Afyon Kocatepe Üniversitesi Fen ve Mühendislik Bilimleri Dergisi
Afyon Kocatepe University – Journal of Science and Engineering
https://dergipark.org.tr/tr/pub/akufemubid

Araştırma Makalesi / Research Article
e-ISSN: 2149-3367
DOI: https://doi.org/10.35414/akufemubid.1964187
CITE: Afyon Kocatepe University Journal of Science and Engineering. **26(4)** (2026) 897-907

Standing Waves of the NLS Equation on Locally Finite Weighted Graphs: A Linking Geometry Approach

***Makale Bilgisi / Article Info**
Alındı/Received: 04.06.2026
Kabul/Accepted: 16.07.2026
Yayımlandı/Published: : 03.08.2026

Lokal Sonlu Ağırlıklı Graflar Üzerinde Doğrusal Olmayan Schrödinger Denkleminin Durağan Dalga Çözümleri: Linking Geometrisi Yaklaşımı

Setenay AKDUMAN*

İzmir Democracy University, Department of Mathematics, İzmir, Turkey

Abstract

We investigate wave solutions of the nonlinear Schrödinger equation $(H-\lambda)u = f(x,u)$ on a locally finite weighted graph, where H is the associated Schrödinger operator. The central hypothesis is a tail summability condition $1/V \in \ell^1_m(\mathcal{V})$ on the effective potential V, which, together with canonical compactifiability of the graph, yields a compact Sobolev-type embedding $E \hookrightarrow \ell^{\mathrm{p}}{}_{\mathrm{m}}(\mathcal{V})$ for the full range $1 \le p < \infty$ and forces the spectrum of H to be purely discrete. This framework does not require the global lower bound $\inf_{x\in\mathcal{V}} m(x) > 0$ and is strictly incomparable with the measure-theoretic potential conditions in the existing literature, thereby covering a genuinely broader class of weighted graphs. Under the Ambrosetti–Rabinowitz super linearity condition on f, we establish the Palais–Smale condition for the associated energy functional for every $\lambda \in \mathbb{R}$ and verify the linking geometry through a spectral decomposition of the energy space. An application of the Linking theorem then yields at least one nontrivial solution for every value of the spectral parameter. When the nonlinearity is additionally odd, the Symmetric Mountain Pass Theorem produces infinitely many pairs of solutions.

Keywords: Nonlinear Schrödinger equation; Locally finite weighted graphs; Standing waves; Linking theorem; Palais-Smale condition; Variational methods.
AMS Subject Classifications (2020): 35Q55, 35A15, 47J30, 39A12, 39A70, 58E05, 78A40

Öz

Bu çalışmada, lokal sonlu ağırlıklı bir graf üzerinde tanımlanan doğrusal olmayan Schrödinger denkleminin $(H-\lambda)u = f(x,u)$ durağan dalga çözümlerini inceliyoruz. Burada H ilgili Schrödinger operatörünü göstermektedir. Çalışmanın temel varsayımı, etkin potansiyel V için $1/V \in \ell^1_m(\mathcal{V})$ biçimindeki kuyruk toplanabilirlik koşuludur. Bu koşul, grafın kanonik kompaktlanabilirliği ile birlikte, $1 \le p < \infty$ aralığındaki tüm p değerleri için $E \hookrightarrow \ell^{\mathrm{p}}{}_{\mathrm{m}}(\mathcal{V})$ şeklinde kompakt bir Sobolev tipi gömülme sağlamakta ve H operatörünün spektrumunun tamamen diskrit olmasına yol açmaktadır. Bu çerçeve, $\inf_{x\in\mathcal{V}} m(x) > 0$ biçimindeki global bir alt sınır koşulunu gerektirmemekte olup, mevcut literatürde kullanılan ölçü-teorik potansiyel koşullarıyla kesin olarak karşılaştırılamaz niteliktedir. Bu sayede gerçekten daha geniş bir ağırlıklı graf sınıfını kapsamaktadır. Doğrusal olmayan f fonksiyonu için Ambrosetti–Rabinowitz süper lineerlik koşulu altında, ilişkili enerji fonksiyonelinin her $\lambda \in \mathbb{R}$ değeri için Palais–Smale koşulunu sağladığını gösteriyor ve enerji uzayının spektral ayrışımı aracılığıyla Linking geometrisini doğruluyoruz. Daha sonra Linking teoremi uygulanarak, spektral parametrenin her değeri için en az bir adet aşikar olmayan çözümün varlığı elde edilmektedir. Doğrusal olmayan f fonksiyonu tek fonksiyon olduğunda ise Simetrik Mountain Pass teoremi sonsuz sayıda çözüm çifti üretmektedir.

Anahtar Kelimeler: Doğrusal olmayan Schrödinger denklemi; Lokal sonlu ağırlıklı graflar; Durağan dalgalar; Linking teoremi; Palais-Smale koşulu; Varyasyonel yöntemler.
AMS Konu Sınıflandırması (2020): 35Q55, 35A15, 47J30, 39A12, 39A70, 58E05, 78A40

1. Introduction

The nonlinear Schrödinger (NLS) equation on weighted graphs models wave propagation in a variety of physical systems, including nonlinear optical waveguide arrays, Bose–Einstein condensates in optical lattices, and energy localisation in biomolecular chains; see, for example, (Ablowitz et al., 2003; Lederer et al., 2008; Pelinovsky et al., 2005) and the references therein. In its standing wave reduction, the equation takes the form

$$(H-\lambda)u(x) = \kappa f\big(x,u(x)\big), \qquad x \in \mathcal{V}. \tag{1}$$

Here $\kappa = \pm 1$ and $\mathcal{V}$ is the vertex set of a locally finite weighted graph, $H = L + I$ is the associated Schrödinger operator, $\lambda \in R$ is a spectral parameter, and f is a

*Sorumlu Yazar/Corresponding Author: Setenay AKDUMAN e-posta/e-mail: setenay.akduman@idu.edu.tr

nonlinearity. The natural variational framework for equation (1) leads to the energy functional

$$J_\lambda(u)=\frac{1}{2}Q(u)-\frac{\lambda}{2}\parallel u\parallel_{\ell_m^2}^2-\kappa\sum_{x\in\mathcal{V}}m(x)\,F(x,u(x)),$$

whose critical points are precisely the solutions of (1). The central analytic difficulty is to establish sufficient compactness properties of the energy space E to carry out the minimax arguments of critical point theory. This is a non-trivial task on a general weighted graph, where neither a Poincaré inequality nor a coercive potential growth condition is assumed a priori.

Background and motivation

Variational methods for NLS-type equations on discrete structures have been developed since the foundational work of Rabinowitz on minimax methods in critical point theory (Rabinowitz, 1986), which provides the abstract framework underlying both the Mountain Pass Theorem and the Linking Theorem. The application of these tools to NLS equations requires compact Sobolev-type embeddings of the energy space into weighted sequence spaces, and the precise mechanism by which compactness is achieved depends sensitively on the growth behaviour of the potential and the geometric properties of the underlying graph.

On the one-dimensional lattice $\mathcal{V}=\mathbb{Z}$, Zhang and Pankov (Zhang and Pankov, 2010) proved the existence of standing wave solutions for every $\lambda\in\mathbb{R}$ via the Linking Theorem, under a coercive potential condition ($V(n)\to+\infty$ as $|n|\to\infty$) combined with the boundedness of the graph Laplacian on $\ell^2(\mathbb{Z})$. The coercive growth of the potential provides the compact embedding of the energy space into $\ell^2(\mathbb{Z})$ directly, and is the key analytic ingredient in their Palais–Smale argument.

On locally finite graphs, Grigor'yan, Lin, and Yang (Grigor'yan et al., 2017) established the existence of positive solutions via the Mountain Pass Theorem under the global lower bound $\inf_{x\in\mathcal{V}}m(x)>0$ and a summability condition on the inverse potential. The lower bound implies that every subset of finite m-measure is finite, which in turn gives a continuous embedding $\ell_m^1(\mathcal{V})\hookrightarrow\ell^\infty(\mathcal{V})$; together with a compact embedding of the energy space into $\ell_m^1(\mathcal{V})$, this yields compact embeddings $E\hookrightarrow\ell^q(\mathcal{V})$ for all $1\le q\le\infty$.

