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Journal of Scientific Reports-C

journal homepage: *https://dergipark.org.tr/en/pub/jsrc*

E-ISSN: 2717-8633 ***Sayı(Number) 13, Ağustos(August) 2026***

ARAŞTIRMA MAKALESİ/RESEARCH ARTICLE

Geliş Tarihi(Receive Date): 08.06.2026 *Kabul Tarihi(Accepted Date): 22.06.2026*

Kinematic tubular surface at a constant distance from the edge of regression

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Abstract

In this study, a rigid-body motion represented by quaternions is applied to a tubular surface in E^3. Starting from a given tubular surface, the corresponding kinematic surface is obtained by using a quaternionic rotation and a prescribed constant displacement. It is shown that, in general, the resulting kinematic surface is not a tubular surface. By considering special choices of the rotation axis, the rotation angle, and the translation vector of the motion, several special cases of the kinematic surface are obtained. For these cases, necessary and sufficient conditions are derived for the resulting kinematic surface to be a tubular surface. In the most general special case in which the kinematic surface is again a tubular surface, the coefficients of the first and second fundamental forms, the Gaussian curvature, the mean curvature, the principal curvatures, and the shape operator are computed. Moreover, criteria are given for the parameter curves of this kinematic tubular surface to be lines of curvature, asymptotic curves, and geodesics. Finally, for a unit speed curve in E^3, the associated tubular surface is first constructed, and then its kinematic surface at a constant distance from the edge of regression is obtained. The relevant computations are carried out, and the corresponding figures are presented.

Keywords: rigid-body motion; tubular surface; kinematic surface

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Kulu, A. & Ata, E., (2026), Journal of Scientific Reports-C, 013, 1-23

Sabit sırt uzaklıklı kinematik tubular yüzey

Öz

Bu çalışmada E^3 uzayında alınan bir tubular yüzeye kuaterniyonlarla ifade edilen katı cisim hareketi uygulanmaktadır. Tubular yüzeyin sabit sırt uzaklıklı kinematik yüzeyi elde edilmektedir. Elde edilen kinematik yüzeyin genel durumda bir tubular yüzey olmadığı gösterilmektedir. Kinematik hareketin dönme ekseni, dönme açısı ve öteleme vektörünün özel seçimleriyle kinematik yüzeyin özel durumları elde edilmektedir. Bu özel durumlarda kinematik yüzeyin tubular yüzey olması için gerekli ve yeterli şartlar incelenmektedir. Kinematik yüzeyin yine tubular yüzey olduğu en genel durum olan birinci durumda, temel diferansiyel geometrik özellikler olan temel formlar, Gauss eğriliği, ortalama eğrilik, asli eğrilikler ve şekil operatörü hesaplanmaktadır. Ayrıca bu kinematik tubular yüzeyin parametrik eğrilerinin eğrilik çizgisi, asimptotik eğri ve geodezik eğri olması için gerek ve yeter şartlar verilmektedir. Makalenin sonunda E^3 uzayında alınan birim hızlı bir eğri için önce tubular yüzey ve sonra sabit sırt uzaklıklı kinematik yüzeyi elde edilmektedir. İlgili hesaplamalar yapılmakta ve grafikler sunulmaktadır.

Anahtar Kelimeler: Katı cisim hareketi; tubular yüzey; kinematik yüzeyler

1. Introduction

A canal surface may be described as the envelope generated by a family of spheres whose centers move along a prescribed spine curve. When the radius is fixed along the curve, this construction gives a tube, also referred to as a tubular surface [1-3].

Tubular surfaces are important examples of surface families in differential geometry and geometric modelling. Because of their circular cross-sections along a spine curve, they naturally occur in CAD/CAM, engineering design, architectural modelling, medical visualization, and the modelling of pipe-like, rope-like, or cable-like structures [1].

The differential-geometric structure of tubular surfaces has been studied in several ambient spaces by using different moving frames and surface-theoretic approaches [2-5].

Quaternion-based techniques have also been used to describe rotations and related geometric transformations in the study of canal and tubular surfaces [6].

In this paper, we consider a tubular surface under a quaternionic rigid-body motion in Euclidean 3-space. The transformed surface is obtained by combining a rotation with a constant displacement determined along the surface frame. We then investigate the conditions under which this transformed surface keeps the tubular form.

The construction yields several special cases according to the choices of the rotation angle, the rotation axis, and the translation vector. These cases include the tubular case, the screw-motion case, the constant-distance tubular case, and the parallel surface case.

For the general tubular case obtained from this construction, we compute the fundamental form coefficients, the Gaussian and mean curvatures, the principal curvatures, and the shape operator. Moreover, we give necessary and sufficient conditions for the parameter curves to be lines of curvature, asymptotic curves, and geodesics.

The main contribution of this study is to determine how quaternionic rigid-body motions change the differential-geometric invariants of tubular surfaces and to characterize the cases in which the transformed surface remains tubular.

2. Preliminaries

Throughout this paper, E^3 denotes the three-dimensional Euclidean space endowed with the standard Euclidean metric. Let M be a regular surface parametrized by $\phi(s,\theta)$. The tangent vectors ϕ_s and ϕ_θ determine the tangent plane of M at regular points, and the unit normal vector field is denoted by n.

For this parametrization, we denote the coefficients of the first fundamental form by E, F and G, and the coefficients of the second fundamental form by e, f and g. The Gaussian curvature K, the mean curvature H, the principal curvatures and the shape operator are used in their classical differential-geometric sense. The standard formulas for these quantities can be found in Struik [7].

Let $\alpha: I \subset \mathbb{R} \rightarrow E^3$ be an arc-length parametrized regular curve. Its Frenet frame is denoted by $\{T, N, B\}$, where T, N and B are the tangent, principal normal and binormal vector fields, respectively. The curvature and torsion of α are denoted by κ and τ. Throughout the paper, the usual Frenet equations and the standard relations between curvature, torsion and the Frenet frame are used.

2.1. Quaternions

Let $q = q_0 + q_1 i + q_2 j + q_3 k$ be a quaternion, where $q_0, q_1, q_2, q_3 \in \mathbb{R}$ and i, j, k are the standard quaternionic units satisfying $i^2 = j^2 = k^2 = ijk = -1$. We write $q = S_q + V_q$, where $S_q = q_0$ is the scalar part and $V_q = q_1 i + q_2 j + q_3 k$ is the vector part. The conjugate of q is denoted by $\bar{q} = S_q - V_q$, and its norm is given by $||q|| = \sqrt{q\,\bar{q}}$.

If ||q||=1, then q is called a unit quaternion. Such a quaternion can be written as $q = \cos\left(\frac{\alpha}{2}\right) + \sin\left(\frac{\alpha}{2}\right) V$, where V is a unit vector and α is the rotation angle. Thus, q represents a rotation about the axis V. We refer to Kuipers [8] for the standard algebraic properties of quaternions and their use in rotations.

2.2. Rigid-body motion (Kinematic motion)

In this study, rigid-body motions in E^3 are represented by unit quaternions. A point $p \in E^3$ is identified with a pure quaternion. If q is the unit quaternion corresponding to a rotation of angle α about the unit axis V, then the rotated point is $p' = qp\overline{q}$. After adding a translation vector $t \in \mathbb{R}^3$, the rigid-body motion is written as $p'' = qp\overline{q} + t$.

When the translation direction coincides with the rotation axis, namely $t = V$, this motion gives a screw motion in E^3. The screw motion used in the construction is illustrated in Fig. 1.

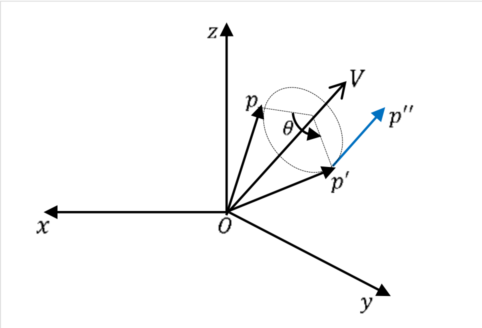

Fig. 1. Screw motion.

2.3. Tubular surface (Tube surface)

Let $C(s)$ be an arc-length parametrized regular curve in E^3 with Frenet frame $\{T(s), N(s), B(s)\}$. For a fixed positive radius r, a tubular surface around C is formed by placing, at each point of the curve, a circle of radius r in the normal plane spanned by $N(s)$ and $B(s)$. Therefore, the corresponding parametrization is written as

$$\phi: I \times [0,2\pi) \longrightarrow E^3$$

$$(s,\theta) \longrightarrow \phi(s,\theta) = C(s) + r\big(cos\theta N(s) + sin\theta B(s)\big),$$

where $r > 0$ and $\phi\big(I \times [0,2\pi)\big) = M$ [10].

