y}\bar{\tau}=0\end{aligned}\right\} \qquad (4.1)$$

Multiplying the first, second and third equations in (4.1) with $\bar{x}$, $\bar{y}$ and $\bar{z}$, respectively, adding the results, and using equation (3.3), we find $\bar{x}=0$. By substituting $\bar{x}=0$ into system (4.1), the solution is found as follows

$$\left\{\bar{x}(\bar{s})=0,\ \bar{y}(\bar{s})=\sin\left(\int\bar{\tau}\,d\bar{s}\right),\ \bar{z}(\bar{s})=\cos\left(\int\bar{\tau}\,d\bar{s}\right)\right\}.$$

From Proposition 4.2, we have a method to construct a unit speed dual evolute curve of a unit speed dual curve. Moreover, this construction can be achieved just by using dual Frenet vectors $\tilde{N},\tilde{B}$ and dual torsion $\bar{\tau}$ of a given dual curve $\tilde{\alpha}$.

Now we can give the relationships between dual curvatures of the dual curve $\tilde{\alpha}$ and its dual evolute-direction curve $\tilde{\alpha}^*$.

Theorem 4.3. Let $\tilde{\alpha}$ be a unit speed dual curve in dual space and $\tilde{\alpha}^*$ be dual evolute-direction curve of $\tilde{\alpha}$. The dual curvature and dual torsion of the dual evolute-direction curve $\tilde{\alpha}^*$ can be given in terms of the dual curvatures of $\tilde{\alpha}$ as follows

$$\bar{\kappa}^* = -\bar{\kappa}\sin\left(\int \bar{\tau}\, d\bar{s}\right)$$

and

$$\bar{\tau}^* = \bar{\kappa}\cos\left(\int \bar{\tau}\, d\bar{s}\right)$$

respectively.

Proof. From the first equation of system (4.1), we can see that $\bar{\kappa}^* = -\bar{\kappa}\sin\left(\int \bar{\tau}\, d\bar{s}\right)$. From Proposition (4.2), we find $\tilde{X} = \tilde{T}^* = \sin\left(\int \bar{\tau}\, d\bar{s}\right)\tilde{N} + \cos\left(\int \bar{\tau}\, d\bar{s}\right)\tilde{B}$, and from Theorem 2.6, we have $\tilde{N}^* = \tilde{T}$. The dual unit binormal vector $\tilde{B}^*$ of the dual curve $\tilde{\alpha}^*$ can be found as

$$\tilde{B}^* = \tilde{T}^* \times \tilde{N}^* = \cos\left(\int \bar{\tau}\, d\bar{s}\right)\tilde{N} - \sin\left(\int \bar{\tau}\, d\bar{s}\right)\tilde{B}.$$

By differentiating dual unit binormal vector and using the third equality of (2.1), we get $\bar{\tau}^* = \bar{\kappa}\cos\left(\int \bar{\tau}\, d\bar{s}\right)$.

On the other hand, the dual curvatures of the dual involute-donor curve $\tilde{\alpha}$ can be given in terms of the dual curvatures of $\tilde{\alpha}^*$ as follows.

Theorem 4.4. Let $\tilde{\alpha}$ be a unit speed dual curve in dual space and $\tilde{\alpha}^*$ be dual evolute-direction curve of $\tilde{\alpha}$. The dual curvature and dual torsion of the dual involute-donor curve $\tilde{\alpha}$ can be given in terms of the dual curvatures of $\tilde{\alpha}^*$ as follows

$$\bar{\kappa} = \sqrt{\bar{\kappa}^{*2} + \bar{\tau}^{*2}}$$

and

$$\bar{\tau} = \frac{\bar{\kappa}^{*2}}{\bar{\kappa}^{*2} + \bar{\tau}^{*2}}\left(\frac{\bar{\tau}^*}{\bar{\kappa}^*}\right)'$$

respectively.