More recently, Akduman, Hofmann, and Karakılıç (Akduman et al., 2025) extended the variational theory to locally finite graphs by replacing the coercive condition with the measure-theoretic potential condition

$$\sup_{\substack{\mathcal{K}\Subset\mathcal{V}\\ m(\mathcal{K})<\infty}}\ \inf_{x\in\mathcal{V}\setminus\mathcal{K}}V(x)=+\infty,\tag{2}$$

and used the generalised Nehari manifold approach (Szulkin and Weth, 2010) to produce ground state solutions. Condition (2) requires the potential to grow to infinity outside every canonically compactifiable subset of finite m-measure; it is strictly weaker than pointwise coercivity, and it yields compact embeddings $E\hookrightarrow\ell_m^p(\mathcal{V})$ for $p\ge2$.

Present contributions

The present paper works under a genuinely different set of hypotheses, which we now describe. We consider a locally finite weighted graph $\Gamma=(\mathcal{V},b,c)$ subject to two structural conditions:

(A1) *Canonical compactifiability: This provides* the required continuous embedding $E\hookrightarrow\ell^\infty(\mathcal{V})$.

(A2) *Tail summability:* $1/V\in\ell_m^1(\mathcal{V})$, equivalently,

$$\inf_{\substack{\mathcal{K}\subset\mathcal{V}\\ m(\mathcal{K})<\infty}}\ \sum_{x\in\mathcal{V}\setminus\mathcal{K}}\frac{m(x)^2}{m(x)+c(x)}<\infty.$$

Crucially, we do *not* require $\inf_{x\in\mathcal{V}}m(x)>0$. As a consequence, infinite subsets of $\mathcal{V}$ with finite m-measure are permitted, which accommodates vertices with infinitesimally small weights and a substantially broader class of weighted graphs than those covered by (Grigor'yan et al., 2017).

The condition (A2) is strictly incomparable with condition (2) of (Akduman et al., 2025): there exist weighted graphs satisfying (A2) for which (2) fails, and conversely (see Examples 10–11 in Section 3). The compactness mechanism developed here is therefore genuinely different from those of both (Zhang and Pankov, 2010) and (Akduman et al., 2025).

A key advantage of the tail summability condition (A2) is that it is particularly well suited to variational methods based on Palais–Smale compactness and linking or mountain-pass geometry, where compact embeddings into finite-exponent spaces suffice. Indeed, condition (A2) yields compact embeddings $E\hookrightarrow\ell_m^p(\mathcal{V})$ for the *full* range $1\le p<\infty$, including the low-exponent regime $1\le p<2$ that is not covered by (2) in (Akduman et al., 2025). This stronger family of embeddings provides additional flexibility in the linking geometry and is also expected to be useful in the study of multiplicity results and bifurcation phenomena on weighted graphs.

Our main results are as follows.

i. *Strict incomparability* (Section 3). The tail summability condition (A2) and the measure-theoretic potential condition (2) of (Akduman et al., 2025) are strictly incomparable: neither implies the other in general (Examples 10–11).

ii. *Compact Sobolev embedding and discrete spectrum* (Sections 3–4). Under (A1) and (A2), the embedding $E \hookrightarrow \ell^p_m(\mathcal{V})$ is compact for every $1 \le p < \infty$ (Lemma 12), and the Schrödinger operator H has compact resolvent and purely discrete spectrum (Proposition 16).

iii. *Palais–Smale condition* (Section 6). The energy functional J_λ satisfies the Palais–Smale condition for every $\lambda \in \mathbb{R}$ (Theorem 18). The proof adapts the Ambrosetti–Rabinowitz condition to replace the coercive splitting argument of (Zhang and Pankov, 2010), which is unavailable in our setting.

iv. *Existence and multiplicity for $\kappa = 1$* (Section 8). For every $\lambda \in \mathbb{R}$, equation (1) has at least one nontrivial solution $u \in E$. When $f(x,\cdot)$ is additionally odd, the equation has infinitely many pairs of nontrivial solutions $\pm u_m \in E$ (Theorem 27).

Organisation of the paper

Section 2 introduces the functional framework, the weighted graph Laplacian, the quadratic form and energy space, and the assumptions on the nonlinearity and the graph. Section 3 establishes the compact Sobolev embedding and compares our framework with those of (Grigor'yan et al., 2017) and (Akduman et al., 2025). Section 4 proves that H has purely discrete spectrum. Section 5 establishes the C^1-regularity of the energy functional. Section 6 proves the Palais–Smale condition. Section 7 develops the spectral decomposition and linking geometry. Section 8 contains the main existence and multiplicity theorem for the self-focusing case $\kappa = 1$. Section 9 collects remarks on related open problems.

2. Functional Setting and Assumptions

2.1 Weighted graphs

Let $\mathcal{V}$ be an infinite countable set and $m: \mathcal{V} \to (0, \infty)$ a measure with full support. A *weighted graph* $\Gamma = (\mathcal{V}, b, c)$ over $(\mathcal{V}, m)$ consists of a killing weight $c: \mathcal{V} \to [0, \infty)$ and an edge weight $b: \mathcal{V} \times \mathcal{V} \to [0, \infty)$ satisfying

$$b(x,x) = 0, \qquad b(x,y) = b(y,x) \qquad \text{for all } x, y \in \mathcal{V},$$

and the local finiteness condition

$$\sum_{y \sim x} b\,(x,y) < \infty \qquad \text{for all } x \in \mathcal{V},$$

where we write $y \sim x$ to indicate that x and y are adjacent, i.e., $b(x,y) > 0$. This framework is standard; we refer the reader to (Keller et al., 2021), (Georgakopoulos et al., 2015) and (Akduman et al., 2025) for a systematic treatment.

2.2 Function spaces

For $1 \le p < \infty$, the weighted Lebesgue space $\ell^p_m(\mathcal{V})$ is defined by

$$\ell^p_m(\mathcal{V}) := \left\{ u: \mathcal{V} \to \mathbb{R} \,\middle|\, \| u \|^p_{\ell^p_m} := \sum_{x \in \mathcal{V}} m\,(x) |u(x)|^p < \infty \right\},$$

and $\ell^\infty(\mathcal{V})$ comprises all bounded functions on $\mathcal{V}$, normed by $\| u \|_\infty := \sup_{x \in \mathcal{V}} |u(x)|$. The space $\ell^2_m(\mathcal{V})$ is a Hilbert space with

$$(u, v)_{\ell^2_m} := \sum_{x \in \mathcal{V}} m\,(x)\, u(x)\, v(x).$$

The weighted graph Laplacian takes the form

$$\Delta u(x) := \frac{1}{m(x)} \sum_{y \sim x} b\,(x,y)(u(y) - u(x)), \quad x \in \mathcal{V},$$

where $u: \mathcal{V} \to \mathbb{R}$. Let $C_c(\mathcal{V})$ denotes the set of functions whose support is finite on $\mathcal{V}$.

2.3 Quadratic forms and the Schrödinger operator

Following the Dirichlet form approach of (Keller et al., 2021), we define q_0 on $C_c(\mathcal{V})$ by

$$q_0(u) := \frac{1}{2} \sum_{\substack{x,y \in \mathcal{V} \\ y \sim x}} b\,(x,y)(u(x) - u(y))^2 + \sum_{x \in \mathcal{V}} c\,(x)\, u(x)^2.$$

The form q_0 is closable on $\ell^2_m(\mathcal{V})$; we denote its closure by q, which is the Dirichlet form with the form domain $D(q)$. The Neumann form $\tilde{q}$ is the maximal closed extension of q_0, given on

$$D(\tilde{q}) := \{ u \in \ell^2_m(\mathcal{V}) : \tilde{q}(u) < \infty \}.$$

where $\tilde{q}(u)$ denotes the same expression as q_0 but evaluated on all $u \in \ell^2_m(\mathcal{V})$ for which it is finite; see (Akduman et al. 2025). By construction, $D(q) \subset D(\tilde{q})$ and $\tilde{q} = q$ on $D(q)$.