Assuming that $C(s)$ is unit-speed, we have

$$\phi_s(s,\theta) = (1 - r\kappa cos\theta)T(s) + r\tau\big(-sin\theta N(s) + cos\theta B(s)\big) = (1 - r\kappa cos\theta)T(s) + \tau\phi_\theta(s,\theta)$$

$$\phi_\theta(s,\theta) = -rsin\theta N(s) + rcos\theta B(s)$$

and the unit normal vector field is given by

$$n(s,\theta) = \frac{\phi_s \times \phi_\theta}{\|\phi_s \times \phi_\theta\|} = -cos\theta N(s) - sin\theta B(s).$$

Moreover, the Gaussian curvature and mean curvature of M are given by

$$K = \frac{-\kappa cos\theta}{r(1 - r\kappa cos\theta)}, \quad H = \frac{1 - 2r\kappa cos\theta}{2r(1 - r\kappa cos\theta)} = \frac{1}{2}\left(\frac{1}{r} + rK\right),$$

and the principal curvatures are

$$k_1 = \frac{-\kappa cos\theta}{1 - r\kappa cos\theta}, \; k_2 = \frac{1}{r}.$$

The tubular surface generated by the spine curve $C(s)$ is illustrated in Fig. 2.

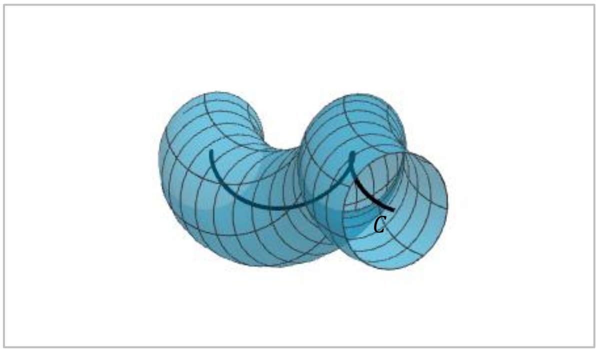

Fig. 2. Tubular surface.

Quaternionic constructions of canal and tubular surfaces have also been considered in [11,12].

2.4. Kinematic surface at a constant distance from the edge of regression

Let M and M^g be two surfaces in E^3 and let n_p be the unit normal vector at a point $p \in M$. For a unit vector $Z_p \in E^3$ at p, consider the mapping

$$g: M \longrightarrow M^g, \; g(p) = qp\overline{q} + \lambda Z_p \,; \; \lambda \in \mathbb{R}.$$

If this mapping is well defined, then M^g is called the kinematic surface generated from M by the above constant-distance construction [9,10].

Here $q = cos\frac{\alpha}{2} + sin\frac{\alpha}{2}V$ is a unit quaternion representing a positive rotation about the axis V. For $\alpha = 0$, the rotational part disappears and the construction reduces to the corresponding constant-distance surface of M.

If Z_p is chosen to coincide with the rotation axis V (i.e., $Z_p = V$), then M^g is called the screw surface of M.

If $\alpha = 0$ and Z_p is chosen to be the unit normal vector field of the surface (i.e., $Z_p = n_p$), then M^g is called the parallel surface of M.

In the next section, this construction is applied to a tubular surface, and the conditions under which the resulting kinematic surface remains tubular are investigated.

3. Kinematic tubular surface at a constant distance from the edge of regression

Let $M \subset E^3$ be a tubular surface

$$M = \{P = \phi(s,\theta) | \ \phi(s,\theta) = C(s) + r\big(cos\theta N(s) + sin\theta B(s)\big); \ \ s \in \mathbb{R}, \theta \in [0,2\pi)\}.$$

At a point $p = \phi(s,\theta) \in M$, the coordinate tangent vectors $\{\phi_s, \phi_\theta\}$, span the tangent space T_pM. Let n be the unit normal vector field of M. Then $\{\phi_s, \phi_\theta, n\}$ forms a positively oriented frame in E^3. Since

$\|\phi_s\| = \sqrt{(1 - r\kappa cos\theta)^2 + r^2\tau^2} = A$ and $\|\phi_\theta\| = r$, we set

$$X = \frac{\phi_s}{\|\phi_s\|} = \frac{(1 - r\kappa cos\theta)}{A}T(s) - \frac{r\tau sin\theta}{A}N(s) + \frac{r\tau cos\theta}{A}B(s)$$

$$Y = \frac{\phi_\theta}{\|\phi_\theta\|} = -sin\theta N(s) + cos\theta B(s)$$

$$n = \frac{\phi_s \times \phi_\theta}{\|\phi_s \times \phi_\theta\|} = -cos\theta N(s) - sin\theta B(s).$$

Hence, $\{X, Y, n\}$ is an orthonormal frame along the tubular surface M. Accordingly, the kinematic surface associated with M is defined by the following mapping

$$g: M \longrightarrow M^g, \ \ g(p) = qp\bar{q} + \lambda Z_p; \ \ \lambda \in \mathbb{R}$$

and

$$M^g = \{g(p) : g(p) = qp\bar{q} + \lambda Z_p; \ \ \lambda \in \mathbb{R} \text{ and } Z_p \in T_pE^3\}.$$

Here, quaternion multiplication is used in the definition of g, and Z_p is a unit vector. Carrying out the necessary computations in $g(p) = qp\bar{q} + \lambda Z_p$, for $p = (p_1, p_2, p_3) \in E^3$ and $V = (v_1, v_2, v_3) \in \mathbb{R}^3$ with $\|V\| = 1$, we obtain

$$g(p) = \ qp\bar{q} + \lambda Z_p$$

$$= \left(cos\frac{\alpha}{2} + sin\frac{\alpha}{2}\vec{V}\right)\vec{p}\left(cos\frac{\alpha}{2} - sin\frac{\alpha}{2}\vec{V}\right) + \lambda Z_p$$

$$= \cos\alpha\, p + (1-\cos\alpha)\langle V,p\rangle V + \sin\alpha\,(V\times p) + \lambda Z_p$$

$$\begin{aligned}= (&\cos\alpha\, p_1 + (1-\cos\alpha)\langle V,p\rangle v_1 + \sin\alpha\, v_2p_3 - \sin\alpha\, v_3p_2 + \lambda z_1,\\ &\cos\alpha\, p_2 + (1-\cos\alpha)\langle V,p\rangle v_2 + \sin\alpha\, v_3p_1 - \sin\alpha\, v_1p_3 + \lambda z_2\,,\\ &\cos\alpha\, p_3 + (1-\cos\alpha)\langle V,p\rangle v_3 + \sin\alpha\, v_1p_2 - \sin\alpha\, v_2p_1 + \lambda z_3)\end{aligned}$$

$$g(p) = \begin{bmatrix} v_1^2(1-\cos\alpha)+\cos\alpha & v_1v_2(1-\cos\alpha)-v_3\sin\alpha & v_1v_3(1-\cos\alpha)+v_2\sin\alpha \\ v_1v_2(1-\cos\alpha)+v_3\sin\alpha & v_2^2(1-\cos\alpha)+\cos\alpha & v_2v_3(1-\cos\alpha)-v_1\sin\alpha \\ v_1v_3(1-\cos\alpha)-v_2\sin\alpha & v_2v_3(1-\cos\alpha)+v_1\sin\alpha & v_3^2(1-\cos\alpha)+\cos\alpha \end{bmatrix}\begin{bmatrix}p_1\\p_2\\p_3\end{bmatrix} + \lambda\vec{Z}_p$$

and hence

$$g(p) = Rp + \lambda Z_p,$$

where

$$R = \begin{bmatrix} v_1^2(1-\cos\alpha)+\cos\alpha & v_1v_2(1-\cos\alpha)-v_3\sin\alpha & v_1v_3(1-\cos\alpha)+v_2\sin\alpha \\ v_1v_2(1-\cos\alpha)+v_3\sin\alpha & v_2^2(1-\cos\alpha)+\cos\alpha & v_2v_3(1-\cos\alpha)-v_1\sin\alpha \\ v_1v_3(1-\cos\alpha)-v_2\sin\alpha & v_2v_3(1-\cos\alpha)+v_1\sin\alpha & v_3^2(1-\cos\alpha)+\cos\alpha \end{bmatrix}$$

is the orthogonal rotation matrix representing the rotation through an angle α about the unit vector $V=(v_1,v_2,v_3)$. For $p=\phi(s,\theta)$, the parametric representation of the surface M^g is therefore

$$\Psi(s,\theta) = R\phi(s,\theta) + \lambda Z_p.$$

If we set $\varphi(s,\theta) = R\phi(s,\theta)$, then the parametrization of the kinematic surface can be written as

$$\Psi(s,\theta) = \varphi(s,\theta) + \lambda Z_p.$$

We first determine the parametric expression of $\varphi(s,\theta)$.

$$\varphi(s,\theta) = R\phi(s,\theta) = RC(s) + r\big(\cos\theta N(s) + \sin\theta B(s)\big)$$

$$\varphi(s,\theta) = RC(s) + r\cos\theta\big(RN(s)\big) + r\sin\theta\big(RB(s)\big).$$