Proof. By differentiating the equality $\tilde{N}^* = \tilde{T}$ and using (2.1), we get $\bar{\kappa} = \sqrt{\bar{\kappa}^{*2} + \bar{\tau}^{*2}}$. By using the dual curvatures of the dual curve $\tilde{\alpha}^*$ found in Theorem 4.3, we have $\frac{\bar{\tau}^*}{\bar{\kappa}^*} = -\cot\left(\int \bar{\tau}\, d\bar{s}\right)$, and so $\int \bar{\tau}\, d\bar{s} = -\operatorname{arc}\cot\left(\frac{\bar{\tau}^*}{\bar{\kappa}^*}\right)$. By differentiating the last equality, we get $\bar{\tau} = \frac{\bar{\kappa}^{*2}}{\bar{\kappa}^{*2} + \bar{\tau}^{*2}}\left(\frac{\bar{\tau}^*}{\bar{\kappa}^*}\right)'$.

From Theorem 4.4, we have the following corollary.

Corollary 4.5. Let $\tilde{\alpha}$ be a unit speed dual curve in dual space and $\tilde{\alpha}^*$ be dual evolute-direction curve of $\tilde{\alpha}$. Then the following relation satisfies

$$\frac{\bar{\tau}}{\bar{\kappa}} = \frac{\bar{\kappa}^{*2}}{\left(\bar{\kappa}^{*2} + \bar{\tau}^{*2}\right)^{3/2}}\left(\frac{\bar{\tau}^*}{\bar{\kappa}^*}\right)'.$$

Thus, we can give the following theorem which can be used to construct a unit speed dual slant helix from a unit speed dual helix by using dual involute-evolute-direction curves.

Theorem 4.6. Let $\tilde{\alpha}$ be a unit speed dual curve with nonzero curvatures in dual space and $\tilde{\alpha}^*$ be dual evolute-direction curve of $\tilde{\alpha}$. $\tilde{\alpha}$ is a dual helix if and only if $\tilde{\alpha}^*$ is a dual slant helix.

Proof. The proof is clear from Corollary 4.5, Theorem 2.2, and Theorem 2.4.

5. Dual Bertrand-Direction Curves

In this section, we define dual Bertrand-direction curves and obtain some relations between these curves.

Definition 5.1. Let $\tilde{\alpha}: ID \to ID^3$ be a dual curve in dual space and $\tilde{\alpha}^*: ID \to ID^3$ be dual $\tilde{X}$ -direction curve of $\tilde{\alpha}$. If $\tilde{\alpha}^*$ is a dual Bertrand curve of $\tilde{\alpha}$, then $\tilde{\alpha}^*$ is called dual Bertrand-direction curve of $\tilde{\alpha}$. Moreover, $\tilde{\alpha}$ is also called dual Bertrand-donor curve of $\tilde{\alpha}^*$.

Proposition 5.2. Let $\tilde{\alpha}: ID \to ID^3$ be a unit speed dual curve in dual space. $\tilde{\alpha}^*$ is dual Bertrand-direction curve of $\tilde{\alpha}$ if and only if the dual functions in (3.1) are as follows

$$\bar{x}(\bar{s}) = \cos\bar{\theta}\,,\;\; \bar{y}(\bar{s}) = 0\,,\; \bar{z}(\bar{s}) = \sin\bar{\theta}\,,$$

where $\bar{\theta}$ is a constant dual angle.

Proof. From Definition 2.7, we know that $\tilde{N}^* = \tilde{N}$. By using equality (3.4), we have the following system of differential equations

$$\left.\begin{aligned} &\bar{x}' - \bar{y}\bar{\kappa} = 0 \\ &\bar{y}' + \bar{x}\bar{\kappa} - \bar{z}\bar{\tau} = \bar{\kappa}^* \\ &\bar{z}' + \bar{y}\bar{\tau} = 0 \end{aligned}\right\} \qquad (5.1)$$