Since q is closed and non-negative, there exists a unique non-negative self-adjoint operator L on $\ell^2_m(\mathcal{V})$ satisfying

$$q(u,v) = (u, Lv)_{\ell^2_m} \qquad \text{for all } u \in D(q),\ v \in D(L),$$

with $D(q) = D(L^{1/2})$ and $q(u,u) = \| L^{1/2} u \|^2_{\ell^2_m}$; see (Keller et al., 2021, Appendix B).

We define the *Schrödinger operator* $H := L + I$. A direct computation shows that

$$Hu(x) = -\Delta u(x) + V(x)\,u(x),$$

$$V(x) := \frac{c(x)+m(x)}{m(x)} \geq 1.$$

The quadratic form corresponding to the operator H is $Q(u) := q(u) + \| u \|^2_{\ell^2_m}$, which may be written explicitly as

$$Q(u) = \frac{1}{2}\sum_{\substack{x,y\in\mathcal{V}\\ y\sim x}} b\,(x,y)(u(x)-u(y))^2 + \sum_{x\in\mathcal{V}} m\,(x)\,V(x)\,u(x)^2.$$

In particular, $D(Q) = D(q)$ by definition. We refer to the Hilbert space

$$E := D(Q) = D(H^{1/2})$$

as the *energy space*, endowed with

$$(u,v)_E := \frac{1}{2}\sum_{x,y\in\mathcal{V}} b\,(x,y)(u(x)-u(y))(v(x)-v(y)) + \sum_{x\in\mathcal{V}} m\,(x)\,V(x)\,u(x)\,v(x)$$

and corresponding norm $\| u \|^2_E := Q(u) = \| H^{1/2}u \|^2_{\ell^2_m}$. We note that the identity $Q(u) = (Hu,u)_{\ell^2_m}$ holds for $u \in D(H)$; for general $u \in E = D(H^{1/2})$, the correct statement is $Q(u) = \| H^{1/2}u \|^2_{\ell^2_m}$.

2.4 Assumptions on the nonlinearity

Assumption 1 (Nonlinearity). *The function $f\colon \mathcal{V}\times\mathbb{R}\to\mathbb{R}$ satisfies the following conditions.*

(f1) *The nonlinearity $f(x,u)$ is measurable in x and continuous in u, with $f(x,0) = 0$, $\forall x \in \mathcal{V}$.*

(f2) $|f(x,u)| \leq \mu(R)|u|$, $\forall x \in \mathcal{V}$ *and* $|u| \leq R$, *with $\mu(R)$ non-decreasing, $\mu(R) > 0$ for $R > 0$, $\mu(R)\to 0$ as $R \to 0$.*

(f3) *There exist constants $C > 0$ and $2 < p < \infty$ such that $|f(x,u)| \leq C(1+|u|^{p-1})$ for all $x \in \mathcal{V}$ and $u \in \mathbb{R}$.*

(f4) *(Ambrosetti–Rabinowitz condition.) There exists $2 < \theta < \infty$ such that*

$0 < \theta\,F(x,u) \leq u\,f(x,u)$ *for all $x \in \mathcal{V}$, $u \neq 0$,*

where $F(x,u) := \int_0^u f\,(x,s)\,ds$.

Remark 2. *Condition (f4) has several useful consequences. Integrating the differential inequality $(\ln F(x,\cdot))' \geq \theta/u$ for $u > 0$ yields the lower bound*

$$F(x,u) \geq F(x,u_0)(\tfrac{u}{u_0})^\theta \qquad \text{for } u \geq u_0 > 0, \tag{3}$$

so that $F(x,u) \geq C_\theta |u|^\theta$ for large $|u|$, uniformly in x, and in particular $F(x,u) > 0$ for all $u \neq 0$. Conditions (f2) and (f3) together imply that for every $\varepsilon > 0$ there exists a constant $A(\varepsilon) > 0$, independent of x, such that

$$0 \leq F(x,u) \leq \tfrac{\varepsilon}{2}|u|^2 + A(\varepsilon)|u|^p. \tag{4}$$

Finally, combining (f3) and (f4) gives the exponent constraint $2 < \theta \leq p$.

Definition 3. (Akduman et al. (2025)) *Any given subset $\mathcal{K}$ in $\mathcal{V}$ is said to be canonically compactifiable if the map $r_{\mathcal{K}}\colon D(\tilde{q}) \to \ell^\infty(\mathcal{V})$ defined by $r_{\mathcal{K}}(u) = \chi_{\mathcal{K}} u$ is continuous. In this case we write $\mathcal{K} \Subset \mathcal{V}$.*

Assumption 4 (Canonical compactifiability). *The weighted graph Γ is canonically compactifiable, that is, the embedding $D(\tilde{q}) \hookrightarrow \ell^\infty(\mathcal{V})$ is continuous.*

Remark 5. Our assumption Assumption 4 implies a continuous embedding $E = D(Q) \hookrightarrow \ell^\infty(\mathcal{V})$, which is sufficient for controlling the nonlinear term and for the variational arguments below.

Assumption 6 (Potential condition).

$$\inf_{\substack{\mathcal{K}\subset\mathcal{V}\\ m(\mathcal{K})<\infty}} \sum_{x\in\mathcal{V}\setminus\mathcal{K}} \frac{m(x)^2}{m(x)+c(x)} < \infty.$$

2.5 Formulation of the problem

We consider the NLS equation (1), which may be written equivalently as

$$-\Delta u(x) + V(x)\,u(x) - \lambda\,u(x) = \kappa\,f(x,u(x)), \quad x \in \mathcal{V},$$

where $u\colon \mathcal{V} \to \mathbb{R}$, the self-focusing case for $\kappa = 1$ and the defocusing case for $\kappa = -1$, and λ is a real spectral parameter. We aim to establish the existence of solutions by variational methods under the assumptions introduced above.

2.6 The energy functional

The energy functional associated with equation (1) is

$$J_\lambda(u) := \tfrac{1}{2}\,Q(u) - \tfrac{\lambda}{2}\,\| u \|^2_{\ell^2_m} - \kappa\,\textstyle\sum_{x\in\mathcal{V}} m\,(x)\,F(x,u(x)). \tag{5}$$

As we shall verify in Section 5, J_λ is of class C^1 on E, and its critical points are precisely the solutions of (1), in both the weak and the pointwise sense.

3. Compact Sobolev Embedding

3.1 Equivalence with the summability condition

Lemma 7. *Assumption 6 is equivalent to the summability condition $1/V \in \ell^1_m(\mathcal{V})$, i.e., $\sum_{x\in\mathcal{V}} m\,(x)/V(x) < \infty$.*

Proof. Since $m(x)/V(x) = m(x)^2/(m(x)+c(x))$ and $V(x) \geq 1$, for any finite-measure subset $\mathcal{K} \subset \mathcal{V}$ we have

$$\sum_{x\in\mathcal{K}} \frac{m(x)^2}{m(x)+c(x)} \leq m(\mathcal{K}) < \infty.$$

Therefore Assumption 6 holds if and only if the full sum $\sum_{x\in\mathcal{V}} m(x)/V(x)$ is finite. □

Remark 8. *Compared with the framework of (Grigor'yan et al., 2017), the principal relaxation in the present work does not stem from the weighted summability condition $1/V \in \ell^1_m(\mathcal{V})$ itself, but from dropping the global lower bound $\inf_{x\in\mathcal{V}} m(x) > 0$, which is a rather restrictive geometric assumption on the weighted graph. Indeed, under this lower bound every subset of finite measure is necessarily finite. Once the assumption is removed, infinite subsets of finite measure become possible, allowing vertices with infinitesimally small weights and accommodating a substantially broader class of weighted graphs.*

The following example illustrates that the present framework applies to weighted graphs for which the global lower bound on the measure fails, thereby highlighting the restrictive nature of the condition $\inf_{x\in\mathcal{V}} m(x) > 0$.