Since $C(s)$ is a unit-speed curve and R is an orthogonal matrix with constant entries, the curve $RC(s)=\beta(s)$ is also a unit-speed curve; so that $\|\beta'(s)\|=1$. Let $\{T_\beta, N_\beta, B_\beta\}$ be the Frenet frame of β, and let κ_β and τ_β denote the curvature and torsion of β, respectively. Then, because R is constant, we obtain

$$\beta(s) = R\big(C(s)\big) \Rightarrow \beta'(s) = \Big(R\big(C(s)\big)\Big)' = R(C'(s)) = RT(s) = T_\beta(s)$$

$$N_\beta(s) = \frac{\beta''(s)}{\|\beta''(s)\|} = \frac{\big(RT(s)\big)'}{\big\|\big(RT(s)\big)'\big\|} = \frac{R\kappa(s)N(s)}{\|R\kappa(s)N(s)\|} = \frac{R\kappa(s)N(s)}{\kappa(s)\|RN(s)\|} = \frac{R\kappa(s)N(s)}{\kappa(s)} = RN(s)$$

$$B_\beta(s) = T_\beta(s)\times N_\beta(s) = RT(s)\times RN(s) = R(T(s)\times N(s)) = RB(s)$$

$$\kappa_\beta(s) = \|\beta''(s)\| = \|R\kappa(s)N(s)\| = \kappa(s)\|RN(s)\| = \kappa(s)\|N_\beta(s)\| = \kappa(s)$$

and

$$\tau_\beta(s) = \frac{det\big(\beta'(s),\beta''(s),\beta'''(s)\big)}{\|\beta''(s)\|^2} = \frac{det\big(RT,RKN,R(\kappa'N-\kappa^2T+\kappa\tau B)\big)}{\kappa^2}$$

$$\tau_\beta(s) = \frac{det(R)\,det\big(T,KN,(\kappa'N-\kappa^2T+\kappa\tau B)\big)}{\kappa^2} = \frac{det\big(T,KN,(\kappa'N-\kappa^2T+\kappa\tau B)\big)}{\kappa^2} = \tau(s).$$

Hence,

$$\varphi(s,\theta) = RC(s) + rcos\theta\big(RN(s)\big) + rsin\theta\big(RB(s)\big) = \beta(s) + r\left(cos\theta N_\beta(s) + sin\theta B_\beta(s)\right).$$

Therefore, $\varphi(s,\theta)$ is a tubular surface whose spine curve is $\beta(s)$; in other words, it is the rotated tubular surface obtained from $\varphi(s,\theta)$ by the orthogonal matrix R.

The partial derivatives of $\varphi(s,\theta)$ are

$$\varphi_\theta(s,\theta) = -rsin\theta N_\beta(s) + rcos\theta B_\beta(s)$$

$$\varphi_s(s,\theta) = \big(1 - r\kappa_\beta cos\theta\big)T_\beta(s) - r\tau_\beta sin\theta N_\beta(s) + r\tau_\beta cos\theta B_\beta(s)$$

Hence,

$$\|\varphi_s(s,\theta)\| = \sqrt{\big(1 - r\kappa_\beta cos\theta\big)^2 + r^2{\tau_\beta}^2} = A^\varphi$$

$$\|\varphi_\theta(s,\theta)\| = r.$$

Define

$$X^\varphi = \frac{\varphi_s(s,\theta)}{\|\varphi_s(s,\theta)\|} = \frac{\big(1-r\kappa_\beta cos\theta\big)}{A^\varphi}T_\beta(s) - \frac{r\tau_\beta sin\theta}{A^\varphi}N_\beta(s) + \frac{r\tau_\beta cos\theta}{A^\varphi}B_\beta(s)$$

$$Y^\varphi = \frac{\varphi_\theta(s,\theta)}{\|\varphi_\theta(s,\theta)\|} = -sin\theta N_\beta(s) + cos\theta B_\beta(s)$$

$$n^\varphi = \frac{\varphi_s \times \varphi_\theta}{\|\varphi_s \times \varphi_\theta\|} = -cos\theta N_\beta(s) - sin\theta B_\beta(s).$$

Then $\{X^\varphi, Y^\varphi, n^\varphi\}$ is an orthonormal frame along the surface $\varphi(s,\theta)$. Let the unit vector $Z(s,\theta)$ be expressed in this frame as

$$Z(s,\theta) = z_1X^\varphi + z_2Y^\varphi + z_3n^\varphi\colon\ z_1,z_2,z_3 \in \mathbb{R} \text{ and } z_1^2 + z_2^2 + z_3^2 = 1.$$

Accordingly,

$$\Psi(s,\theta)=\varphi(s,\theta)+\lambda Z(s,\theta)=\beta(s)+r\left(cos\theta N_\beta(s)+sin\theta B_\beta(s)\right)+\lambda(z_1X^\varphi+z_2Y^\varphi+z_3n^\varphi).$$

Setting $\lambda z_1=\lambda_1,\ \lambda z_2=\lambda_2,\ \lambda z_3=\lambda_3$, and after simplification, we obtain

$$\Psi(s,\theta)=\beta(s)+\left(\frac{\lambda_1(1-r\kappa_\beta cos\theta)}{A^\varphi}\right)T_\beta(s)+\left(rcos\theta-\lambda_1\frac{r\tau_\beta sin\theta}{A^\varphi}-\lambda_2 sin\theta-\lambda_3 cos\theta\right)N_\beta(s)$$
$$+\left(rsin\theta+\lambda_1\frac{r\tau_\beta cos\theta}{A^\varphi}+\lambda_2cos\theta-\lambda_3sin\theta\right)B_\beta(s).$$

The above parametrization defines the transformed surface M^g obtained from the tubular surface M. In general, this surface is not tubular. We now consider special choices of the rotation angle, the rotation axis, and the translation vector in the kinematic motion, and study the corresponding special cases of the kinematic surface. The following cases are classified according to the position of the translation vector with respect to the surface frame, the choice of the rotation axis, and the value of the rotation angle. This classification makes it possible to determine separately the geometric conditions under which the transformed surface preserves the tubular structure.

Case 1. Assume that $z_1=0$ in the translation vector $Z(s,\theta)=z_1X^\varphi+z_2Y^\varphi+z_3n^\varphi$. Then the parametrization $\Psi(s,\theta)=\varphi(s,\theta)+\lambda Z(s,\theta)$ takes the form

$$\Psi(s,\theta)=\beta(s)+\left((r-\lambda_3)cos\theta-\lambda_2sin\theta\right)N_\beta(s)+\left((r-\lambda_3)sin\theta+\lambda_2cos\theta\right)B_\beta(s),$$

$$\Psi(s,\theta)=\beta(s)+\left(\frac{(r-\lambda_3)}{\sqrt{(r-\lambda_3)^2+\lambda_2^2}}cos\theta-\frac{\lambda_2}{\sqrt{(r-\lambda_3)^2+\lambda_2^2}}sin\theta\right)N_\beta(s)\sqrt{(r-\lambda_3)^2+\lambda_2^2}$$
$$+\left(\frac{(r-\lambda_3)}{\sqrt{(r-\lambda_3)^2+\lambda_2^2}}sin\theta+\frac{\lambda_2}{\sqrt{(r-\lambda_3)^2+\lambda_2^2}}cos\theta\right)B_\beta(s)\sqrt{(r-\lambda_3)^2+\lambda_2^2}\,.$$

where $\lambda_2=\lambda z_2$ and $\lambda_3=\lambda z_3$. Let

$$\sqrt{(r-\lambda_3)^2+\lambda_2^2}=r_1,\quad \frac{(r-\lambda_3)}{\sqrt{(r-\lambda_3)^2+\lambda_2^2}}=cosu,\quad \frac{\lambda_2}{\sqrt{(r-\lambda_3)^2+\lambda_2^2}}=sinu. \tag{1}$$

Using these definitions, we obtain

$$\Psi(s,\theta)=\beta(s)+r_1\left((cosucos\theta-sinusin\theta)N_\beta(s)+(cosusin\theta+sinucos\theta)B_\beta(s)\right).$$

Setting $\gamma=u+\theta$, the parametrization becomes

$$\Psi(s,\gamma)=\beta(s)+r_1\left(cos\gamma N_\beta(s)+sin\gamma B_\beta(s)\right).$$

Thus, in Case 1, the resulting kinematic surface is again a tubular surface with spine curve $\beta(s)$ and radius r_1. This means that, when $z_1=0$, the displacement vector has no component in the T_β direction, and hence the transformed surface preserves the tubular surface structure.

Case 2. Assume that the translation vector is chosen to coincide with the rotation axis, i.e., $Z=V$. Then the parametrization of the surface is given by

$$\Psi(s,\theta)=\varphi(s,\theta)+\lambda V$$

$$\Psi(s,\theta) = \beta(s) + r\left(cos\theta N_\beta(s) + sin\theta B_\beta(s)\right) + \lambda V$$

$$\Psi(s,\theta) = \beta_1(s) + r\left(cos\theta N_{\beta_1}(s) + sin\theta B_{\beta_1}(s)\right),$$

where $\beta_1(s) = \beta(s) + \lambda V$. Hence, this kinematic surface is a tubular surface with spine curve $\beta_1(s)$ and radius r. Moreover, this tubular surface is also a screw surface.

