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Research Article

Topological Algebras of Bounded Operators with Locally Solid Riesz Spaces

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Abstract
Suppose X is a vector lattice and (E,τ) is a locally solid vector lattice. An operator $T:X\to E$ is said to be ob-bounded if, for every order bounded set B, $T(B)$ is topological bounded in E. In this paper, we study on algebraic properties of ob-bounded operators with the equicontinuous convergence topology and uniform convergence topology.

Keywords: Locally Solid Riesz Space, Operator, Order Bounded Operator, Vector Lattice

Yerel Solid Riesz Uzayları ile Sınırlı Operatörlerin Cebirsel Topolojisi

Öz
X bir kafes uzayı ve (E,τ) de bir yerel solid kafes uzayı olsun. Eğer X 'deki her sıra sınırlı B alt kümesi için, $T(B)$ E 'de topolojik olarak sınırlı oluyorsa; $T:X\to E$ operatörüne ob-sınırlı operatör denir. Bu çalışmamızda, ob-sınırlılığın cebirsel özelliklerini denk-süreklilik yakınsamasındaki ve düzgün yakınsamasındaki topolojilere göre inceledik.

Anahtar Kelimeler: Yerel Solid Riesz Uzayı, Operatör, Sıra Sınırlı Operatör, Kafes Uzayı

1. Introduction and Preliminaries

Bounded operators have an important role in the operator theory and functional analysis. Our aim is to introduce ob-bounded operators from vector lattice to locally solid vector lattices or between locally solid vector lattices, which attracted the attention of several authors in series of recent papers; for example see (Aydın et al., 2018a and b; Troitsky, 2004; Zabeti, 2017). In this paper, an *operator* means that it is a linear operator between vector spaces. In the (Troitsky, 2004), the bounded operator is studied on topological linear space. Also, by using order boundedness, the bounded operator is defined on locally solid vector lattices in (Zabeti, 2017). In this paper, we define the concept of bounded operator from vector lattice to locally solid vector lattice. Let give some basic notation and terminology that will be used in this paper. Every linear topology τ on a vector space E has a base $\aleph$ for the zero neighborhoods satisfying the followings; for each $V\in\aleph$, we have $\lambda V\subseteq V$ for all scalar $|\lambda|\leq 1$; for any $V_1,V_2\in\aleph$ there is another $V\in\aleph$ such that $V\subseteq V_1\cap V_2$; for each $V\in\aleph$ there exists another $U\in\aleph$ with $U+U\subseteq V$; for any scalar λ and for each $V\in\aleph$, the set λV is also in $\aleph$. In this article, unless otherwise, when we mention a zero neighborhood, it means that it always belongs to a base that holds the above properties. In a vector lattice E, a subset A is called as *solid* if, for $y\in A$ and $x\in E$ with $|x|\leq|y|$, we have $x\in A$. Let E be vector lattice and τ be a linear topology on it. Then (E,τ) is said a *locally solid vector lattice* (or, *locally solid Riesz*

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space) if τ has a base which takes place with solid sets, for more details on these notions see (Aliprantis and Burkinshaw, 2003 and 2006).

Let E be a vector lattice. E is said to be *order complete* if each subset of E that is bounded above has a supremum. For $a \in E_+$, we give an *order interval* defined by $[-a,a]=\{x \in E : -a \leq x \leq a\}$, if any subset in E is included in an order interval then it is called *order bounded*. Similarly, in a topological vector lattice (E,τ), a subset B is said to be called *topological bounded* (or, *bounded*) if, for every zero neighborhood V in E, there is a positive scalar λ with $B \subseteq \lambda V$. For more information about some relation and result for mentioned types of bounded sets, we refer the reader to (Aliprantis and Burkinshaw, 2003 and 2006; Hong, 2016; Troitsky, 2004). An *order bounded operator* between vector lattices sends order bounded subsets to order bounded subsets. Since there exists an order relation on order bounded sets, there is not exist a topological structure on order bounded operators. But, the space of all order bounded sets $B_b(E)$ on a Banach lattices E forms a Banach lattice. Through this paper, we study on a space that is a subspace of linear operators between vector lattices. Also, we assume that all vector spaces are real, and topological vector spaces are locally solid vector lattices. Now, we give some known types of bounded operators. Assume T is an operator between topological vector spaces E and F, then it is called bb*-bounded* if it sends each the topological bounded set into a topological bounded set. Also, T is said to be nb*-bounded* if T sends some zero neighborhood to a topological bounded set; see (Troitsky, 2004). For locally solid vector lattices E and F, T is called as bo*-bounded* if T sends topological bounded set to order bounded set, and called no*-bounded* if T maps some zero neighborhood U in E into order bounded set; see (Zabeti, 2017). Motivated by these definitions, we give the following notion.