Example 9. *Let $\mathcal{V} = \mathbb{N}$, $m(n) = 2^{-n}$, and $c(n) = 2^{-2n}$. Then $\inf_{n\in\mathbb{N}} m(n) = 0$, so the global lower bound condition of (Grigor'yan et al., 2017) fails. Nevertheless,*

$$V(n) = \frac{m(n)+c(n)}{m(n)} = 1 + 2^{-n},$$

and therefore

$$\sum_{n\in\mathbb{N}} \frac{m(n)}{V(n)} = \sum_{n\in\mathbb{N}} \frac{2^{-n}}{1+2^{-n}} < \infty, \text{ so } 1/V \in \ell^1_m(\mathcal{V})$$

and Assumption 6 holds. To see this directly, take $\mathcal{K} = \emptyset$, for which $m(\mathcal{K}) = 0 < \infty$ and

$$\sum_{n\in\mathcal{V}\setminus\mathcal{K}} \frac{m(n)^2}{m(n)+c(n)} = \sum_{n\in\mathbb{N}} \frac{2^{-2n}}{2^{-n}+2^{-2n}} = \sum_{n\in\mathbb{N}} \frac{2^{-n}}{1+2^{-n}} < \infty,$$

so the infimum in Assumption 6 is finite.

3.2 Comparison with existing frameworks

Example 10 (Assumption 6 does not imply the potential condition of (Akduman et al., 2025)). *Let $\mathcal{V} = \mathbb{N}$, $m(n) = 2^{-n}$, and $c(n) = 2^{-2n}$. Then $V(n) = 1 + 2^{-n} \leq 2$ for all n. For any canonically compactifiable subset $\mathcal{K} \Subset \mathcal{V}$ with non-empty complement, one has $\inf_{n\in\mathcal{V}\setminus\mathcal{K}} V(n) \leq 2$, and consequently*

$$\sup_{\substack{\mathcal{K}\Subset\mathcal{V}\\ m(\mathcal{K})<\infty}} \inf_{n\in\mathcal{V}\setminus\mathcal{K}} V(n) < \infty,$$

so condition (2) of (Akduman et al., 2025) fails. On the other hand, $\sum_{n=1}^{\infty} m(n)/V(n) = \sum_{n=1}^{\infty} 2^{-n}/(1+2^{-n}) < \infty$, so $1/V \in \ell^1_m(\mathcal{V})$ and Assumption 6 holds.

Example 11 (The potential condition (2) of (Akduman et al., 2025) does not imply Assumption 6). *Let $\mathcal{V} = \mathbb{Z}$, $m \equiv 1$, and $c(n) = |n|$. Then $V(n) = 1 + |n|$. Since $m \equiv 1$ $m(\mathcal{K}) < \infty$ if and only if $\mathcal{K} \subset \mathcal{V}$ is finite. Taking $\mathcal{K}_N = \{-N, \dots, N\}$, we have $m(\mathcal{K}_N) < \infty$ and $\mathcal{K}_N \Subset \mathcal{V}$. Moreover, $\inf_{n\in\mathcal{V}\setminus\mathcal{K}_N} V(n) = N + 2 \to +\infty$, so condition (2) is satisfied. On the other hand, for any finite subset $\mathcal{K} \subset \mathcal{V}$, the complement $\mathcal{V}\setminus\mathcal{K}$ contains all integers of sufficiently large absolute value, and thus $\sum_{n\in\mathcal{V}\setminus\mathcal{K}} \frac{m(n)^2}{m(n)+c(n)} = \sum_{n\in\mathcal{V}\setminus\mathcal{K}} \frac{1}{1+|n|} = \infty$,*

so Assumption 6 fails.

The two conditions are therefore strictly incomparable. The summability condition $1/V \in \ell^1_m(\mathcal{V})$ controls the total m-weighted mass of $1/V$, while condition (2) of (Akduman et al., 2025) requires V to be large outside canonically compactifiable subsets of finite m-measure. The compactness mechanism developed in this paper is consequently distinct from those of both (Zhang and Pankov, 2010) and (Akduman et al., 2025).

3.3 The compact embedding theorem

Lemma 12 (Compact Sobolev embedding). *Under Assumptions 4 and 6, the embedding $E \hookrightarrow \ell^p_m(\mathcal{V})$ is compact for every $1 \leq p < \infty$.*

Proof. Let (u_n) be a bounded sequence in E. Since E is a Hilbert space, we may pass to a subsequence and assume $u_n \rightharpoonup u$ weakly in E. By Assumption 4, the sequence is uniformly bounded in $\ell^\infty(\mathcal{V})$:

$\| u_n \|_\infty \leq C_\infty \sup_n \| u_n \|_E \leq C_\infty M$ for some constant $M > 0$.

Step 1: Strong convergence in $\ell^1_m(\mathcal{V})$. Let $\varepsilon > 0$. By Lemma 7, choose $\mathcal{K} \subset \mathcal{V}$ with a finite measure and

$$\sum_{x\in\mathcal{V}\setminus\mathcal{K}} \frac{m(x)}{V(x)} < \frac{\varepsilon^2}{4M^2}.$$

For any $w \in E$ with $\| w \|_E \leq 2M$, the Cauchy–Schwarz inequality and the identity $m(x)/V(x) = m(x)^2/(m(x)+c(x))$ give

$$\begin{aligned}\sum_{x\in\mathcal{V}\setminus\mathcal{K}} m(x)|w(x)| &\leq \Big(\sum_{x\in\mathcal{V}\setminus\mathcal{K}} \frac{m(x)^2}{m(x)+c(x)}\Big)^{1/2} \Big(\sum_{x\in\mathcal{V}\setminus\mathcal{K}} m(x)\,V(x)|w(x)|^2\Big)^{1/2}\\ &\leq \Big(\sum_{x\in\mathcal{V}\setminus\mathcal{K}} \frac{m(x)^2}{m(x)+c(x)}\Big)^{1/2} \| w \|_E\\ &< \frac{\varepsilon}{2M}\cdot 2M = \varepsilon.\end{aligned}$$

Applying this to $w = u_n - u$, which is bounded in E by $2M$, shows that the tail contribution is uniformly smaller than ε.

For the finite part, on $\mathcal{K}$, note that for each $x \in \mathcal{V}$, the evaluation functional $\delta_x: E \to \mathbb{R}$, $\delta_x(u) = u(x)$, is bounded and linear by Assumption 4. Hence weak convergence $u_n \rightharpoonup u$ in E implies $u_n(x) \to u(x)$ for every $x \in \mathcal{V}$. Since $m(\mathcal{K}) < \infty$ and $|u_n(x) - u(x)| \le 2C_\infty M$ uniformly, the discrete dominated convergence theorem yields $\| \chi_{\mathcal{K}}(u_n - u) \|_{\ell_m^1} \to 0$.

Combining the two estimates, we conclude that $u_n \to u$ strongly in $\ell_m^1(\mathcal{V})$.

Step 2: Strong convergence in $\ell_m^p(\mathcal{V})$ for $p \in (1, \infty)$. Since $|u_n(x) - u(x)|^p \le \| u_n - u \|_\infty^{p-1} |u_n(x) - u(x)|$ and $(u_n - u)$ is uniformly bounded in $\ell^\infty(\mathcal{V})$, we obtain

$$\|u_n - u\|_{\ell_m^p}^p \le (2C_\infty M)^{p-1} \|u_n - u\|_{\ell_m^1} \to 0$$

□

Remark 13. *The convergence on $\mathcal{K}$ established in Step 1 can alternatively be derived from the local compactness result of (Akduman et al., 2025, Lemma 3.9), which provides a stronger form of compactness on canonically compactifiable subsets. Specifically, since $m(\mathcal{K}) < \infty$, the restriction operator $r_{\mathcal{K}}: \ell^\infty(\mathcal{V}) \to \ell_m^1(\mathcal{K})$ is compact by that lemma, and hence, after passing to a subsequence, $\chi_{\mathcal{K}} u_n \to \chi_{\mathcal{K}} u$ strongly in $\ell_m^1(\mathcal{K})$. The argument given above relies only on the weaker consequence that weak convergence in E, combined with canonical compactifiability, implies pointwise convergence and hence strong convergence in $\ell_m^1(\mathcal{K})$ by dominated convergence.*