Multiplying the first, second and third equations in (5.1) with $\bar{x}$, $\bar{y}$ and $\bar{z}$, respectively, adding the results, and using equation (3.3), we find $\bar{y} = 0$. By substituting $\bar{y} = 0$ into system (5.1), we get $\bar{x} = \bar{c}_1$ and $\bar{z} = \bar{c}_2$, where $\bar{c}_1$ and $\bar{c}_2$ are constant dual numbers. Since $\tilde{X}$ is unit, we can give the solution of system (5.1) as follows

$$\left\{\, \bar{x}(\bar{s}) = \cos\bar{\theta}, \bar{y}(\bar{s}) = 0, \bar{z}(\bar{s}) = \sin\bar{\theta} \,\right\},$$

where $\bar{\theta}$ is the dual constant angle between dual unit tangent vectors of the dual curves. From Proposition 5.2, we have a method to construct a unit speed dual Bertrand curve of a unit speed dual curve. Moreover, this construction can be achieved just by using dual Frenet vectors $\tilde{T}, \tilde{B}$ of a given dual curve $\tilde{\alpha}$, and the dual constant angle $\bar{\theta}$ which is dual angle between dual unit tangent vectors of the dual curves. Now we can give the relationships between dual curvatures of the dual curve $\tilde{\alpha}$ and its dual Bertrand-direction curve $\tilde{\alpha}^*$.

Theorem 5.3. Let $\tilde{\alpha}$ be a unit speed dual curve in dual space and $\tilde{\alpha}^*$ be dual Bertrand-direction curve of $\tilde{\alpha}$. The dual curvature and dual torsion of the dual Bertrand-direction curve $\tilde{\alpha}^*$ can be given in terms of the dual curvatures of $\tilde{\alpha}$ as follows

$$\bar{\kappa}^* = \bar{\kappa}\cos\bar{\theta} - \bar{\tau}\sin\bar{\theta}$$

and

$$\bar{\tau}^* = \bar{\kappa}\sin\bar{\theta} + \bar{\tau}\cos\bar{\theta}$$

respectively.

Proof. From the second equation of system (5.1), we can see that $\bar{\kappa}^* = \bar{\kappa}\cos\bar{\theta} - \bar{\tau}\sin\bar{\theta}$. From Proposition (5.2), we find $\tilde{X} = \tilde{T}^* = \cos\bar{\theta}\tilde{T} + \sin\bar{\theta}\tilde{B}$, and from Definition 2.7, we have $\tilde{N}^* = \tilde{N}$. Thus, the dual unit binormal vector $\tilde{B}^*$ of the dual curve $\tilde{\alpha}^*$ can be found as $\tilde{B}^* = \tilde{T}^* \times \tilde{N}^* = -\sin\bar{\theta}\tilde{T} + \cos\bar{\theta}\tilde{B}$. By differentiating dual unit binormal vector and using the third equality of (2.1), we find the dual torsion of the dual curve $\tilde{\alpha}^*$ as $\bar{\tau}^* = \bar{\kappa}\sin\bar{\theta} + \bar{\tau}\cos\bar{\theta}$.

On the other hand, the dual curvatures of the dual Bertrand-donor curve $\tilde{\alpha}$ can be found in terms of the dual curvatures of $\tilde{\alpha}^*$ as follows

$$\bar{\kappa} = \bar{\kappa}^*\cos\bar{\theta} + \bar{\tau}^*\sin\bar{\theta}$$

and

$$\bar{\tau} = -\bar{\kappa}^*\sin\bar{\theta} + \bar{\tau}^*\cos\bar{\theta}\,.$$

From Theorem 5.3, we have the following corollary.

Corollary 5.4. Let $\tilde{\alpha}$ be a unit speed dual curve in dual space and $\tilde{\alpha}^*$ be dual Bertrand-direction curve of $\tilde{\alpha}$. Then the following relations satisfy

i) $$\frac{\bar{\tau}}{\bar{\kappa}} = \frac{-\sin\theta + \dfrac{\bar{\tau}^*}{\bar{\kappa}^*}\cos\bar{\theta}}{\cos\theta + \dfrac{\bar{\tau}^*}{\bar{\kappa}^*}\sin\bar{\theta}},$$

ii) $$\frac{\bar{\kappa}^2}{\left(\bar{\kappa}^2+\bar{\tau}^2\right)^{3/2}}\left(\frac{\bar{\tau}}{\bar{\kappa}}\right)' = \frac{\bar{\kappa}^{*2}}{\left(\bar{\kappa}^{*2}+\bar{\tau}^{*2}\right)^{3/2}}\left(\frac{\bar{\tau}^*}{\bar{\kappa}^*}\right)'.$$

Thus, we can give the following theorem which can be used to construct unit speed dual helices and dual slant helices by using dual Bertrand-direction curves.