Case 3. If the rotation angle in the kinematic motion is taken as $\alpha = 0$ and $z_1 = 0$ in the translation vector $Z(s,\theta)$, then in the rotation $\varphi(s,\theta) = R\phi(s,\theta)$ we have $R = I$. Hence,

$$\varphi(s,\theta) = \phi(s,\theta).$$

Therefore,

$$\Psi(s,\theta) = \phi(s,\theta) + \lambda Z(s,\theta)$$

$$\Psi(s,\theta) = C(s) + (rcos\theta - \lambda_2 sin\theta - \lambda_3 cos\theta)N(s) + (rsin\theta + \lambda_2 cos\theta - \lambda_3 sin\theta)B(s)$$

$$= C(s) + \left((r-\lambda_3)cos\theta - \lambda_2 sin\theta\right)N(s) + \left((r-\lambda_3)sin\theta + \lambda_2 cos\theta\right)B(s)$$

$$\Psi(s,\theta) = C(s) + \left(\frac{(r-\lambda_3)}{\sqrt{(r-\lambda_3)^2+\lambda_2^2}}cos\theta - \frac{\lambda_2}{\sqrt{(r-\lambda_3)^2+\lambda_2^2}}sin\theta\right)N(s)\sqrt{(r-\lambda_3)^2+\lambda_2^2} + \left(\frac{(r-\lambda_3)}{\sqrt{(r-\lambda_3)^2+\lambda_2^2}}sin\theta + \frac{\lambda_2}{\sqrt{(r-\lambda_3)^2+\lambda_2^2}}cos\theta\right)B(s)\sqrt{(r-\lambda_3)^2+\lambda_2^2}.$$

Let $\sqrt{(r-\lambda_3)^2+\lambda_2^2} = r_2$, $\frac{(r-\lambda_3)}{\sqrt{(r-\lambda_3)^2+\lambda_2^2}} = cosv$, $\frac{\lambda_2}{\sqrt{(r-\lambda_3)^2+\lambda_2^2}} = sinv$.

Using these definitions, we obtain

$$\Psi(s,\theta) = C(s) + r_2\left((cosvcos\theta - sinvsin\theta)N(s) + (cosvsin\theta + sinvcos\theta)B(s)\right).$$

Setting $\omega = v + \theta$, the parametrization of the kinematic surface at a constant distance from the edge of regression becomes

$$\Psi(s,\omega) = C(s) + r_2\left(cos\omega N(s) + sin\omega B(s)\right).$$

Thus, this surface is a tubular surface with spine curve $C(s)$ and radius r_2. Hence, under the above conditions, the kinematic surface is also a tubular surface.

Case 4. If the rotation angle in the kinematic motion is taken as $\alpha = 0$ and $z_1 = z_2 = 0$ in the translation vector $Z(s,\theta)$, then $R = I$. Since $Z(s,\theta) = \lambda n$, the parametrization of the surface at a constant distance from the edge of regression is

$$\Psi(s,\theta) = \phi(s,\theta) + \lambda n$$

$$\Psi(s,\theta) = C(s) + r\left(cos\theta N(s) + sin\theta B(s)\right) + \lambda\left(-cos\theta N(s) - sin\theta B(s)\right)$$

$$\Psi(s,\theta) = C(s) + |r-\lambda|\big(cos\theta N(s) + sin\theta B(s)\big).$$

Thus, this surface is a tubular surface with spine curve $C(s)$ and radius $|r-\lambda|$. Hence, in this case, the kinematic surface is a parallel tubular surface.

After examining the above special cases, we conclude that the most general case in which the kinematic surface of a tubular surface is again a tubular surface is Case 1, namely $z_1 = 0$. In what follows, for the kinematic tubular surface given by the parametrization

$$\Psi(s,\gamma) = \beta(s) + r_1\left(cos\gamma N_\beta(s) + sin\gamma B_\beta(s)\right),$$

we compute the coefficients of the first and second fundamental forms, the Gaussian and mean curvatures, the principal curvatures, and the shape operator. In this part, κ_β and τ_β denote the curvature and torsion of the spine curve $\beta(s)$, respectively, whereas K_Ψ and H_Ψ denote the Gaussian and mean curvatures of the kinematic tubular surface M^g.

First partial derivatives and norms

$$\Psi_\gamma(s,\gamma) = -r_1 sin\gamma N_\beta(s) + r_1 cos\gamma B_\beta(s)$$

$$\Psi_s(s,\gamma) = \big(1 - r_1\kappa_\beta cos\gamma\big)T_\beta(s) - r_1\tau_\beta sin\gamma N_\beta(s) + r_1\tau_\beta cos\gamma B_\beta(s)$$

$$\|\Psi_s(s,\gamma)\| = \sqrt{\big(1 - r_1\kappa_\beta cos\gamma\big)^2 + {r_1}^2{\tau_\beta}^2} = A^\varphi$$

$$\left\|\Psi_\gamma(s,\gamma)\right\| = r_1.$$

Orthonormal frame

$$X^g = \frac{\Psi_s(s,\gamma)}{\|\Psi_s(s,\gamma)\|} = \frac{\big(1 - r_1\kappa_\beta cos\gamma\big)}{A^\varphi}T_\beta(s) - \frac{r_1\tau_\beta sin\gamma}{A^\varphi}N_\beta(s) + \frac{r_1\tau_\beta cos\gamma}{A^\varphi}B_\beta(s)$$

$$Y^g = \frac{\Psi_\gamma(s,\gamma)}{\left\|\Psi_\gamma(s,\gamma)\right\|} = -sin\gamma N_\beta(s) + cos\gamma B_\beta(s)$$

$$n^g = \frac{\Psi_s \times \Psi_\gamma}{\left\|\Psi_s \times \Psi_\gamma\right\|} = -cos\gamma N_\beta(s) - sin\gamma B_\beta(s).$$

Second derivatives

$$\begin{aligned}\Psi_{ss}(s,\gamma) = {} & \big(1 - r_1\kappa_\beta cos\gamma\big)T'_\beta(s) - r_1\tau_\beta sin\gamma N'_\beta(s) + r_1\tau_\beta cos\gamma B'_\beta(s) + \big(r_1\kappa'_\beta cos\gamma\big)T_\beta(s) \\ & - \big(r_1\tau'_\beta sin\gamma\big)\vec{N}_\beta(s) + \big(r_1\tau'_\beta cos\gamma\big)\vec{B}_\beta(s)\end{aligned}$$

$$\begin{aligned}\Psi_{ss}(s,\gamma) = {} & \big(-r_1\kappa'_\beta cos\gamma + r_1\kappa_\beta\tau_\beta sin\gamma\big)T_\beta(s) + \big(\kappa_\beta - r_1\big(\kappa_\beta^2 + \tau_\beta^2\big)cos\gamma - r_1\tau'_\beta sin\gamma\big)N_\beta(s) \\ & + \big(-r_1\tau_\beta^2 sin\gamma + r_1\tau'_\beta cos\gamma\big)B_\beta(s)\end{aligned}$$

$$\Psi_{s\gamma}(s,\gamma) = r_1\kappa_\beta sin\gamma T_\beta(s) - r_1\tau_\beta cos\gamma N_\beta(s) - r_1\tau_\beta sin\gamma B_\beta(s)$$

$$\Psi_{\gamma\gamma}(s,\gamma) = r_1\left(-sin\gamma N_\beta(s) - cos\gamma B_\beta(s)\right) = r_1 n^t(s).$$

First fundamental form coefficients

$$E = \langle \Psi_s(s,\gamma), \Psi_s(s,\gamma)\rangle = \left(1 - r_1\kappa_\beta cos\gamma\right)^2 + r_1^2\tau_\beta^2$$

$$F = \langle \Psi_s(s,\gamma), \Psi_\gamma(s,\gamma)\rangle = r_1^2\tau_\beta$$

$$G = \langle \Psi_\gamma(s,\gamma), \Psi_\gamma(s,\gamma)\rangle = r_1^2.$$

Second fundamental form coefficients

$$e = \langle \Psi_{ss}(s,\gamma), n^g(s)\rangle = -\kappa_\beta cos\gamma\left(1 - r_1\kappa_\beta cos\gamma\right) + r_1\tau_\beta^2$$

$$f = \langle \Psi_{s\gamma}(s,\gamma), n^g(s)\rangle = r_1\tau_\beta$$

$$g = \langle \Psi_{\gamma\gamma}(s,\gamma), n^g(s)\rangle = r_1.$$

Gaussian and mean curvatures

$$K_\Psi = \frac{eg-f^2}{EG-F^2} = -\frac{\kappa_\beta cos\gamma}{r_1(1-r_1\kappa_\beta cos\gamma)}$$

$$H_\Psi = \frac{Eg-2Ff+Ge}{2(EG-F^2)} = \frac{\kappa_\beta r_1 cos\gamma-\frac{1}{2}}{r_1(r_1\kappa_\beta cos\gamma-1)} = \frac{1}{2}\left(\frac{1}{r_1} + K_\Psi r_1\right)$$

Principal curvatures

$$k_1^\Psi = -\frac{\kappa_\beta cos\gamma}{1-r_1\kappa_\beta cos\gamma}, \quad k_2^\Psi = \frac{1}{r_1}.$$

Shape operator matrix

$$S = \frac{1}{EG-F^2}\begin{bmatrix} E & -F \\ -F & G\end{bmatrix}\begin{bmatrix} e & f \\ f & g\end{bmatrix} = \frac{1}{r_1^2(r_1\kappa_\beta cos\gamma-1)^2}\begin{bmatrix} r_1^2 & -r_1^2\tau_\beta \\ -r_1^2\tau_\beta & \left(1 - r_1\kappa_\beta cos\gamma\right)^2 + r_1^2\tau_\beta^2\end{bmatrix}.$$

By taking the identities in (1) into account, the Gaussian and mean curvatures and the principal curvatures can also be expressed as follows:

$$K_\Psi = -\frac{Kk_2(1-\lambda_3k_2)+\lambda_2k_2^2\sqrt{\kappa^2(k_1-k_2)-K^2}}{\sqrt{(1-\lambda_3k_2)^2+\lambda_2^2k_2^2}\left(\lambda_3K-k_2+\lambda_2\sqrt{\kappa^2(k_1-k_2)-K^2}\right)}$$

$$H_\Psi = \frac{1}{2}\left(\frac{-K-k_2^2+2k_2\left(\lambda_3K+\lambda_2\sqrt{\kappa^2(k_1-k_2)-K^2}\right)}{\sqrt{(1-\lambda_3k_2)^2+\lambda_2^2k_2^2}\left(-k_2+\lambda_3K+\lambda_2\sqrt{\kappa^2(k_1-k_2)-K^2}\right)}\right)$$

$$k_1^{\Psi} = -\frac{K(1-\lambda_3 k_2)+\lambda_2 k_2\sqrt{\kappa^2(k_1-k_2)-K^2}}{\sqrt{(1-\lambda_3 k_2)^2+\lambda_2^2 k_2^2}\left(\lambda_3 K-k_2+\lambda_2\sqrt{\kappa^2(k_1-k_2)-K^2}\right)}$$

$$k_2^{\Psi} = \frac{1}{r_1}-\frac{k_2}{\sqrt{(1-\lambda_3 k_2)^2+\lambda_2^2 k_2^2}}.$$

Based on the above computations, we can state the following two theorems.