Remark 14. *Assumption 6 yields compact embeddings for the full range $1 \le p < \infty$, including the low-exponent regime $1 \le p < 2$. In contrast, condition (2) of (Akduman et al., 2025) yields compact embeddings only for $p \ge 2$.*

Remark 15. *In (Grigor'yan et al., 2017), the assumption $\inf_{x\in\mathcal{V}} m(x) > 0$ implies that every set of finite measure is finite, and one obtains the continuous embedding $\ell_m^1(\mathcal{V}) \hookrightarrow \ell^\infty(\mathcal{V})$. Combined with the compact embedding into $\ell_m^1(\mathcal{V})$, this yields compact embeddings $E \hookrightarrow \ell^q(\mathcal{V})$ for the full range $1 \le q \le \infty$.*

As noted above, the condition $\inf_{x\in\mathcal{V}} m(x) > 0$ is quite restrictive and excludes many natural weighted graphs. In the present work, we replace it by canonical compactifiability (Assumption 4), and under this weaker hypothesis we still obtain compact embeddings

$E \hookrightarrow \ell_m^q(\mathcal{V}), 1 \le q < \infty.$

Although compactness of the embedding $E \hookrightarrow \ell^\infty(\mathcal{V})$ is no longer available in general, this causes no difficulty for our variational analysis. The existence results in this paper require only the continuous embedding $E \hookrightarrow \ell^\infty(\mathcal{V})$, which follows directly from Assumption 4 and is sufficient to control the nonlinear term and to carry out the Palais–Smale and linking arguments in the sections that follow.

4. Discreteness of the Spectrum

Proposition 16. *Under Assumptions 4 and 6, the Schrödinger operator H has compact resolvent. In particular, the spectrum of H is purely discrete: $\sigma(H) = \{\lambda_1, \lambda_2, \dots\}$ with $1 \le \lambda_1 \le \lambda_2 \le \dots \to +\infty$, where each eigenvalue is repeated according to its multiplicity, and $\ell_m^2(\mathcal{V})$ admits an orthonormal basis $\{\phi_i\}_{i=1}^\infty$ of eigenfunctions of H.*

Proof. By Lemma 12 applied with $p = 2$, the embedding $\iota: E \hookrightarrow \ell_m^2(\mathcal{V})$ is compact. We claim that $H^{-1}: \ell_m^2(\mathcal{V}) \to E$ is bounded. Indeed, for $f \in \ell_m^2(\mathcal{V})$, let $u = H^{-1} f \in D(H) \subset E$. Using $Q(v) = (Hv, v)_{\ell_m^2}$ for $v \in D(H)$ and $H \ge \lambda_1 I$:

$$\| H^{-1} f \|_E^2 = Q(u) = (f, H^{-1} f)_{\ell_m^2} \le \| f \|_{\ell_m^2} \| H^{-1} f \|_{\ell_m^2} \le \lambda_1^{-1} \| f \|_{\ell_m^2}^2,$$

so $H^{-1}: \ell_m^2(\mathcal{V}) \to E$ is bounded with norm at most $\lambda_1^{-1/2}$. The composition $\iota \circ H^{-1}: \ell_m^2(\mathcal{V}) \to \ell_m^2(\mathcal{V})$ is then compact, being the composition of a bounded operator with the compact embedding ι. The standard spectral theorem for compact self-adjoint operators (see, e.g., (Birman and Solomjak, 1987; Reed and Simon, 1975)) then gives the stated conclusion: the spectrum is purely discrete, consisting of eigenvalues of finite multiplicity accumulating only at $+\infty$. □

Since $H \ge I$, we have $\lambda_1 \ge 1$. We write $H\phi_1 = \lambda_1 \phi_1$ with $\phi_1 \in E$ normalised, and refer to λ_1 as the *first eigenvalue* of H throughout.

5. C^1-Regularity of the Energy Functional

Proposition 17. *Under Assumptions 1, 4, and 6, the functional $J_\lambda: E \to \mathbb{R}$ is of class C^1. For every $u \in E$, the derivative $J'_\lambda(u) \in E^*$ acts on $v \in E$ by*

$\langle J'_\lambda(u), v \rangle =$

$Q(u, v) - \lambda(u, v)_{\ell_m^2} - \kappa \sum_{x\in\mathcal{V}} m(x) f\big(x, u(x)\big) v(x),$

where $\langle ., . \rangle$ denotes the duality pairing between E^* and E.

Proof. By Assumption 4, every $u \in E$ satisfies $\| u \|_\infty \le C_\infty \| u \|_E < \infty$. Using the estimate (4), the Nemytskii functional $\Psi(u) := \sum_{x\in\mathcal{V}} m(x) F(x, u(x))$ satisfies

$$|\Psi(u)| \le \left(\frac{\mu(\| u \|_\infty)}{2} + A(\varepsilon) \| u \|_\infty^{p-2} \right) \| u \|_{\ell_m^2}^2 < \infty,$$

so Ψ is well defined on E. By Lemma 12 with $p = 1$, the embedding $E \hookrightarrow \ell_m^1(\mathcal{V})$ is continuous (compactness is not needed at this stage), and for each fixed $u \in E$ the linear map $v \mapsto \sum_x m(x) f(x, u(x)) v(x)$ defines a bounded functional on E. The C^1-regularity of Ψ and the derivative formula then follow by the standard Nemytskii operator argument, using the uniform bound $\| u \|_\infty \le C_\infty \| u \|_E$; see (Akduman et al., 2025, Proposition 2.8) and (Drábek and Milota, 2013) for the details. □

6. Palais–Smale Condition

Definition. *A sequence $(u_n) \subset E$ is called a Palais–Smale sequence for J_λ at level $c \in \mathbb{R}$ if $J_\lambda(u_n) \to c$ and $J'_\lambda(u_n) \to 0$ in E^*. We say that J_λ satisfies the Palais–Smale condition (PS condition) if every such sequence possesses a convergent subsequence in E.*

Theorem 18 (Palais–Smale condition). *Under Assumptions 1, 4, and 6, the functional J_λ satisfies the PS condition for every $\kappa \in \{1, -1\}$ and every $\lambda \in \mathbb{R}$.*

Proof. Let $(u_n) \subset E$ satisfy $|J_\lambda(u_n)| \le C$ and $J'_\lambda(u_n) \to 0$ in E^*.

Step 1: Boundedness. Suppose for contradiction that $\| u_n \|_E \to \infty$. Set $\tilde{u}_n := u_n / \| u_n \|_E$, so $\| \tilde{u}_n \|_E = 1$. By Lemma 12 with $p = 2$, after passing to a subsequence we have $\tilde{u}_n \to \tilde{u}$ strongly in $\ell^2_m(\mathcal{V})$, and consequently $\tilde{u}_n(x) \to \tilde{u}(x)$ for every $x \in \mathcal{V}$

(since $m(x)|\tilde{u}_n(x) - \tilde{u}(x)|^2 \le \| \tilde{u}_n - \tilde{u} \|^2_{\ell^2_m} \to 0$ and $m(x) > 0$).

Dividing the relation $J_\lambda(u_n) \to c$ by $\| u_n \|^2_E$ and using $\| u_n \|_E \to \infty$, we obtain

$$\frac{\sum_x m(x) F(x, u_n)}{\| u_n \|^2_E} \to \begin{cases} \alpha := 1/2 \left(1 - \lambda \| \tilde{u} \|^2_{\ell^2_m}\right) & \text{if } \kappa = 1, \\ \beta := 1/2 \left(\lambda \| \tilde{u} \|^2_{\ell^2_m} - 1\right) & \text{if } \kappa = -1. \end{cases} \quad (6)$$

Since $F \ge 0$ by (f4), both $\alpha \ge 0$ and $\beta \ge 0$.

Dividing $\langle J'_\lambda(u_n), u_n \rangle$ by $\| u_n \|^2_E$ and passing to the limit yields

$$\frac{\sum_x m(x) f(x, u_n) u_n}{\| u_n \|^2_E} \to 1 - \lambda \| \tilde{u} \|^2_{\ell^2_m}. \quad (7)$$

By (f4), $f(x, u) u \ge \theta F(x, u) \ge 0$ for $\theta > 2$.