Definition 1.1. An operator T from a vector lattice X to a locally solid vector lattice (E,τ) is called ob*-bounded* if T sends every order bounded set in X into topological bounded set in E.

In locally solid vector lattices, each order bounded subset is topologically bounded; see Theorem 2.19(i) of (Aliprantis and Burkinshaw, 2003), and so we have the following result.

Remark 1.2.

1. bo-bounded and bb-bounded operators between locally solid vector lattices are ob-bounded.
2. Every order bounded operator from a vector lattice to a locally solid vector lattice is ob-bounded.

It can be asked that whether an -bounded operator is order bounded or -bounded, or not. By considering Example 2.4 of (Hong, 2016), we have the negative answer for order boundedness since bounded set may be not order bounded in locally solid vector lattice. Also, by considering Example 2 of (Zabeti, 2017) and the Theorem 2.19(i) of (Aliprantis and Burkinshaw, 2003), the identity operator in the example is -bounded but it can not be -bounded. But, on the other hand, we have a partial positive answer. In locally solid vector lattice, each bounded set is order bounded when the space has an order bounded zero neighborhood; see Theorem 2.2 of (Hong, 2016). So, every -bounded operator from a vector lattice to a locally solid vector lattice with an order bounded zero neighborhood is order bounded, and also every -bounded operator between locally solid vector lattices with order bounded zero neighborhoods is -bounded, and lastly every -bounded operator from a locally solid vector lattice with an order bounded zero neighborhood to a locally solid vector lattice is -bounded.

Remark 1.3. In a topological vector space, summation of finite topological bounded sets are bounded. So, summation of finite ob-bounded operators are also ob-bounded. The product of two ob-bounded operators may not be ob-bounded. However, the product of an ob-bounded operator T with a bo-bounded operator S from the left hand (i.e $S \circ T$) is order bounded, so ob-bounded. A net (S_α) of ob-bounded operators is called *uniform convergent to zero* on an order bounded set B if, for each zero neighborhood V in $E,$ there exists an index α_0 such that $S_\alpha(B) \subseteq V$ for all $\alpha > \alpha_0$; see 2.16 of (Troitsky, 2004). The class of all ob-bounded operators from a vector lattice X to a locally solid vector lattice (E,τ) is denoted by $B_{ob}(X,E)$. Thus, this space can be considered on bounded sets with the topology of uniform convergence. Also, ob-bounded operators will be considered with the topology of equicontinuous convergence with order bounded sets. A net (S_α) of ob-bounded operators is called *equicontinuous convergent to zero* with order bounded sets if, for each zero neighborhood V in $E,$ there is an order bounded set B in X such that, for every $\varepsilon > 0,$ there exists an index α_0 such that $S_\alpha(B) \subseteq \varepsilon V$ for all $\alpha > \alpha_0$. We say that (S_α) converges to S equicontinuously with order bounded sets if $(S_\alpha - S)$ converges to zero equicontinuously with order bounded sets; see 2.18 of (Troitsky, 2004).

2. Main Results

We give some rerults about ob-bounded operator. In the following two results, we show the continuity of addition and scalar multiplication with the uniform convergence topology and the equicontinuous convergence topology, respectively.

Theorem 2.1. Consider the uniform convergence topology on order bounded sets. Then the operations of addition and scalar multiplication are continuous in $B_{ob}(X,E)$.

