

Well-Conditioned Volume-Surface Integral Equation for Scatterers with Positive-Negative Permittivity Regions

Sadeed Bin Sayed¹, Abdulkadir C. Yücel^{2,3}

ABSTRACT

Heterogeneous scatterers containing positive- and negative-permittivity regions remain challenging for traditional surface integral equation (SIE) and volume integral equation (VIE) solvers, since SIE generally requires piecewise-homogeneous regions, whereas VIE can yield poorly conditioned systems. This work proposes a multi-region internally combined volume-surface integral equation (ICVSIE) solver for accurate electromagnetic analysis of such scatterers. In the proposed formulation, positive- and negative-permittivity regions are enclosed by equivalent surfaces, while polarization currents radiate in fictitious media selected to reduce permittivity contrast. Exterior and interior equations are combined through Galerkin testing. Numerical results show that ICVSIE remains well-conditioned and nearly insensitive to permittivity contrast, with condition numbers around 10^2 compared with 10^5 – 10^6 for VIE. It converges within a few hundred iterations, while VIE fails to converge within 15,000 iterations, and computed radar cross sections agree with FEKO and COMSOL, demonstrating a stable solver for mixed-sign-permittivity scatterers.

Keywords: *Electromagnetic scattering; negative permittivity; volume integral equation; volume-surface integral equation; well-conditioned formulations.*

1. INTRODUCTION

Many emerging applications, including nanostructure design and communication systems mounted on hypersonic re-entry vehicles, involve heterogeneous regions with both positive and negative permittivity [1-3]. The design, development, and deployment of such systems require accurate and efficient electromagnetic (EM) solvers. However, EM analysis of these multi-region problems with traditional formulations may yield inaccurate or unreliable solutions. Therefore, robust EM solvers capable of handling scattering from multi-region scatterers with positive and negative permittivity are called for.

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Among the various classes of EM solvers available to designers, integral equation (IE) solvers offer several advantages over their differential-equation counterparts [4]. Based on the domain of the unknown equivalent currents, IE solvers can be broadly classified as surface integral equation (SIE) and volume integral equation (VIE) solvers [5, 6]. Although SIE formulations can yield well-conditioned systems for multi-region scatterers, they generally require that all regions be piecewise homogeneous. In contrast, VIE formulations can analyze heterogeneous scatterers, but they may result in poorly conditioned systems when the scatterer contains multiple regions with large permittivity contrast. Consequently, the iterative solution of VIE may fail to converge or converge very slowly, limiting its use for practical problems. Carefully designed preconditioners have been developed to improve the convergence of VIE systems [7, 8]. However, such formalisms have primarily been developed and demonstrated for positive-permittivity media and do not directly address the negative-permittivity breakdown considered in this work [9].

In this work, a stable and accurate simulation tool based on a combined volume-surface integral equation is proposed to analyze scatterers with both positive and negative permittivity regions. The proposed solver builds on the internally combined volume-surface integral equation (ICVSIE) technique, which was developed for the EM analysis of a strictly negative-permittivity plasma medium embedded in a positive-permittivity background medium [9]. The main idea behind ICVSIE is to artificially change the sign of the background medium's permittivity where polarization currents radiate. This dramatically reduces the contrast between the plasma permittivity and that of the background medium in which the polarization currents radiate. Inspired by this work, we propose to separately wrap the positive- and negative-permittivity regions in the multi-region problem. This permits polarization currents to radiate into a background medium with reduced permittivity contrast relative to the region. The permittivity of the artificial medium in the positive and negative permittivity regions is set to be lower than the smallest and higher than the largest possible permittivity in the pertinent region, respectively. By doing so, the proposed work extends the ICVSIE to scatterers comprising multiple regions with both positive and negative permittivity by using per-region effective media.

The rest of the paper is organized as follows. Section 2 outlines the proposed multi-region ICVSIE solver. Sections 2.1–2.4 describe the exterior and interior scenarios, the construction of the matrix system, and the treatment of junctions involving different domains, respectively. Section 3 demonstrates the stability and accuracy of the proposed multi-region ICVSIE solver through numerical experiments.