Theorem 5.5. Let $\tilde{\alpha}$ be a unit speed dual curve with nonzero curvatures in dual space and $\tilde{\alpha}^*$ be dual Bertrand-direction curve of $\tilde{\alpha}$. Then

i) $\tilde{\alpha}$ is a dual helix if and only if $\tilde{\alpha}^*$ is a dual helix.

ii) $\tilde{\alpha}$ is a dual slant helix if and only if $\tilde{\alpha}^*$ is a dual slant helix.

Proof. The proof is clear from Corollary 5.4, Theorem 2.2, and Theorem 2.4.

6. Dual Mannheim-Direction Curves

In this section, we define dual Mannheim-direction curves and obtain some relations between these curves.

Definition 6.1. Let $\tilde{\alpha}: ID \to ID^3$ be a dual curve in dual space and $\tilde{\alpha}^*: ID \to ID^3$ be dual $\tilde{X}$ -direction curve of $\tilde{\alpha}$. If $\tilde{\alpha}^*$ is a dual Mannheim curve of $\tilde{\alpha}$, then $\tilde{\alpha}^*$ is called dual Mannheim-direction curve of $\tilde{\alpha}$. $\tilde{\alpha}$ is also called dual Mannheim-donor curve of $\tilde{\alpha}^*$.

Proposition 6.2. Let $\tilde{\alpha}: ID \to ID^3$ be a unit speed dual curve in dual space. $\tilde{\alpha}^*$ is dual Mannheim-direction curve of $\tilde{\alpha}$ if and only if the dual functions in (3.1) are as follows

$$\bar{x}(\bar{s}) = \sin\left(\int \bar{\kappa}\,d\bar{s}\right),$$

$$\bar{y}(\bar{s}) = \cos\left(\int \bar{\kappa}\,d\bar{s}\right),\ \bar{z}(\bar{s}) = 0.$$

Proof. From Definition 2.8, we know that $\tilde{N}^* = \tilde{B}$. By using equality (3.4), we have the following system of differential equations

$$\left.\begin{aligned} &\bar{x}' - \bar{y}\bar{\kappa} = 0 \\ &\bar{y}' + \bar{x}\bar{\kappa} - \bar{z}\bar{\tau} = 0 \\ &\bar{z}' + \bar{y}\bar{\tau} = \bar{\kappa}^* \end{aligned}\right\} \qquad (6.1)$$

Multiplying the first, second and third equations in (6.1) with $\bar{x}$, $\bar{y}$ and $\bar{z}$, respectively, adding the results, and using equation (3.3), we find $\bar{z} = 0$.

By substituting $\bar{z} = 0$ into system (6.1), the solution can be obtained as follows

$$\left\{\bar{x}(\bar{s}) = \sin\left(\int \bar{\kappa}\,d\bar{s}\right), \bar{y}(\bar{s}) = \cos\left(\int \bar{\kappa}\,d\bar{s}\right), \bar{z}(\bar{s}) = 0\right\}.$$

From Proposition 6.2, we have a method to construct a unit speed dual Mannheim curve of a unit speed dual curve. Moreover, this construction can be achieved just by using dual unit Frenet vectors $\tilde{T}$ and $\tilde{N}$, and dual curvature $\bar{\kappa}$ of a given dual curve $\tilde{\alpha}$. Now we can give the relationships between dual curvatures of the dual curve $\tilde{\alpha}$ and its dual Mannheim-direction curve $\tilde{\alpha}^*$.

Theorem 6.3. Let $\tilde{\alpha}$ be a unit speed dual curve in dual space and $\tilde{\alpha}^*$ be dual Mannheim-direction curve of $\tilde{\alpha}$. The dual curvature and dual torsion of the dual Mannheim-direction curve $\tilde{\alpha}^*$ can be given in terms of the dual curvatures of $\tilde{\alpha}$ as follows

$$\bar{\kappa}^* = \bar{\tau}\cos\left(\int \bar{\kappa}\,d\bar{s}\right)$$

and

$$\bar{\tau}^* = \bar{\tau}\sin\left(\int \bar{\kappa} d\bar{s}\right)$$

respectively.