Proof: Let's take the nets (S_α) and (T_α) of ob-bounded operators that converges to zero uniformly on order bounded sets. Fix an arbitrary order bounded set B in X. Consider a zero neighborhood V in E then there exists another zero neighborhood U such that $U+U \subseteq V.$ So that there are some indexes α_1 and α_2 such that $(T_\alpha)(B) \subseteq U$ for every $\alpha > \alpha_1$ and $(S_\alpha)(B) \subseteq U$ for each $\alpha > \alpha_2$. Since index set is directed, there is another index α_0 such that $\alpha_0 > \alpha_1$ and $\alpha_0 > \alpha_2$. Hence, $(T_\alpha)(B) \subseteq U$ and $(S_\alpha)(B) \subseteq U$ for all $\alpha > \alpha_0$. Then we have $(T_\alpha + S_\alpha)(B) \subseteq T(U)_\alpha + S_\alpha(B) \subset U + U \subseteq V$ for each $\alpha > \alpha_0$. Therefore, since B is arbitrary, we get that addition is continuous in $B_{ob}(X,E)$. Next, we show that the scalar multiplication with the topology of uniform convergence is continuous. Consider the order bounded set B in E. Take a sequence of reals λ_n and assume it converges to zero. Since (T_α) is uniform convergent to zero on order bounded sets, for every zero neighborhood V in $E,$ there exists an index α_0 such that $(T_\alpha)(B) \subseteq V$ for all $\alpha > \alpha_0$. We know that for enough large $n,$ we have $|\lambda_n| \leq 1,$ and so $\lambda_n V \subseteq V.$ Then, for all $\alpha > \alpha_0$ and sufficiently large n, we have $\lambda_n T_\alpha(B) = T(\lambda_n B) \subseteq \lambda_n V \subseteq V$. Therefore, we get the desired result.

Theorem 2.2. The operations of addition and scalar multiplication are continuous in $B_{ob}(X,E)$ with the topology of equicontinuous convergence with order bounded sets.*Proof:* Suppose (S_α) and (T_α) are nets of ob-bounded operators and they converge to zero equicontinuously with order bounded sets. Fix an arbitrary zero neighborhood V. Then there is another zero neighborhood U with $U+U\subseteq V$. So, there exist order bounded sets B_1 and B_2 in X such that, for every $\varepsilon>0$, there are indexes α_1 and α_2 such that $T_\alpha(B_1)\subseteq\varepsilon U$ for all $\alpha>\alpha_1$ and $S_\alpha(B_2)\subseteq\varepsilon U$ for each $\alpha>\alpha_2$. Take an index α_0 such that $\alpha_0>\alpha_1$ and $\alpha_0>\alpha_2$. Hence, $T_\alpha(B_1)\subseteq\varepsilon U$ and $S_\alpha(B_2)\subseteq\varepsilon U$ for all $\alpha>\alpha_0$. Choose the order bounded set $B=B_1\cap B_2$ then

$$\begin{aligned}(T_\alpha+S_\alpha)(B)&\subseteq T_\alpha(B_1)+S_\alpha(B_2)\\ &\subseteq\varepsilon U+\varepsilon U\subseteq\varepsilon V\end{aligned}$$

for each $\alpha>\alpha_0$. Therefore, since V is arbitrary, we get the desired result.

Next, we show the continuity of scalar multiplication. Consider the zero neighborhood V and a sequence of reals λ_n such that $\lambda_n\to 0$. Since (T_α) is equicontinuous convergent to zero with order bounded sets, there is an order bounded set B such that, for every $\varepsilon>0$, there is an index α_0 such that $(T_\alpha)(B)\subseteq\varepsilon V$ for each $\alpha>\alpha_0$. Also, for sufficiently large n, we have $|\lambda_n|\le 1$, so that $\lambda_n V\subseteq V$. Then, for all $\alpha>\alpha_0$ and for fixed sufficiently large n, we have $\lambda_n T_\alpha(B)=T_\alpha(\lambda_n B)\subseteq\lambda_n\varepsilon V\subseteq\varepsilon V$. Thus, we get the desired result. The next two results give the continuity of the product of ob-bounded operators with the topology of uniform convergence and the topology of equicontinuous convergence, respectively.

Theorem 2.3. Let (E,τ) be locally solid vector lattice with an order bounded zero neighborhood. Then the product of ob-bounded operators is continuous in $B_{ob}(X,E)$ with the topology of uniform convergence on order bounded sets.

Proof: Assume (S_α) and (T_α) are nets of ob-bounded operators and they converge to zero uniformly on order bounded sets. Let's fix a zero neighborhood V in E, so there is $a\in E_+$ such that $V\subseteq[-a,a]$; see Theorem 2.2 of (Hong, 2016). Since (T_α) is uniform convergent to zero, there exists an index α_1 such that $T_\alpha([-a,a])\subseteq V$ for all $\alpha>\alpha_1$. Also, there is an index α_2 such that $S_\alpha([-a,a])\subseteq V$ for all $\alpha>\alpha_2$. Since there is an index α_0 such that $\alpha_0>\alpha_1$ and $\alpha_0>\alpha_2$, we have

$S_\alpha\left(T_\alpha([-a,a])\right)\subseteq S_\alpha(V)\subseteq S_\alpha([-a,a])\subseteq V$

for each $\alpha>\alpha_0$. Therefore, we get the desired result.