Case $\kappa = 1$. From (6)–(7), we get $2\alpha = 1 - \lambda \| \tilde{u} \|^2_{\ell^2_m} \ge \theta\alpha$, so $(2 - \theta)\alpha \ge 0$. Since $\theta > 2$, this forces $\alpha = 0$.

Case $\kappa = -1$. Similarly, $(\theta - 2)\beta \le 0$, and since $\theta > 2$, we conclude $\beta = 0$.

In both cases, $\lambda \| \tilde{u} \|^2_{\ell^2_m} = 1$.

For $\lambda \le 0$ this is impossible, since $\alpha \ge 0$ implies $\lambda \| \tilde{u} \|^2_{\ell^2_m} \le 1$, which together with $\lambda \le 0$ contradicts $\lambda \| \tilde{u} \|^2_{\ell^2_m} = 1 > 0$. For $\lambda > 0$: the identity $\lambda \| \tilde{u} \|^2_{\ell^2_m} = 1$ forces $\tilde{u} \ne 0$, so there exists $x_0 \in \mathcal{V}$ with $\tilde{u}(x_0) \ne 0$. Then $|u_n(x_0)| = \| u_n \|_E \, |\tilde{u}_n(x_0)| \to +\infty$. By (3), $f(x_0, u) u \ge C_{x_0} |u|^\theta$ for large $|u|$, giving

$$\frac{m(x_0) f(x_0, u_n(x_0)) u_n(x_0)}{\| u_n \|^2_E} \ge C_{x_0} m(x_0) \| u_n \|^{\theta - 2}_E \, |\tilde{u}_n(x_0)|^\theta \to +\infty,$$

which contradicts (7). Hence (u_n) is bounded in E.

Step 2: Strong convergence. Since (u_n) is bounded in E, we may extract a subsequence with $u_n \rightharpoonup u$ weakly in E. By Lemma 12, $u_n \to u$ strongly in $\ell^p_m(\mathcal{V})$ for all $p < \infty$, and in particular in $\ell^2_m(\mathcal{V})$ and $\ell^1_m(\mathcal{V})$. Assumption 4 gives $\| u_n \|_\infty \le M$ uniformly.

Expanding $\langle J'_\lambda(u_n) - J'_\lambda(u), \, u_n - u \rangle$:

$$\| u_n - u \|^2_E = \langle J'_\lambda(u_n) - J'_\lambda(u), \, u_n - u \rangle + \lambda \| u_n - u \|^2_{\ell^2_m} + \kappa \sum_{x \in \mathcal{V}} m(x) (f(x, u_n) - f(x, u))(u_n - u). \quad (8)$$

Each term on the right-hand side tends to zero:

1. $\langle J'_\lambda(u_n), u_n - u \rangle \to 0$: since $\| J'_\lambda(u_n) \|_{E^*} \to 0$ and $(u_n - u)$ is bounded in E.
2. $\langle J'_\lambda(u), u_n - u \rangle \to 0$: since $u_n \rightharpoonup u$ weakly and $J'_\lambda(u) \in E^*$ is a fixed element.
3. $\lambda \| u_n - u \|^2_{\ell^2_m} \to 0$: by strong convergence in $\ell^2_m(\mathcal{V})$.
4. The nonlinear term: by (f3) and the uniform bound $|u_n|, |u| \le M$, we have $|f(x, u_n) - f(x, u)| \le C_M$ uniformly, so $|\sum_x m(x)(f(x, u_n) - f(x, u))(u_n - u)| \le C_M \| u_n - u \|_{\ell^1_m} \to 0$.

Therefore $\| u_n - u \|^2_E \to 0$, i.e., $u_n \to u$ strongly in E. □

Remark 19. *The main departure from Pankov–Zhang (Zhang and Pankov, 2010) occurs in Step 1. Their proof exploits the coercivity of the potential on $\mathbb{Z}$ (meaning $V(n) \to +\infty$) to decompose $\| u \|^2_{\ell^2(\mathbb{Z})}$ into a controlled finite-set contribution and a small tail; this splitting is unavailable under our assumptions. We replace it with three ingredients: (i) the compact embedding $E \hookrightarrow \ell^2_m(\mathcal{V})$ to extract a pointwise-convergent subsequence; (ii) the Ambrosetti–Rabinowitz condition (f4) to derive $\alpha = 0$ (or $\beta = 0$); (iii) the polynomial lower bound (3) to produce the final contradiction. In particular, the strengthened assumption $F(x, u)/|u|^2 \to \infty$ used in (Zhang and Pankov, 2010) is not required.*

7. Linking Geometry

Throughout this section we take $\kappa = 1$.

7.1 Spectral decomposition and abstract critical point theorems

Fix $\lambda \in \mathbb{R}$. If $\lambda \geq \lambda_1$, let $k \geq 1$ be the unique integer satisfying $\lambda_k \leq \lambda < \lambda_{k+1}$; if $\lambda < \lambda_1$, set $k = 0$. Define

$$Y := \overline{\operatorname{span}\{\phi_1, \dots, \phi_k\}}^E,$$

$$Z := Y^\perp = \overline{\operatorname{span}\{\phi_{k+1}, \phi_{k+2}, \dots\}}^E, \qquad (9)$$

so that $E = Y \oplus Z$ orthogonally in both E and $\ell_m^2(\mathcal{V})$ (with $Y = \{0\}$ when $k = 0$). Set

$$\lambda_+ := \max(\lambda, 0), \qquad \delta_Z := \frac{\lambda_{k+1} - \lambda_+}{\lambda_{k+1}} \in (0,1]. \quad (10)$$

Fix $z_0 \in Z$ with $\| z_0 \|_E = 1$. For $0 < r < \rho$, define

$$N := \{u \in Z : \| u \|_E = r\},$$
$$M := \{u = y + sz_0 : y \in Y,\ s \geq 0,\ \| u \|_E \leq \rho\},$$
$$\partial M := \{u = y + sz_0 : y \in Y,\ s \geq 0,\ \| u \|_E = \rho\} \cup \{y \in Y : \| y \|_E \leq \rho\}.$$

We shall use the following two abstract results.

Theorem 20 (Linking Theorem (Rabinowitz, 1986, Theorem 5.3)). *Let $J \in C^1(E, \mathbb{R})$ satisfy the PS condition and* $\inf_{u \in N} J(u) > \sup_{u \in \partial M} J(u)$.

Setting $\Gamma := \{\gamma \in C(M, E) : \gamma|_{\partial M} = \mathrm{id}\}$*, the value*
$$c := \inf_{\gamma \in \Gamma} \sup_{u \in M} J(\gamma(u))$$
is a critical value of J with $c \geq \inf_{u \in N} J(u)$.

For $r > 0$ let $B_r(0) := \{u \in E : \| u \|_E < r\}$.

Theorem 21 (Symmetric Mountain Pass Theorem (Rabinowitz, 1986, Theorem 9.12)). *Let $J \in C^1(E, \mathbb{R})$ be even with $J(0) = 0$, satisfying the PS condition and $E = Y \oplus Z$ with $\dim Y < \infty$. Suppose:*

(B1) *there exist $r, \alpha > 0$ such that $J|_{\partial B_r \cap Z} \geq \alpha$;*

(B2) *for a nested dense sequence $E_1 \subset E_2 \subset \cdots$ of finite-dimensional subspaces, there exist radii $\rho_m > 0$ such that $J \leq 0$ on $E_m \backslash B_{\rho_m}(0)$ for each $m \geq 1$.*

Then J has an unbounded sequence of critical values $c_1 \leq c_2 \leq \cdots \to +\infty$.

7.2 Spectral bounds

Lemma 22 (Spectral bounds). *For the decomposition (9) with $\lambda_k \leq \lambda < \lambda_{k+1}$:*
$$((H - \lambda)y, y)_{\ell_m^2} \leq 0 \qquad \text{for all } y \in Y,$$
$$((H - \lambda)z, z)_{\ell_m^2} \geq \delta_Z \| z \|_E^2 \qquad \text{for all } z \in Z.$$

Proof. Upper bound on Y. Expanding in the eigenbasis: $((H - \lambda)y, y)_{\ell_m^2} = \sum_{i \leq k} (\lambda_i - \lambda)|y_i|^2 \leq 0$, since $\lambda_i \leq \lambda_k \leq \lambda$ for all $i \leq k$.