Theorem 1. Let M be a tubular surface in E^3. Under the condition $z_1 = 0$, let M^g be the kinematic tubular surface of M. Then the Gaussian curvature K_Ψ and the mean curvature H_Ψ of M^g are given, respectively, by

$$K_{\Psi} = -\frac{Kk_2(1-\lambda_3 k_2)+\lambda_2 k_2^2\sqrt{\kappa^2(k_1-k_2)-K^2}}{\sqrt{(1-\lambda_3 k_2)^2+\lambda_2^2 k_2^2}\left(\lambda_3 K-k_2+\lambda_2\sqrt{\kappa^2(k_1-k_2)-K^2}\right)}$$

$$H_{\Psi} = \frac{1}{2}\left(\frac{-K-k_2^2+2k_2\left(\lambda_3 K+\lambda_2\sqrt{\kappa^2(k_1-k_2)-K^2}\right)}{\sqrt{(1-\lambda_3 k_2)^2+\lambda_2^2 k_2^2}\left(-k_2+\lambda_3 K+\lambda_2\sqrt{\kappa^2(k_1-k_2)-K^2}\right)}\right)$$

where k_1 and k_2 are the principal curvatures of M.

Theorem 2. Let M be a tubular surface in E^3. Under the condition $z_1 = 0$, let M^g be the kinematic tubular surface of M. Then the principal curvatures of M^g are given, respectively, by

$$k_1^{\Psi} = -\frac{K(1-\lambda_3 k_2)+\lambda_2 k_2\sqrt{\kappa^2(k_1-k_2)-K^2}}{\sqrt{(1-\lambda_3 k_2)^2+\lambda_2^2 k_2^2}\left(\lambda_3 K-k_2+\lambda_2\sqrt{\kappa^2(k_1-k_2)-K^2}\right)}$$

$$k_2^{\Psi} = \frac{1}{r_1}-\frac{k_2}{\sqrt{(1-\lambda_3 k_2)^2+\lambda_2^2 k_2^2}}.$$

These formulas show how the Gaussian curvature, mean curvature, and principal curvatures of the kinematic tubular surface depend on the curvature and torsion of the transformed spine curve and on the parameters of the constant displacement.

Theorem 3. Let M be a tubular surface in E^3. Assume that $z_1 = 0$, and let M^g be the kinematic tubular surface of M. Then the parameter curves of M^g are lines of curvature if and only if the spine curve $\beta(s)$ is planar.

Proof. Under the condition $z_1 = 0$, it is known that the parametric equation of the kinematic tubular surface M^g is given by

$$\Psi(s,\gamma) = \beta(s) + r_1\left(cos\gamma N_\beta(s) + sin\gamma B_\beta(s)\right).$$

For the parametric curves of M^g to be lines of curvature, it is necessary that

$$F = \langle \Psi_s(s,\gamma), \Psi_\gamma(s,\gamma)\rangle = 0 \text{ and } f = \langle \Psi_{s\gamma}(s,\gamma), n^g(s)\rangle = 0.$$

Accordingly, we obtain

$$F = \langle \Psi_s(s,\gamma), \Psi_\gamma(s,\gamma)\rangle = r_1^2\tau_\beta = 0 \text{ and}$$

$$f = \langle \Psi_{s\gamma}(s,\gamma), n^g(s) \rangle = r_1 \tau_\beta = 0.$$

From these equalities, it *follows* that $r_1 \tau_\beta = 0$.

Since $r_1 = \sqrt{(r-\lambda_3)^2 + \lambda_2^2} \neq 0$, we must have $\tau_\beta = 0$. Hence, the curve $\beta(s)$ is a planar curve. Conversely, if the curve $\beta(s)$ is planar, then $\tau_\beta = 0$, and therefore $F = f = 0$. Thus, the parameter curves of the surface M^g are lines of curvature.

Thus, the line of curvature property of the parameter curves is directly related to the planarity of the spine curve.

Theorem 4. Let M be a tubular surface in E^3. Under the condition $z_1 = 0$, let M^g be the kinematic tubular surface at a constant distance from the edge of regression of M. Then the s-parameter curves of M^g are asymptotic curves if and only if

$$\frac{\tau^2}{\kappa} = \left(\frac{(r-\lambda_3)cos\theta - \lambda_2 sin\theta}{(r-\lambda_3)^2+\lambda_2^2}\right)\left(1 - K_\beta\big((r-\lambda_3)cos\theta - \lambda_2 sin\theta\big)\right).$$

Proof. Under the condition $z_1 = 0$, it is known that the parametric equation of the kinematic tubular surface M^g is

$$\Psi(s,\gamma) = \beta(s) + r_1\left(cos\gamma N_\beta(s) + sin\gamma B_\beta(s)\right).$$

For the s-parameter curves of the surface M^g to be asymptotic curves, it is necessary and sufficient that

$$e = \langle \Psi_{ss}(s,\gamma), n^g(s,\gamma) \rangle = 0.$$

Accordingly, we have

$$e = \langle \Psi_{ss}(s,\gamma), n^g(s,\gamma) \rangle = -\kappa_\beta cos\gamma\big(1 - r_1\kappa_\beta cos\gamma\big) + r_1\tau_\beta^2 = 0.$$

Substituting the identities in (1) into this equation and simplifying, we obtain

$$\frac{\tau^2}{\kappa} = \left(\frac{(r-\lambda_3)cos\theta - \lambda_2 sin\theta}{(r-\lambda_3)^2+\lambda_2^2}\right)\left(1 - K_\beta\big((r-\lambda_3)cos\theta - \lambda_2 sin\theta\big)\right).$$

Hence, the s-parameter curves of M^g are asymptotic curves if and only if

$$\frac{\tau^2}{\kappa} = \left(\frac{(r-\lambda_3)cos\theta - \lambda_2 sin\theta}{(r-\lambda_3)^2+\lambda_2^2}\right)\left(1 - K_\beta\big((r-\lambda_3)cos\theta - \lambda_2 sin\theta\big)\right).$$

This condition characterizes precisely when the corresponding parameter curves have zero normal curvature, and hence become asymptotic curves on the kinematic tubular surface.

Theorem 5. Let M *be a* tubular surface in E^3. Under the condition $z_1 = 0$, let M^g be the kinematic tubular surface of M. Then the γ-parameter curves of M^g are not asymptotic curves.

Proof. Under the condition $z_1 = 0$, it is known that the parametric equation of the kinematic tubular surface M^g is

$$\Psi(s,\gamma) = \beta(s) + r_1\left(cos\gamma N_\beta(s) + sin\gamma B_\beta(s)\right).$$

For the γ-parameter curves of the surface M^g to be asymptotic curves, the condition

$g = \langle \Psi_{\gamma\gamma}(s,\gamma), n^g(s) \rangle = 0$ must be satisfied. However,

$$g = \langle \Psi_{\gamma\gamma}(s,\gamma), n^g(s) \rangle = r_1.$$

Since $r_1 = \sqrt{(r-\lambda_3)^2 + \lambda_2^2} \neq 0$, we obtain $g \neq 0$. Therefore, the γ-parameter curves of M^g are not asymptotic curves.

Theorem 6. Let M be a tubular surface in E^3. Under the condition $z_1 = 0$, let M^g be the kinematic tubular surface of M. Then the s-parameter curves of M^g are geodesics if and only if there exists a constant $a \in \mathbb{R}$ such that

$$\sqrt{(r-\lambda_3)^2+\lambda_2^2}\left(\frac{(r-\lambda_3)cos\theta-\lambda_2 sin\theta}{\sqrt{(r-\lambda_3)^2+\lambda_2^2}}\right)^2\kappa_\beta^2 - 2\left(\frac{(r-\lambda_3)cos\theta-\lambda_2 sin\theta}{\sqrt{(r-\lambda_3)^2+\lambda_2^2}}\right)\kappa_\beta + \sqrt{(r-\lambda_3)^2+\lambda_2^2}\tau_\beta^2 = a$$

Proof. Assume that $z_1 = 0$. Then the kinematic tubular surface M^g of M is given by

$$\Psi(s,\gamma) = \beta(s) + r_1\left(cos\gamma N_\beta(s) + sin\gamma B_\beta(s)\right).$$

The s-parameter curves of M^g are of the form

$$\Psi(s,\gamma_0) = \beta(s) + r_1\left(cos\gamma_0 N_\beta(s) + sin\gamma_0 B_\beta(s)\right) = a_\Psi(s),$$

where γ_0 is constant.