Theorem 2.4. Suppose (E,τ) is a locally solid vector lattice with an order bounded zero neighborhood. Then the product of ob-bounded operators is continuous in $B_{ob}(X,E)$ with the topology of equicontinuous convergence with order bounded sets.

Proof: Let (S_α) and (T_α) be nets of ob-bounded operators that are equicontinuous convergent to zero with order bounded sets. Let's fix a zero neighborhood V in E and, by applying Theorem 2.2 of (Hong, 2016), there is $a\in E_+$ such that $V\subseteq[-a,a]$. On the other hand, there exist order bounded sets B_1 and B_2 in X such that, for every $\varepsilon>0$, there are indexes α_1 and α_2 such that $T_\alpha(B_1)\subseteq\varepsilon V$ for each $\alpha>\alpha_1$ and $S_\alpha(B_2)\subseteq\varepsilon V$ for all $\alpha>\alpha_2$. Take an index α_0 with $\alpha_0>\alpha_1$ and $\alpha_0>\alpha_2$, and order bounded set $B=B_1\cap B_2$. Then, for each $\alpha>\alpha_0$,

$$\begin{aligned}S_\alpha\left(T_\alpha(B)\right)&\subseteq S_\alpha(\varepsilon V)=\varepsilon S_\alpha(V)\\ &\subseteq\varepsilon S_\alpha([-a,a]).\end{aligned}$$

By Theorem 2.19(i) of (Aliprantis and Burkinshaw, 2003), there is a positive scalar

$\lambda > 0$ such that $[-a,a] \subseteq \lambda V$. Then, for given positive scalar $\frac{\varepsilon}{\lambda}$, there exists an index α_3 such that $T_\alpha(B) \subseteq \frac{\varepsilon}{\lambda} V$ and $S_\alpha(B) \subseteq \frac{\varepsilon}{\lambda} V$ for each $\alpha > \alpha_3$. Thus,

$$S_\alpha\left(T_\alpha(B)\right) \subseteq S_\alpha(\frac{\varepsilon}{\lambda}V) \subseteq \frac{\varepsilon}{\lambda} S_\alpha([-a,a]) \subseteq \frac{\varepsilon}{\lambda}\lambda V = \varepsilon V$$

for all $\alpha > \alpha_3$.

In the following two works, we show the lattice operations are continuous with the topology of uniform convergence and the topology of equicontinuous convergence, respectively.

Theorem 2.5. Suppose (E,τ) is a locally solid vector lattice and X is a vector lattice. If E has an order bounded zero neighborhood and order complete property, then the lattice operations with the topology of uniform convergence are continuous in $B_{ob}(X,E)$ on order bounded sets.

Proof: Assume (S_α) and (T_α) are nets of *ob*-bounded operators, and suppose they converge to the linear operators S and T uniformly on order bounded sets, respectively. For each $x \in X_+$, by applying the Riesz–Kantorovich formula, we have

$$(S \vee T)x = \sup\left\{Sv + Tu : u + v = x, 0 \le u, v\right\}.$$

Fix an order bounded set B and fix $x \in B_+$. Pick positive elements u and v with $u + v = x$, and so that $u, v \in B_+$. Also, for subsets A_1 and A_2, we have $\sup(A_1) - \sup(A_2) \le \sup(A_1 - A_2)$ in a vector lattice. Then we get

$$\begin{aligned}(S_\alpha \vee T_\alpha - S \vee T)(x) &= \sup\left\{S_\alpha u + T_\alpha v : u + v = x, 0 \le u, v\right\} \\ &\quad - \sup\left\{Su + Tv : u + v = x, 0 \le u, v\right\} \\ &\le \sup\left\{(S_\alpha - S)u + (T_\alpha - T)v : u + v = x, 0 \le u, v\right\}.\end{aligned}$$