Lower bound on Z. $((H - \lambda)z, z)_{\ell_m^2} = \sum_{i \geq k+1} (1 - \lambda/\lambda_i)\lambda_i |z_i|^2$. For $\lambda \geq 0$, the factor $1 - \lambda/\lambda_i$ is increasing in λ_i, so $\inf_{i \geq k+1}(1 - \lambda/\lambda_i) = 1 - \lambda_+/\lambda_{k+1} = \delta_Z$. For $\lambda < 0$ (which occurs only when $k = 0$): $((H - \lambda)z, z)_{\ell_m^2} = \| z \|_E^2 - \lambda \| z \|_{\ell_m^2}^2 \geq \| z \|_E^2 = \delta_Z \| z \|_E^2$. □

7.3 Ambrosetti–Rabinowitz growth on finite-dimensional subspaces

Lemma 23 (AR-monotonicity). *Let $W \subset E$ be a finite-dimensional subspace spanned by finitely many eigenfunctions of H, so that $W \subset \ell_m^\theta(\mathcal{V})$. Define $G(v) := \sum_{x \in \mathcal{V}} m(x) F(x, v(x))$ for v in the unit sphere S_W of W. Then*

$$C_4 := \inf_{v \in S_W} G(v) > 0,$$

and consequently

$$\sum_{x \in \mathcal{V}} m(x) F(x, u(x)) \geq C_4 \rho^\theta$$

for all $u \in W$ with $\| u \|_E = \rho \geq 1$.

Proof. The unit sphere S_W is compact (since W is finite-dimensional) and G is continuous on S_W. For any $v \in S_W$, there exists $x_0 \in \mathcal{V}$ with $v(x_0) \neq 0$ (since $\| v \|_E = 1$ and $m > 0$ everywhere), and condition (f4) then gives $F(x_0, v(x_0)) > 0$. Hence $G > 0$ on S_W, and the infimum $C_4 = \inf G > 0$ is attained.

For the lower bound, write $u = \rho v$ with $v = u/\rho \in S_W$. By (f4), the map $t \mapsto F(x,t)/|t|^\theta$ is non-decreasing on $(0, \infty)$ (since its derivative equals $(f(x,t)t - \theta F(x,t))/|t|^{\theta+1} \geq 0$). Hence $F(x, \rho v(x)) \geq \rho^\theta F(x, v(x))$ for $\rho \geq 1$ and $v(x) \neq 0$. Summing over x with weights:

$$\sum_{x \in \mathcal{V}} m(x) F(x, \rho v(x)) \geq \rho^\theta G(v) \geq C_4 \rho^\theta. \ □$$

7.4 Linking geometry

Lemma 24 (Linking geometry). *Let $\lambda \geq \lambda_1$ and set $z_0 := \phi_{k+1}/\| \phi_{k+1} \|_E$. There exist $0 < r < \rho$ such that* $\inf_{u \in N} J_\lambda(u) > 0 \geq \sup_{u \in \partial M} J_\lambda(u)$.

Proof. **Lower bound on N.** For $u \in Z$ with $\| u \|_E = r$, Lemma 22 and (4) yield

$$J_\lambda(u) \geq \frac{\delta_Z}{2} \| u \|_E^2 - \frac{\varepsilon}{2} \| u \|_{\ell_m^2}^2 - A(\varepsilon) \| u \|_{\ell_m^p}^p.$$

Since $H \geq \lambda_1 I$, we have $\| u \|_{\ell_m^2}^2 \leq \lambda_1^{-1} \| u \|_E^2$. By Assumption 4 and Hölder's inequality:

$\| u \|_{\ell_m^p}^p \leq C_\infty^{p-2} r^{p-2} \lambda_1^{-1} r^2$. Substituting:

$$J_\lambda(u) \geq r^2 \left[\frac{\delta_Z}{2} - \frac{\varepsilon}{2\lambda_1} - \frac{A(\varepsilon) C_\infty^{p-2} r^{p-2}}{\lambda_1} \right].$$

Choose $\varepsilon = \delta_Z \lambda_1/4$. For $r > 0$ sufficiently small, the bracket is at least $\delta_Z/4 > 0$, giving $\inf_N J_\lambda \geq (\delta_Z/4) r^2 > 0$.

Upper bound on ∂M, Part 1 ($s = 0$, $u = y \in Y$).

Lemma 22 and $F \geq 0$ together give $J_\lambda(y) \leq 0$.

Upper bound on ∂M, Part 2 ($u = y + sz_0 \in W := Y \oplus \mathrm{span}\{z_0\}$, $s > 0$, $\| u \|_E = \rho$). By Lemma 12 with $p = \theta$, the eigenfunctions satisfy $\phi_i \in \ell^\theta_m(\mathcal{V})$, so $W \subset \ell^\theta_m(\mathcal{V})$. Lemma 23 applied to W gives

$\sum_x m(x)F(x, u(x)) \geq C_4\rho^\theta$.

For the quadratic part, using $H\phi_{k+1} = \lambda_{k+1}\phi_{k+1}$ and the continuity of $E \hookrightarrow \ell^2_m(\mathcal{V})$:

$$((H-\lambda)(sz_0), sz_0)_{\ell^2_m} = s^2(\lambda_{k+1} - \lambda)\frac{\| \phi_{k+1} \|^2_{\ell^2_m}}{\| \phi_{k+1} \|^2_E}$$
$$\leq C(\lambda_{k+1} - \lambda)s^2 \leq C(\lambda_{k+1} - \lambda)\rho^2.$$

Combined with $((H-\lambda)y, y)_{\ell^2_m} \leq 0$, this yields

$\frac{1}{2}((H-\lambda)u, u)_{\ell^2_m} \leq C_W\rho^2$ for some constant $C_W > 0$. Hence $J_\lambda(u) \leq C_W\rho^2 - C_4\rho^\theta \leq 0$ for $\rho \geq \rho_0 := (C_W/C_4)^{1/(\theta-2)}$.

Choosing $\rho \geq \max(r+1, \rho_0)$ completes the proof. □

Remark 25 (Mountain pass as a special case). *When $\lambda < \lambda_1$, we have $Y = \{0\}$, $M = \{sz_0 : 0 \leq s \leq \rho\}$, $N = \partial B_r(0) \cap E$, and $\delta_Z = (\lambda_1 - \lambda_+)/\lambda_1 \in (0,1]$. Lemma 22 provides the lower bound on N, and $J_\lambda(sz_0) \leq C_W s^2 - C_4 s^\theta \to -\infty$ as $s \to \infty$. In this case the Linking Theorem reduces to the standard Mountain Pass Theorem and yields a nontrivial critical point.*

Lemma 26 (Finite-dimensional estimate). *For $m \geq 1$, let $E_m := \mathrm{span}\{\phi_1, \dots, \phi_{k+m}\}$. There exist constants $c_1, c_2 > 0$ such that*

$J_\lambda(u) \leq c_1 \| u \|^2_E - c_2 \| u \|^\theta_E$ for all $u \in E_m$.

In particular, $J_\lambda \leq 0$ on $E_m \backslash B_{R_m}(0)$ for all sufficiently large R_m.

Proof. Since E_m is finite-dimensional, all norms are equivalent; in particular, $((H-\lambda)u, u)_{\ell^2_m} \leq C_m \| u \|^2_E$. Lemma 23 applied to E_m gives $\sum_x m(x)F(x, u) \geq C_5 \| u \|^\theta_E$ for $\| u \|_E \geq 1$. Since $\theta > 2$, the estimate $J_\lambda(u) \leq c_1 \| u \|^2_E - c_2 \| u \|^\theta_E$ follows immediately, and the right-hand side is non-positive for $\| u \|_E \geq R_m := (c_1/c_2)^{1/(\theta-2)}$. □

8. Existence of Solutions for $\kappa = 1$

Theorem 27 (Existence and multiplicity, $\kappa = 1$). *Under Assumptions 1, 4, and 6 with $\kappa = 1$, for every $\lambda \in \mathbb{R}$ the following hold.*

i. *There exists at least one nontrivial solution $u \in E$ of $(H-\lambda)u = f(x, u)$.*
ii. *If in addition $f(x,\cdot)$ is odd for every $x \in \mathcal{V}$, then J_λ possesses an unbounded sequence of critical values, and the equation has infinitely many pairs of nontrivial solutions $\pm u_m \in E$.*

Proof. By Theorem 18, J_λ satisfies the PS condition.