For the curve $a_\Psi(s)$ to be a geodesic on the surface M^g, it is necessary and sufficient that

$$a_\Psi''(s) \times n^g(s,\gamma_0) = 0.$$

Since

$$a_\Psi''(s) = \Psi_{ss}(s,\gamma_0),$$

it is enough to compute $\Psi_{ss}(s,\gamma_0)$ and the unit normal vector field $n^g(s,\gamma_0)$. By direct differentiation, we obtain

$$\begin{aligned}\Psi_{ss}(s,\gamma_0) = &\left(-r_1\kappa_\beta' cos\gamma_0 + r_1\kappa_\beta\tau_\beta sin\gamma_0\right)T_\beta(s) + \left(\kappa_\beta - r_1(\kappa_\beta^2+\tau_\beta^2)cos\gamma_0 - r_1\tau_\beta' sin\gamma_0\right)N_\beta(s)\\ &+ \left(-r_1\tau_\beta^2 sin\gamma_0 + r_1\tau_\beta' cos\gamma_0\right)B_\beta(s).\end{aligned}$$

On the other hand, the unit normal vector field of M^g is

$$n^g(s,\gamma) = \frac{\Psi_s\times\Psi_\gamma}{\|\Psi_s\times\Psi_\gamma\|} = -cos\gamma N_\beta(s) - sin\gamma B_\beta(s).$$

Therefore,

$$\Psi_{ss}(s,\gamma_0)\times n^g(s,\gamma_0) = \left(\kappa_\beta sin\gamma_0 - r_1\kappa_\beta^2 sin\gamma_0 cos\gamma_0\right)T_\beta(s) + \left(r_1\kappa_\beta' sin\gamma_0 cos\gamma_0 - r_1\kappa_\beta\tau_\beta sin^2\gamma_0\right)N_\beta(s)$$

$$+\left(r_1\kappa_\beta' cos^2\gamma_0 + r_1\kappa_\beta\tau_\beta sin\gamma_0 cos\gamma_0\right)B_\beta(s) = 0.$$

Since $a_\Psi(s)$ is geodesic if and only if

$$a''_\Psi(s) \times n^g(s,\gamma_0) = 0,$$

we must have

$$\kappa_\beta sin\gamma_0 - r_1\kappa_\beta^2 sin\gamma_0 cos\gamma_0 = 0,$$

$$r_1\kappa'_\beta sin\gamma_0 cos\gamma_0 - r_1\kappa_\beta\tau_\beta sin^2\gamma_0 = 0,$$

and

$$r_1\kappa'_\beta cos^2\gamma_0 + r_1\kappa_\beta\tau_\beta sin\gamma_0 cos\gamma_0 = 0.$$

Combining the last two equations, we obtain

$$\kappa'_\beta cos\gamma_0 - r_1\kappa_\beta\kappa'_\beta cos\gamma_0 + r_1\tau_\beta\tau'_\beta = 0.$$

Integrating both sides, it follows that

$$r_1\kappa_\beta^2 cos^2\gamma_0 - 2\kappa_\beta cos\gamma_0 + r_1\tau_\beta^2 = a\,, \qquad (a \in \mathbb{R}).$$

Since $\gamma = \theta + u$, we use the relations

$$cos\gamma = cosucos\theta - sinusin\theta$$

$$sin\gamma = cosusin\theta + sinucos\theta$$

together with

$$r_1 = \sqrt{(r-\lambda_3)^2 + \lambda_2^2},\ \frac{(r-\lambda_3)}{r_1} = cosu,\ \frac{\lambda_2}{r_1} = sinu.$$

Substituting these equalities into the relation

$r_1\kappa_\beta^2 cos^2\gamma - 2\kappa_\beta cos\gamma + r_1\tau_\beta^2 = a$, *we obtain*

$$\sqrt{(r-\lambda_3)^2 + \lambda_2^2}\left(\frac{(r-\lambda_3)cos\theta - \lambda_2 sin\theta}{\sqrt{(r-\lambda_3)^2 + \lambda_2^2}}\right)^2 \kappa_\beta^2 - 2\left(\frac{(r-\lambda_3)cos\theta - \lambda_2 sin\theta}{\sqrt{(r-\lambda_3)^2 + \lambda_2^2}}\right)\kappa_\beta + \sqrt{(r-\lambda_3)^2 + \lambda_2^2}\tau_\beta^2 = a.$$

Hence, the s-parameter curves of M^g are geodesics if and only if there exists a constant $a \in \mathbb{R}$ such that

$$\sqrt{(r-\lambda_3)^2 + \lambda_2^2}\left(\frac{(r-\lambda_3)cos\theta - \lambda_2 sin\theta}{\sqrt{(r-\lambda_3)^2 + \lambda_2^2}}\right)^2 \kappa_\beta^2 - 2\left(\frac{(r-\lambda_3)cos\theta - \lambda_2 sin\theta}{\sqrt{(r-\lambda_3)^2 + \lambda_2^2}}\right)\kappa_\beta + \sqrt{(r-\lambda_3)^2 + \lambda_2^2}\tau_\beta^2 = a$$

This completes the proof.

4. Numerical example

Let us consider the curve $C(s)=\left(\frac{s}{\sqrt{2}},\cos\frac{s}{\sqrt{2}},sin\frac{s}{\sqrt{2}}\right)$. Since $C'(s)=\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}sin\frac{s}{\sqrt{2}},\frac{1}{\sqrt{2}}cos\frac{s}{\sqrt{2}}\right)$ and $\|C'(s)\|=1$, the curve is unit-speed. Therefore, Frenet frame and the curvature and torsion $\{T(s),N(s),B(s),\kappa(s),\tau(s)\}$ are obtained as follows:

$$T(s)=C'(s)=\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}sin\frac{s}{\sqrt{2}},\frac{1}{\sqrt{2}}cos\frac{s}{\sqrt{2}}\right)$$

$$N(s)=\frac{C''(s)}{\|C'(s)\|}=\left(0,-cos\frac{s}{\sqrt{2}},-sin\frac{s}{\sqrt{2}}\right)$$

$$B(s)=\vec{T}(s)\times\vec{N}(s)=\left(\frac{1}{\sqrt{2}},\frac{1}{2}sin\frac{s}{\sqrt{2}},-\frac{1}{2}cos\frac{s}{\sqrt{2}}\right)$$

$$\kappa(s)=\|C''(s)\|$$

$$\tau(s)=\frac{det\left(C'(s),C''(s),C''(s)\right)}{\|C''(s)\|^2}=\frac{1}{2}.$$

Let M be the tubular surface generated by the spine curve $C(s)$ with radius $r=2$. Then the parametric equation of M is

$$\phi(s,\theta)=C(s)+r\left(cos\theta N(s)+sin\theta B(s)\right)$$

$$\phi(s,\theta)=\left(\left(\frac{s}{\sqrt{2}}+\sqrt{2}sin\theta\right),\left(cos\frac{s}{\sqrt{2}}-2cos\theta cos\frac{s}{\sqrt{2}}+\sqrt{2}sin\theta sin\frac{s}{\sqrt{2}}\right),\left(sin\frac{s}{\sqrt{2}}-2cos\theta sin\frac{s}{\sqrt{2}}-\sqrt{2}sin\theta cos\frac{s}{\sqrt{2}}\right)\right)$$

The figure of this tubular surface is given below.

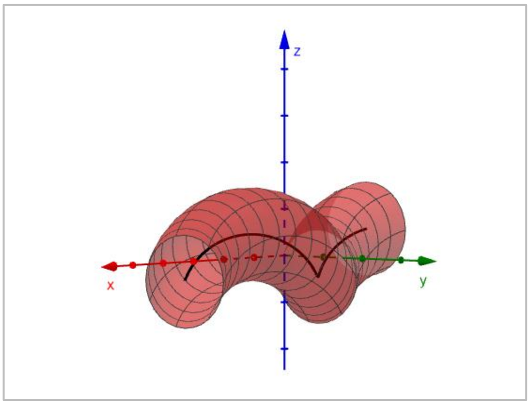

Fig. 3. The tubular surface with spine curve $C(s)$ and radius $r=2$.

Let $P = \phi(s,\theta)$. Then, in order to determine the kinematic surface M^g of the M, we take the rotation angle $\alpha = \pi$ and the rotation axis $V = \left(\frac{1}{2},\frac{1}{2},\frac{1}{\sqrt{2}}\right)$.