For given a zero neighborhood V in E, pick a zero neighborhood U with $U + U \subseteq V$. Thus, there are some indexes α_1 and α_2 such that $(T_\alpha - T)(B) \subseteq U$ for every $\alpha > \alpha_1$ and $(S_\alpha - S)(B) \subseteq U$ for each $\alpha > \alpha_2$. There exists another index α_0 such that $\alpha_0 > \alpha_1$ and $\alpha_0 > \alpha_2$, so that $(T_\alpha - T)(B) \subseteq U$ and $(S_\alpha - S)(B) \subseteq U$ for all $\alpha > \alpha_0$. Hence, for all $\alpha > \alpha_0$ we get

$$(S_\alpha \vee T_\alpha - S \vee T)(x) \le (S_\alpha - S)(x) + (T_\alpha - T)(x) \subseteq U + U \subseteq V.$$

Therefore, we have $(S_\alpha \vee T_\alpha)(B) - (S \vee T)(B) \subseteq V$ for each $\alpha > \alpha_0$, and so we get the desired result.

Question 2.6. Does the Theorem 2.5. hold without order bounded zero neighborhood?

Theorem 2.7. Suppose (E,τ) is a locally solid vector lattice and X is a vector lattice. If E has an order bounded zero neighborhood and order complete property, then the lattice operations with the topology of equicontinuous convergence are continuous in $B_{ob}(X,E)$ with order bounded sets.

Proof: Let (S_α) and (T_α) be nets of *ob*-bounded operators which converge to the linear operators S and T equicontinuously with order bounded sets, respectively. By the Riesz–Kantorovich formula, we have

$$(S \vee T)(x) = \sup\left\{Sv + Tu : u + v = x, 0 \le u, v\right\}$$

for ever $x \in X_+$. Fix a zero neighborhood V in E. Take a zero neighborhood U with $U + U \subseteq V$. There are order bounded sets B_1 and B_2 in X such that, for every $\varepsilon > 0$, there exist indexes α_1 and α_2 such that $T_\alpha(B_1) \subseteq \varepsilon U$ for each $\alpha > \alpha_1$ and $_\alpha(B_2) \subseteq \varepsilon U$ for all $\alpha > \alpha_2$. Pick an index α_0 with $\alpha_0 > \alpha_1$ and $\alpha_0 > \alpha_2$, and order bounded set $B = B_1 \cap B_2$. Hence, $T_\alpha(B) \subseteq \varepsilon U$

and $S_\alpha(B)\subseteq \varepsilon U$ for all $\alpha>\alpha_0$. Fix $x\in B_+$, suppose u and v are positive elements with $u+v=x$, and so that $u,v\in B$. Then we get

$$\begin{aligned}(S_\alpha\vee T_\alpha)(x)-(S\vee T)(x)&=\sup\{S_\alpha u+T_\alpha v: u+v=x, 0\le u,v\}\\ &\quad-\sup\{Su+Tv: u+v=x, 0\le u,v\}\\ &\le \sup\{(S_\alpha-S)(u)+(T_\alpha-T)(v): u+v=x, 0\le u,v\}.\end{aligned}$$

Then, for each , $\alpha>\alpha_0$ we have

$$\begin{aligned}(S_\alpha\vee T_\alpha)(x)-(S\vee T)(x)&=(S_\alpha-S)(x)+(T_\alpha-T)(x)\\ &\subseteq \varepsilon U+\varepsilon U\subseteq \varepsilon V.\end{aligned}$$

Therefore, we get the result.

Question 2.8. Does the Theorem 2.7. hold without order bounded zero neighborhood?

We finish this paper with the following results which show that the topologically complete of $B_{ob}(X,E)$ with the assigned topologies.

Proposition 2.9. If (T_α) be a net of ob-bounded operators in $B_{ob}(X,E)$ which uniform convergent to the linear operators T on order bounded sets then T is ob-bounded.

Proof: Let B be an order bounded set. For given a zero neighborhood V in E, there exists an index α_0 such that $(T-T_\alpha)(B)\subseteq V$ for all $\alpha>\alpha_0$. So, $T(B)\subseteq V+T_\alpha(B)$.

Since T_{α_0} is ob-bounded, $T_{\alpha_0}(B)$ is bounded in E. Also, $T(B)$ is also bounded since the sum of two bounded sets is bounded in topological vector spaces.

Question 2.10. Let (T_α) be a net of ob-bounded bounded operators in $B_{ob}(X,E)$. Is T ob-bounded operator whenever (T_α) equicontinuous convergent to the linear operators T with order bounded sets?