Proof of (i).

Case 1 ($\lambda < \lambda_1$, mountain pass). Set $z_0 = \phi_1/\| \phi_1 \|_E$. By Remark 25, $\inf_N J_\lambda \geq (\delta_Z/4)\, r^2 > 0$ with $\delta_Z = (\lambda_1 - \lambda_+)/\lambda_1 > 0$, and $J_\lambda(sz_0) \to -\infty$ as $s \to \infty$. Since $J_\lambda(0) = 0 < \inf_N J_\lambda$, Theorem 20 (with $Y = \{0\}$) yields a critical value $c \geq (\delta_Z/4)\, r^2 > 0$ and hence a nontrivial critical point.

Case 2 ($\lambda \geq \lambda_1$, linking). By Lemma 24, $\inf_N J_\lambda > 0 \geq \sup_{\partial M} J_\lambda$. Since $J_\lambda(0) = 0 < \inf_N J_\lambda$, Theorem 20 yields a critical value $c \geq \inf_N J_\lambda > 0$ and a nontrivial critical point $u \in E$.

Proof of (ii). The oddness of $f(x,\cdot)$ implies that J_λ is even and satisfies $J_\lambda(0) = 0$. Condition (B1) of Theorem 21 is provided by Lemma 24, which gives $J_\lambda|_{\partial B_r \cap Z} \geq (\delta_Z/4)\, r^2 > 0$. Condition (B2) is provided by Lemma 26 with

$$E_m := \begin{cases} \mathrm{span}\{\phi_1, \dots, \phi_m\} & \text{if } \lambda < \lambda_1, \\ \mathrm{span}\{\phi_1, \dots, \phi_{k+m}\} & \text{if } \lambda_k \leq \lambda < \lambda_{k+1},\ k \geq 1, \end{cases}$$

with corresponding radii $R_m > 0$ such that $J_\lambda \leq 0$ on $E_m \backslash B_{R_m}(0)$ for each $m \geq 1$. Theorem 21 then yields an unbounded sequence of critical values and the pairs $\pm u_m$. □

9. Further Remarks and Open Problems

The results of this paper naturally suggest several directions for further investigation.

The defocusing case $\kappa = -1$. The Palais–Smale condition established in Theorem 18 holds for both signs of κ, but the linking geometry developed in Section 7 is specific to the self-focusing case; adapting the variational framework to treat existence and non-existence of solutions in the defocusing regime requires a different approach and will be pursued separately.

Bifurcation. A natural continuation of Theorem 27 is to study the bifurcation of nontrivial solutions from the trivial one as λ crosses an eigenvalue λ_k of H, a question that the discrete spectrum established in Proposition 16 makes precise.

Exponential decay. The solutions obtained here belong to the energy space $E \subset \ell^2_m(\mathcal{V})$, and it is natural to ask whether they decay exponentially at infinity; under suitable additional assumptions on f near zero, this is expected to follow from the functional calculus of H, in analogy with the one-dimensional argument of (Zhang and Pankov, 2010).

Declaration of Ethical Standards

The authors declare that they comply with all ethical standards.

Credit Authorship Contribution Statement

Author: Conceptualization, investigation, methodology and software, visualization and writing – original draft.

Declaration of Competing Interest

The author declares that there are no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data Availability Statement

All data generated or analyzed during this study are included in this published article.

Acknowledgements

This article is based upon work from COST Action 24122 mSPACE, supported by COST (European Cooperation in Science and Technology), www.cost.eu. I am grateful to Sedef ÖZCAN for her unwavering support and encouragement throughout this work.

10. References

Ablowitz, M. J., Prinari, B. and Trubatch, A. D., (2003). *Discrete and continuous nonlinear Schrödinger systems*. Vol. 302, London Mathematical Society Lecture Note Series, Cambridge University Press.

Akduman, S. and Pankov, A., (2017). Schrödinger operators with locally integrable potentials on infinite metric graphs. *Applicable Analysis,* 96(12), 2149–2161. https://doi.org/10.1080/00036811.2016.1207247

Akduman, S. and Pankov, A., (2019). Nonlinear Schrödinger equations with growing potential on infinite metric graphs. *Nonlinear Analysis,* 184, 258–272. https://doi.org/10.1016/j.na.2019.02.020

Birman, M. S. and Solomjak, M. Z., (1987). *Spectral theory of self-adjoint operators in Hilbert space*. Vol. 5, Springer Science & Business Media.

Drábek, P. and Milota, J., (2013). *Methods of nonlinear analysis: Applications to differential equations*. 2nd ed., Birkhäuser.

Georgakopoulos, A., Haeseler, S., Keller, M., Lenz, D. and Wojciechowski, R. K., (2015). Graphs of finite measure. *Journal de Mathématiques Pures et Appliquées,* 103(5), 1093–1131. https://doi.org/10.1016/j.matpur.2014.10.006

Grigor'yan, A., Lin, Y. and Yang, Y., (2017). Existence of positive solutions to some nonlinear equations on locally finite graphs. *Science China Mathematics*, 60(7), 1311–1324. https://doi.org/10.1007/s11425-016-0422-y

Keller, M. and Lenz, D., (2010). Unbounded Laplacians on graphs: Basic spectral properties and the heat equation. *Mathematical Modelling of Natural Phenomena*, 5(4), 198–224. https://doi.org/10.1051/mmnp/20105409

Keller, M. and Lenz, D., (2012). Dirichlet forms and stochastic completeness of graphs and subgraphs. *Journal für die reine und angewandte Mathematik,* 666, 189–223. https://doi.org/10.1515/CRELLE.2011.122

Keller, M., Lenz, D. and Wojciechowski, R. K., (2021). *Graphs and discrete Dirichlet spaces*. Vol. 358, Springer

Lederer, F., Stegeman, G. I., Christodoulides, D. N., Assanto, G., Segev, M. and Silberberg, Y., (2008). Discrete solitons in optics. *Physics Reports*, 463(1–3),1–126. https://doi.org/10.1016/j.physrep.2008.04.004

Pankov, A., (2013). Standing waves for discrete nonlinear Schrödinger equations: Sign-changing nonlinearities. *Applicable Analysis, 92*(2), 308–317. https://doi.org/10.1080/00036811.2011.609987

Pelinovsky, D. E., Kevrekidis, P. G. and Frantzeskakis, D. J., (2005). Stability of discrete solitons in nonlinear Schrödinger lattices. *Physica D: Nonlinear Phenomena*, 212(1–2), 1–19. https://doi.org/10.1016/j.physd.2005.07.021

Rabinowitz, P. H., (1986). *Minimax methods in critical point theory with applications to differential equations.* Vol. 65, American Mathematical Society.

Reed, M. and Simon, B., (1975). *Methods of modern mathematical physics*, Vol. 2: Fourier analysis, self-adjointness. Academic Press.

Szulkin, A. and Weth, T., (2009). Ground state solutions for some indefinite variational problems. *Journal of Functional Analysis,* 257(12), 3802–3822. https://doi.org/10.1016/j.jfa.2009.09.013

Szulkin, A. and Weth, T., (2010). The method of Nehari manifold. In D. Y. Gao and D. Motreanu (Eds.), *Handbook of nonconvex analysis and applications* (pp. 597–632). International Press.

Zhang, G. and Pankov, A., (2010). Standing wave solutions of the discrete nonlinear Schrödinger equations with unbounded potentials, II. *Applicable Analysis*, *89*(9), 1541–1557. https://doi.org/10.1080/00036810902942234

Internet References

Akduman, S., Hofmann, M. and Karakılıç, S., On the existence of Nehari ground states for the nonlinear Schrödinger equation on discrete graphs, https://arxiv.org/abs/2504.03552, (2025).