Then, the corresponding rotation operator is

$$q = cos\frac{\alpha}{2} + Vsin\frac{\alpha}{2} = cos\frac{\pi}{2} + \left(\frac{1}{2},\frac{1}{2},\frac{1}{\sqrt{2}}\right)sin\frac{\pi}{2}.$$

Accordingly, the corresponding orthogonal rotation matrix is

$$R = \frac{1}{4\sqrt{2}}\begin{bmatrix} \sqrt{2} & \sqrt{2}-4 & 2+2\sqrt{2} \\ \sqrt{2}+4 & \sqrt{2} & 2-2\sqrt{2} \\ 2-2\sqrt{2} & 2+2\sqrt{2} & 2\sqrt{2} \end{bmatrix}.$$

Let the translation vector be $Z = \left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$ and let the translation coefficient be $\lambda = 3 \in \mathbb{R}$. Then the parametric equation of the kinematic surface M^g, obtained by the transformation $g(p) = qp\bar{q} + \lambda Z_p$, is given by

$$\Psi(s,\theta) = R\phi(s,\theta) + \lambda Z(s,\theta).$$

Thus,

$$\Psi(s,\theta) = \beta(s) + \left(\frac{\sqrt{3}(1-cos\theta)}{\sqrt{(1-cos\theta)^2+1}}\right)T_\beta(s) + \left(2cos\theta - \frac{\sqrt{3}sin\theta}{\sqrt{(1-cos\theta)^2+1}} - \sqrt{3}sin\theta - \sqrt{3}cos\theta\right)N_\beta(s)$$
$$+\left(2sin\theta + \frac{\sqrt{3}cos\theta}{\sqrt{(1-cos\theta)^2+1}} + \sqrt{3}cos\theta - \sqrt{3}sin\theta\right)B_\beta(s)$$

Here,

$$\beta(s) = \left(\left(\frac{s}{4\sqrt{2}} + \frac{2+\sqrt{2}}{4}sin\frac{s}{\sqrt{2}} + \frac{1-2\sqrt{2}}{4}cos\frac{s}{\sqrt{2}}\right), \left(\frac{1+2\sqrt{2}}{4\sqrt{2}}s + \frac{\sqrt{2}-2}{4}sin\frac{s}{\sqrt{2}} + \frac{1}{4}cos\frac{s}{\sqrt{2}}\right), \left(\frac{\sqrt{2}-2}{4\sqrt{2}}s + \frac{1}{2}sin\frac{s}{\sqrt{2}} + \frac{2+\sqrt{2}}{4}cos\frac{s}{\sqrt{2}}\right)\right).$$

It should be noted that the kinematic surface M^g of the M is not a tubular surface. The corresponding kinematic surface obtained from the tubular surface is illustrated in Fig. 4.

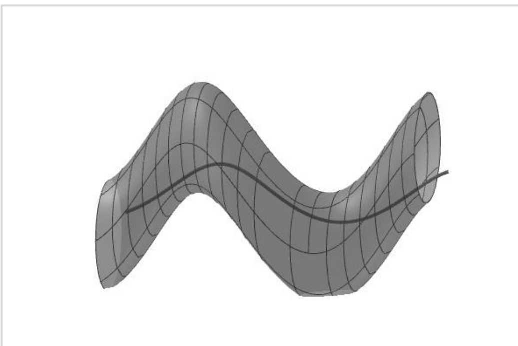

Fig. 4. Kinematic surface of the tubular surface.

Now, let us examine the special cases of the kinematic surface M^g of M. In the following example cases, the constants are chosen according to the translation vector, the rotation axis and the rotation angle considered in each case. These choices make the corresponding kinematic surfaces explicitly computable and suitable for graphical representation.

Example case 1. If, in the kinematic surface M^g, the translation vector is chosen as $Z = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, that is, if $z_1 = 0$ in the translation vector $Z(s,\theta)$, then the parametric equation of the surface M^g is

$$\Psi(s,\theta) = R\phi(s,\theta) + \lambda Z(s,\theta).$$

Thus,

$$\Psi(s,\theta) = \beta(s) + \left(2cos\theta - \tfrac{3}{\sqrt{2}}sin\theta - \tfrac{3}{\sqrt{2}}cos\theta\right)N_\beta(s) + \left(2sin\theta + \tfrac{3}{\sqrt{2}}cos\theta - \tfrac{3}{\sqrt{2}}sin\theta\right)B_\beta(s),$$

$$\Psi(s,\theta) = \beta(s) + \sqrt{13-6\sqrt{2}}\left(\frac{2-\frac{3}{\sqrt{2}}}{\sqrt{13-6\sqrt{2}}}cos\theta - \frac{\frac{3}{\sqrt{2}}}{\sqrt{13-6\sqrt{2}}}sin\theta\right)N_\beta(s)$$
$$+\sqrt{13-6\sqrt{2}}\left(\frac{2-\frac{3}{\sqrt{2}}}{\sqrt{13-6\sqrt{2}}}sin\theta + \frac{\frac{3}{\sqrt{2}}}{\sqrt{13-6\sqrt{2}}}cos\theta\right)B_\beta(s).$$

Here, if we take

$$r_1 = \sqrt{13-6\sqrt{2}}\,,\ \frac{2-\frac{3}{\sqrt{2}}}{\sqrt{13-6\sqrt{2}}} = cosu,\ \frac{\frac{3}{\sqrt{2}}}{\sqrt{13-6\sqrt{2}}} = sinu,$$

then

$$\Psi(s,\theta) = \beta(s) + r_1\left((cosucos\theta - sinusin\theta)N_\beta(s) + (cosusin\theta + sinucos\theta)B_\beta(s)\right).$$

Using the identity $\gamma = \theta + u$, we obtain

$$\Psi(s,\gamma) = \beta(s) + r_1\left(cos\gamma N_\beta(s) + sin\gamma B_\beta(s)\right).$$

This equation represents a tubular surface with spine curve $\beta(s)$ and radius $r_1 = \sqrt{13-6\sqrt{2}}$. In this case, M^g is a kinematic tubular surface. This surface corresponds to the most general case in which the kinematic surface M^g of the M is again a tubular surface. The other special cases can be obtained as particular cases of this one. The kinematic tubular surface obtained in the most general case in which the kinematic surface is again a tubular surface is illustrated in Fig. 5.

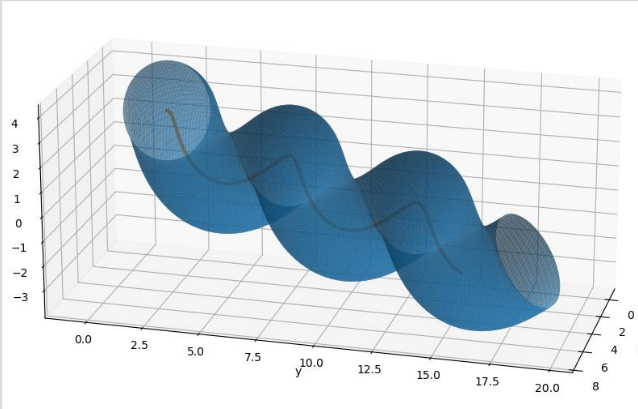

Fig. 5. Kinematic tubular surface with spine curve $\beta(s)$ and radius r_1.

For the kinematic surface M^g, the coefficients of the first and second fundamental forms, the Gaussian and mean curvatures, the principal curvatures, and the matrix of the shape operator can be computed as follows.

Coefficients of the first fundamental form

$$E = \langle \Psi_s(s,\gamma), \Psi_s(s,\gamma) \rangle = \left(1 - r_1\kappa_\beta cos\gamma\right)^2 + r_1^2\tau_\beta^2 = \left(1 - \frac{r_1}{2}cos\gamma\right)^2 + \frac{13-6\sqrt{2}}{4}$$

$$F = \langle \Psi_s(s,\gamma), \Psi_\gamma(s,\gamma) \rangle = r_1^2\tau_\beta = \frac{13-6\sqrt{2}}{2}$$

$$G = \langle \Psi_\gamma(s,\gamma), \Psi_\gamma(s,\gamma) \rangle = r_1^2 = 13 - 6\sqrt{2}$$

Coefficients of the second fundamental form

$$e = \langle \Psi_{ss}(s,\gamma), \vec{n}^g(s) \rangle = -\kappa_\beta cos\gamma\left(1 - r_1\kappa_\beta cos\gamma\right) + r_1\tau_\beta^2 = -\frac{1}{2}cos\gamma\left(1 - \frac{r_1}{2}cos\gamma\right) + \frac{\sqrt{13-6\sqrt{2}}}{4}$$

$$f = \langle \Psi_{s\gamma}(s,\gamma), \vec{n}^g(s) \rangle = r_1\tau_\beta = \frac{\sqrt{13-6\sqrt{2}}}{2}$$

$$g = \langle \Psi_{\gamma\gamma}(s,\gamma), \vec{n}^g(s) \rangle = r_1 = \sqrt{13 - 6\sqrt{2}}$$

Gaussian and mean curvatures

$$K_\Psi = \frac{eg-f^2}{EG-F^2} = -\frac{\kappa_\beta cos\gamma}{r_1(1-r_1\kappa_\beta cos\gamma)} = -\frac{cos\gamma}{2r_1\left(1-\frac{r_1}{2}cos\gamma\right)}$$

$$H_\Psi = \frac{eG-2Ff+Eg}{2(EG-F^2)} = \frac{1}{2}\left(\frac{1}{r_1} + K_\Psi r_1\right) = \frac{1}{2}\left(\frac{1}{r_1} - \frac{cos\gamma}{2\left(1-\frac{r_1}{2}cos\gamma\right)}\right)$$