A net (S_α) in a locally solid vector lattice is said to be *order converges to zero uniformly* on topological bounded set B if, for each order interval $[-a,a]$, there exists an index α_0 such that $S_\alpha(B)\subseteq[-a,a]$ for each $\alpha>\alpha_0$; see (Zabeti, 2017).

Proposition 2.11. Let (T_α) be a net of ob-bounded operators from a locally solid vector lattice $(X,\hat{\tau})$ to a locally solid vector lattice (E,τ). If (T_α) order converges to the linear operator T uniformly on zero neighborhoods then T is ob-bounded.

Proof: Suppose B is an order bounded set. So, B is bounded in X; see Theorem 2.19(i) of (Aliprantis and Burkinshaw, 2003). Then, by assumption, for any $e\in E_+$, there exists an index α_0 with $(T-T_\alpha)(B)\subseteq[-e,e]$ for each $\alpha>\alpha_0$. Thus, we have $T(B)\subseteq[-e,e]+T_{\alpha_0}(B)$. Fix a τ-neighborhood V, and consider another τ-neighborhood W with $W+W\subseteq V$. Since T_{α_0} is ob-bounded, there exists a positive scalar γ_1 such that $T_{\alpha_0}(B)\subseteq\gamma_1 W$. Also, by using Theorem 2.19(i) of (Aliprantis and Burkinshaw, 2003), there exists another positive scalar γ_2 such that $[-e,e]\subseteq\gamma_2 W$. Let's take $\gamma=\max\{\gamma_1,\gamma_2\}$, and so, by solidness of W, we have $\gamma_1 W\subseteq\gamma W$ and $\gamma_2 W\subseteq\gamma W$. Thus, we have

$$\begin{aligned}T(B)&\subseteq[-e,e]+T_{\alpha_0}(B)\subseteq\gamma_1 W+\gamma_2 W\\ &\subseteq\gamma W+\gamma W\subseteq\gamma V.\end{aligned}$$

Therefore we get the desired result.

Theorem 2.12. Let (T_α) be a net of $B_{ob}(X,E)$ which uniform convergent to the linear operator T on order bounded sets. If (E,τ) is topologically complete vector lattice, and each order bounded set is absorbing in E then, with the topology of uniform convergence on order bounded sets, $B_{ob}(X,E)$ is complete.

Proof: Assume (E,τ) is topological complete, and (T_α) is a Cauchy net in $B_{ob}(X,E)$. Consider an order bounded set B in X. Then, for each zero neighborhood V, there exists an index α_0 such that $(T_\alpha - T_\beta)(B) \subseteq V$ for every $\alpha > \alpha_0$ and for all $\beta > \alpha_0$. Let's $x \in X$, there exists a positive real η so that $x \in \eta B$ since B is absorbing. Thus, $(T_\alpha - T_\beta)(x) \subseteq V$ for each $\alpha, \beta > \alpha_0$, so we get that $T_\alpha(x)$ is a Cauchy net in E. Thus, if we take $T(x) = \lim_\alpha T_\alpha(x)$. Therefore, by Proposition 2.9, we get the desired result.

Theorem 2.13. Suppose $(X,\hat{\tau})$ and (E,τ) are locally solid vector lattices, and (T_α) is a net of ob-bounded operators between them. Also, assume that (T_α) order converges to the linear operator T uniformly on zero neighborhoods. If (E,τ) is complete vector lattice with locally solid topology then $B_{ob}(X,E)$ is also complete with the topology of order uniformly convergence on zero neighborhood sets.

Proof: Assume is (E,τ) topological complete and (T_α) is a Cauchy net in $B_{ob}(X,E)$. Consider a bounded set B in X. Thus, for each $a \in E_+$, there is an α_0 with $(T_\alpha - T_\beta)(B) \subseteq [-a,a]$ for each $\alpha, \beta > \alpha_0$. Take $x \in X$, there exists a positive real η so that $x \in \eta B$ since B is solid. Thus, $(T_\alpha - T_\beta)(x) \subseteq V$ for each $\alpha, \beta > \alpha_0$, so we get that $T_\alpha(x)$ is a Cauchy net in E. Pick $T(x) = \lim_\alpha T_\alpha(x)$. Therefore, by Proposition 2.11, we get the desired result.

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