2. FORMULATION

Consider a time-harmonic field $(\mathbf{E}^{\text{inc}}(\mathbf{r}), \mathbf{H}^{\text{inc}}(\mathbf{r}))$ exciting a heterogeneous scatterer consisting of positive- and negative-permittivity regions [Figure 1(a)]. The region u is a medium with permittivity $\varepsilon_u(\mathbf{r})$, that is either strictly positive or strictly negative, occupying a volume V_u . $S_q = \partial V_u \cap \partial V_l$ represents a surface separating the regions u and

t , where t denotes another region. To solve the unknown fields $(\mathbf{E}(\mathbf{r}), \mathbf{H}(\mathbf{r}))$, the original problem is converted into an equivalent exterior scenario [Figure 1(b)] and interior scenarios [Figure 1(d)].

2.1. The Exterior Scenario

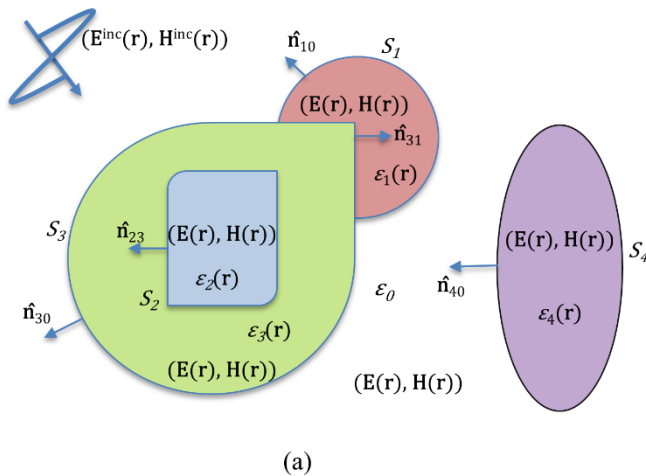
By enforcing boundary conditions on the surface S_q [Figure 1(b)], we obtain the exterior electric field integral equations (EFIE) and magnetic field integral equation (MFIE) for the equivalent currents for the unknown electric $\mathbf{J}_{S_q}^+$ and magnetic polarization currents $\mathbf{K}_{S_q}^+$ as

$$\begin{aligned} \text{EFIE}_{ext}^{uq} : & 0.5\mathbf{K}_{S_q}^+(\mathbf{r}) - \hat{n}_u \times K_u \left(\sum_n \mathbf{K}_{S_n}^+(\mathbf{r}) \right) \\ & + \hat{n}_u \times \eta_u L_u \left(\sum_n \mathbf{J}_{S_n}^+(\mathbf{r}) \right) = -\hat{n}_u \times \mathbf{E}^{inc}(\mathbf{r}); \mathbf{r} \in S_{S_q}^+ \end{aligned} \quad (1)$$

$$\begin{aligned} \text{MFIE}_{ext}^{uq} : & -0.5\mathbf{J}_{S_q}^+(\mathbf{r}) + \hat{n}_u \times \frac{1}{\eta_u} L_u \left(\sum_n \mathbf{K}_{S_n}^+(\mathbf{r}) \right) \\ & + \hat{n}_u \times K_u \left(\sum_n \mathbf{J}_{S_n}^+(\mathbf{r}) \right) = -\hat{n}_u \times \mathbf{H}^{inc}(\mathbf{r}); \mathbf{r} \in S_{S_q}^+ \end{aligned} \quad (2)$$

where L_u is the electric source field operator and K_u is the magnetic source field operator defined for the medium with permittivity ϵ_u and intrinsic impedance η_u , and $\hat{n}_u$ is the unit vector pointing out of V_u . Similarly, by enforcing zero fields inside V_u , we obtain the exterior VIE as

$$\text{VIE}_{ext}^u : -K_u \left(\sum_n \mathbf{K}_{S_n}^+(\mathbf{r}) \right) + \eta_u L_u \left(\sum_n \mathbf{J}_{S_n}^+(\mathbf{r}) \right) = -\mathbf{E}^{inc}(\mathbf{r}); \mathbf{r} \in V_u \quad (3)$$