Principal curvatures

$k_1^{\Psi} = -\frac{\kappa_\beta cos\gamma}{1-r_1\kappa_\beta cos\gamma} = -\frac{cos\gamma}{2\left(1-\frac{r_1}{2}cos\gamma\right)}$

$k_2^{\Psi} = \frac{1}{r_1} = \frac{1}{\sqrt{13-6\sqrt{2}}}$

Matrix of the shape operator

$$S = \frac{1}{EG-F^2}\begin{bmatrix} E & -F \\ -F & G \end{bmatrix}\begin{bmatrix} e & f \\ f & g \end{bmatrix} = \frac{1}{r_1^2\left(r_1\kappa_\beta cos\gamma - 1\right)^2}\begin{bmatrix} r_1^2 & -r_1^2\tau_\beta \\ -r_1^2\tau_\beta & \left(1-r_1\kappa_\beta cos\gamma\right)^2 + r_1^2\tau_\beta^2 \end{bmatrix}$$

that is

$$S = \frac{1}{(13-6\sqrt{2})\left(\left(\frac{r_1}{2}cos\gamma-1\right)^2\right)}\begin{bmatrix} 13-6\sqrt{2} & \frac{-13+6\sqrt{2}}{2} \\ \frac{-13+6\sqrt{2}}{2} & \left(1-\frac{r_1}{2}cos\gamma\right)^2 + \frac{13-6\sqrt{2}}{4} \end{bmatrix}$$

where $r_1 = \sqrt{13-6\sqrt{2}}$.

Example case 2. If, in the parametric equation $\Psi(s,\theta) = \varphi(s,\theta) + \lambda Z(s,\theta)$ of the kinematic surface M^g, the translation vector is chosen to coincide with the rotation axis, that is $Z = V = \left(\frac{1}{2},\frac{1}{2},\frac{1}{\sqrt{2}}\right)$, and $\lambda = 3$, then

$$\Psi(s,\theta) = \beta(s) + \left(\frac{3}{2},\frac{3}{2},\frac{3}{\sqrt{2}}\right) + 2\left(cos\theta N_\beta(s) + sin\theta B_\beta(s)\right)$$

is obtained. This equation represents a tubular surface with spine curve $\beta(s) + \left(\frac{3}{2},\frac{3}{2},\frac{3}{\sqrt{2}}\right)$ and radius $r = 2$. The coincidence of the rotation axis and the translation vector means that the kinematic motion is a screw motion. Therefore, this surface is also a screw surface. Fig. 6 illustrates the tubular surface M together with the corresponding screw surface obtained when the translation vector coincides with the rotation axis.

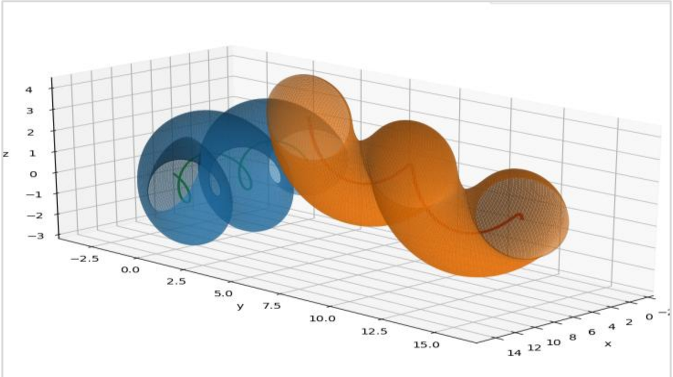

Fig. 6. Tubular surface M and the corresponding screw surface.

Example case 3. If, in the parametric equation $\Psi(s,\theta) = R\phi(s,\theta) + \lambda Z(s,\theta)$ of the kinematic surface M^g, the rotation angle is taken as $\alpha = 0$ and the translation vector is chosen as $Z = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, then we obtain

$$\Psi(s,\theta) = C(s) + \sqrt{13-6\sqrt{2}}\left(\frac{2-\frac{3}{\sqrt{2}}}{\sqrt{13-6\sqrt{2}}} cos\theta - \frac{\frac{3}{\sqrt{2}}}{\sqrt{13-6\sqrt{2}}} sin\theta\right) N(s)$$
$$+\sqrt{13-6\sqrt{2}}\left(\frac{2-\frac{3}{\sqrt{2}}}{\sqrt{13-6\sqrt{2}}} sin\theta + \frac{\frac{3}{\sqrt{2}}}{\sqrt{13-6\sqrt{2}}} cos\theta\right) B(s).$$

Here, let

$$r_2 = \sqrt{13-6\sqrt{2}}\,,\ \frac{2-\frac{3}{\sqrt{2}}}{\sqrt{13-6\sqrt{2}}} = cost\,,\ \frac{\frac{3}{\sqrt{2}}}{\sqrt{13-6\sqrt{2}}} = sint\,.$$

Then,

$$\Psi(s,\theta) = \beta(s) + r_1\big((costcos\theta - sintsin\theta)N(s) + (costsin\theta + sintcos\theta)B(s)\big).$$

For $\omega = \theta + t$, we have

$$\Psi(s,\omega) = C(s) + r_2\big(cos\omega N(s) + sin\omega B(s)\big).$$

This equation represents a tubular surface with spine curve $C(s)$ and radius $r_2 = \sqrt{13-6\sqrt{2}}$. Hence, this surface is called a tubular surface. The tubular surface obtained in Example case 3, where the rotation angle is taken as $(\alpha = 0)$ is shown in Fig. 7.

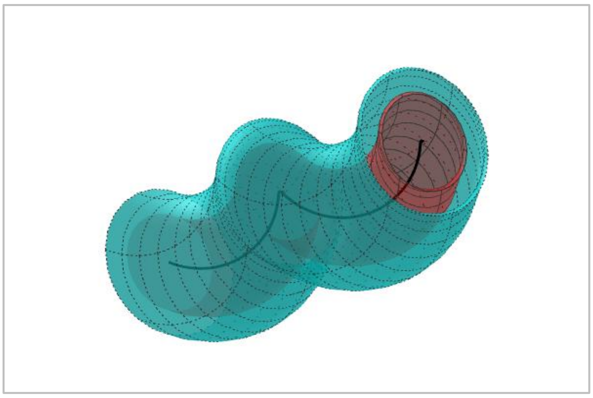

Fig. 7. Tubular surface at a constant distance from the edge of regression with spine curve $C(s)$ and radius r_2.

Example case 4. If, in the parametric equation $\Psi(s,\theta) = R\phi(s,\theta) + \lambda Z(s,\theta)$ of the kinematic surface M^g, the rotation angle is taken as $\alpha = 0$ and the translation vector is chosen as $Z = (0,0,1)$, then Z coincides with the unit normal vector of the tubular surface M. For $\lambda = 3$, we obtain

$$\Psi(s,\theta) = \phi(s,\theta) + \lambda n = C(s) + 2\big(cos\theta B(s) + sin\theta B(s)\big) + 3\big(-cos\theta B(s) - sin\theta B(s)\big)$$

$$\Psi(s,\theta) = C(s) + \big((cos\theta)N(s) + (sin\theta)B(s)\big).$$

This equation represents a tubular surface with spine curve $C(s)$ and radius $r_3 = 1$. This is called a parallel tubular surface. The parallel tubular surface obtained by choosing the translation direction along the unit normal vector is shown in Fig. 8.

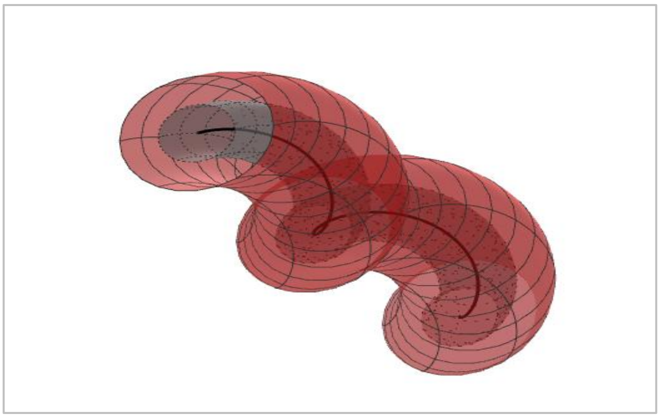

Fig. 8. Parallel tubular surface with spine curve $C(s)$ and radius r_3.

5. Conclusion

Tubular surfaces have long attracted the attention of researchers due to their wide range of applications in various fields such as mathematics, engineering, computer-aided design, and architecture. In particular, many studies have been carried out by mathematicians, especially geometers, on these surfaces in both Euclidean and non-Euclidean spaces from different perspectives.

In this article, in contrast to related studies in the literature, kinematic motions are applied to a tubular surface. The resulting kinematic surfaces are then investigated by means of the tools of differential geometry. This approach provides a different perspective on the study of tubular surfaces and allows us to better understand how their geometric properties change under kinematic motions.

The results obtained in this study show that the preservation of the tubular surface structure under the considered kinematic motion depends essentially on the choice of the translation vector, the rotation axis and the rotation angle. In future studies, similar constructions may be investigated for other surface families and different types of kinematic motions.

Acknowledgements

This study was supported by the Scientific and Technological Research Council of Türkiye (TÜBİTAK) through the 2211-A Domestic Graduate Scholarship Program.

Conflict of interest

The authors stated that there are no conflicts of interest regarding the publication of this article.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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