Proof. From the first equation of system (4.1), we can easily see that $\bar{\kappa}^* = \bar{\tau}\cos\left(\int \bar{\kappa} d\bar{s}\right)$. From Proposition (6.2), we find $\tilde{X} = \tilde{T}^* = \sin\left(\int \bar{\kappa} d\bar{s}\right)\tilde{T} + \cos\left(\int \bar{\kappa} d\bar{s}\right)\tilde{N}$, and from Definition (2.8), we have $\tilde{N}^* = \tilde{B}$. The dual unit binormal vector $\tilde{B}^*$ of the dual curve $\tilde{\alpha}^*$ can be found as $\tilde{B}^* = \tilde{T}^* \times \tilde{N}^* = \cos\left(\int \bar{\kappa} d\bar{s}\right)\tilde{T} - \sin\left(\int \bar{\kappa} d\bar{s}\right)\tilde{N}$. By differentiating dual unit binormal vector and using the third equality of (2.1), we get $\bar{\tau}^* = \bar{\tau}\sin\left(\int \bar{\kappa} d\bar{s}\right)$.

On the other hand, the dual curvatures of the Mannheim-donor curve $\tilde{\alpha}$ can be found in terms of the dual curvatures of $\tilde{\alpha}^*$ as given in the following theorem.

Theorem 6.4. Let $\tilde{\alpha}$ be a unit speed dual curve in dual space and $\tilde{\alpha}^*$ be dual Mannheim-direction curve of $\tilde{\alpha}$. The dual curvature and dual torsion of the dual Mannheim-donor curve $\tilde{\alpha}$ can be given in terms of the dual curvatures of $\tilde{\alpha}^*$ as follows

$$\bar{\kappa} = \frac{\bar{\kappa}^{*2}}{\bar{\kappa}^{*2} + \bar{\tau}^{*2}}\left(\frac{\bar{\tau}^*}{\bar{\kappa}^*}\right)'$$

and

$$\bar{\tau} = \sqrt{\bar{\kappa}^{*2} + \bar{\tau}^{*2}}$$

respectively.

Proof. By a simple computation with dual curvatures in Theorem 6.3, we can see the dual torsion of dual Mannheim-donor curve $\tilde{\alpha}$ as $\bar{\tau} = \sqrt{\bar{\kappa}^{*2} + \bar{\tau}^{*2}}$. Moreover, from Theorem 6.3, we have $\dfrac{\bar{\tau}^*}{\bar{\kappa}^*} = \tan\left(\int \bar{\kappa} d\bar{s}\right)$, and so $\int \bar{\kappa} d\bar{s} = \arctan\left(\dfrac{\bar{\tau}^*}{\bar{\kappa}^*}\right)$. By differentiating the last equality, we get $\bar{\kappa} = \dfrac{\bar{\kappa}^{*2}}{\bar{\kappa}^{*2} + \bar{\tau}^{*2}}\left(\dfrac{\bar{\tau}^*}{\bar{\kappa}^*}\right)'$.

From Theorem 6.4, we have the following corollary.

Corollary 6.5. Let $\tilde{\alpha}$ be a unit speed dual curve in dual space and $\tilde{\alpha}^*$ be dual Mannheim-direction curve of $\tilde{\alpha}$. Then the following relation satisfies

$$\frac{\bar{\tau}}{\bar{\kappa}} = \frac{1}{\dfrac{\bar{\kappa}^{*2}}{\left(\bar{\kappa}^{*2} + \bar{\tau}^{*2}\right)^{3/2}}\left(\dfrac{\bar{\tau}^*}{\bar{\kappa}^*}\right)'}.$$

Thus, we can give the following theorem which can be used to construct a unit speed dual slant helix from a unit speed dual helix by using dual Mannheim-direction curves.

Theorem 6.6. Let $\tilde{\alpha}$ be a unit speed dual curve with nonzero curvatures in dual space and $\tilde{\alpha}^*$ be dual Mannheim-direction curve of $\tilde{\alpha}$. $\tilde{\alpha}$ is a dual helix if and only if $\tilde{\alpha}^*$ is a dual slant helix.

Proof. The proof is clear from Corollary 6.5, Theorem 2.2, and Theorem 2.4.

7. References

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