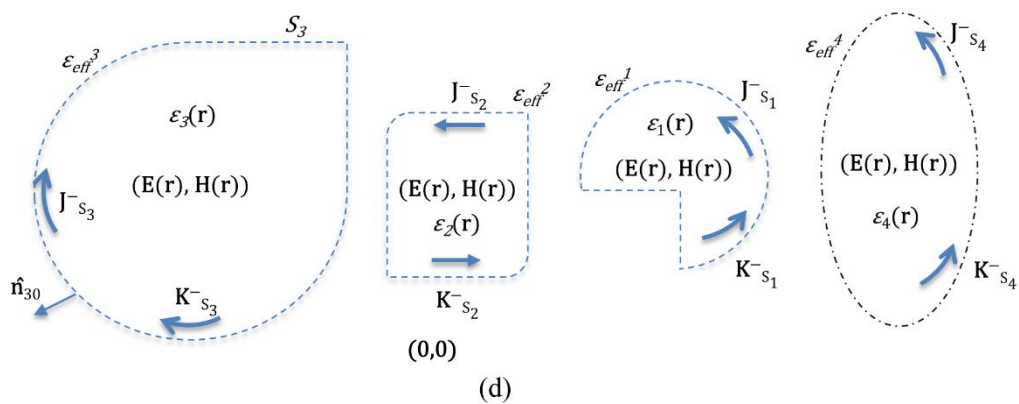
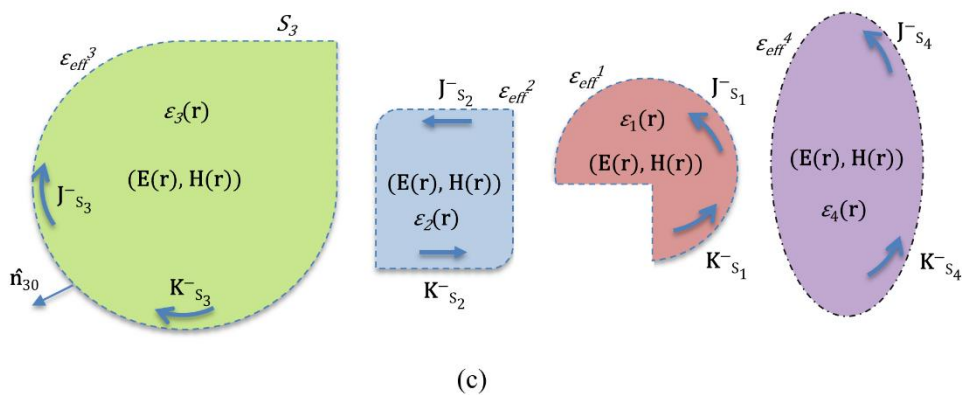
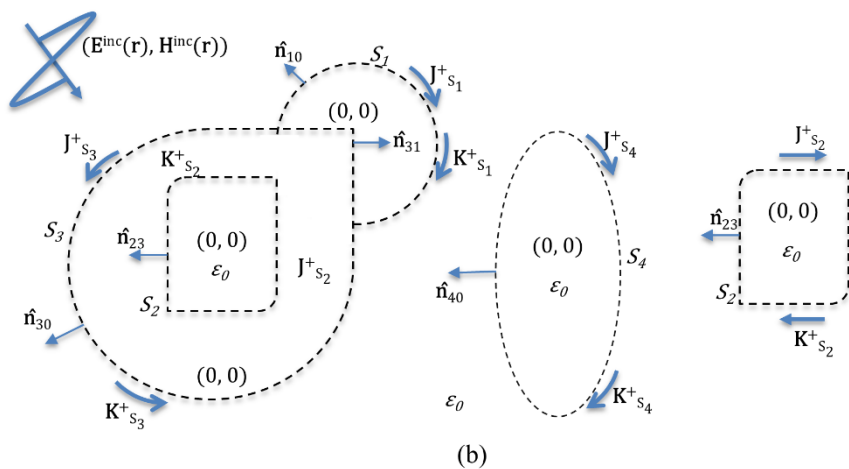


Figure 1. (a) The original problem. The equivalent (b) exterior scenario, (c) interior scenarios for different regions wrapped using respective effective media, and (d) interior scenarios with permittivity inside the regions replaced with equivalent volume currents.

2.2. The Interior Scenario

In the interior scenarios, the V_u is wrapped with a medium with permittivity ϵ_{eff}^u [Figure 1(c)] in which polarization currents $\mathbf{J}_{S_q}^+$ and $\mathbf{K}_{S_q}^+$ are allowed to radiate, noting that $\mathbf{J}_{S_q}^+ = -\mathbf{J}_{S_q}^-$ and $\mathbf{K}_{S_q}^+ = -\mathbf{K}_{S_q}^-$. To minimize the permittivity contrast between the region and the background medium, ϵ_{eff}^u is chosen as $\epsilon_{eff}^u < \min(\epsilon_u(\mathbf{r}))$, when the region is strictly positive, and $\epsilon_{eff}^u > \max(\epsilon_u(\mathbf{r}))$, when the region is strictly negative. Enforcing the boundary conditions on the surface S_q [Figure 1(d)], we obtain the interior EFIE and MFIE as

$$\begin{aligned} \text{EFIE}_{int}^{uq} : & -0.5\mathbf{K}_{S_q}^+(\mathbf{r}) + \hat{n}_u \times \eta_{eff}^u \mathbf{L}_{eff}^u \left(\mathbf{J}_{S_q}^+(\mathbf{r}) \right) \\ & - \hat{n}_p \times K_{eff}^u \left(\mathbf{K}_{S_q}^+(\mathbf{r}) \right) - \hat{n}_u \times \eta_{eff}^u \mathbf{L}_{eff}^u \left(\tilde{\mathbf{J}}_u^v(\mathbf{r}) \right) = 0; \mathbf{r} \in S_{S_q}^+ \end{aligned} \quad (4)$$

$$\begin{aligned} \text{MFIE}_{int}^{uq} : & 0.5\mathbf{J}_{S_q}^+(\mathbf{r}) + \hat{n}_u \times 1/\eta_{eff}^u K_{eff}^u \left(\mathbf{K}_{S_q}^+(\mathbf{r}) \right) \\ & + \hat{n}_u \times \mathbf{L}_{eff}^u \left(\mathbf{J}_{S_q}^+(\mathbf{r}) \right) - \hat{n}_u \times K_{eff}^u \left(\tilde{\mathbf{J}}_u^v(\mathbf{r}) \right) = 0; \mathbf{r} \in S_{S_q}^+ \end{aligned} \quad (5)$$

Using the volume equivalence principle in region u , we obtain the interior VIE for volume equivalent current $\tilde{\mathbf{J}}_u^v$ as

$$\text{VIE}_{int}^u : \mathbf{E}(\mathbf{r}) - K_{eff}^u \left(\mathbf{K}_{S_q}^+(\mathbf{r}) \right) + \eta_{eff}^u \mathbf{L}_{eff}^u \left(\mathbf{J}_{S_q}^+(\mathbf{r}) \right) - \eta_{eff}^u \mathbf{L}_{eff}^u \left(\tilde{\mathbf{J}}_u^v(\mathbf{r}) \right) = 0; \mathbf{r} \in V_u \quad (6)$$

Finally, a multi-region ICVSIE system is obtained by combining the equivalent exterior and interior surface and volume equations

$$\begin{aligned} & \epsilon_0 \text{EFIE}_{ext}^{uq} - \epsilon_{eff}^u \text{EFIE}_{int}^{uq} \\ & \text{MFIE}_{ext}^{uq} - \text{MFIE}_{int}^{uq} \\ & \epsilon_0 \text{VIE}_{ext}^u - \epsilon_{eff}^u \text{VIE}_{int}^u \end{aligned} \quad (7)$$

The weights for each of the EFIE and MFIE are chosen to ensure that SIE blocks in the system are free from spurious resonances. Similarly, VIE weights are chosen to ensure low-

frequency stability. These weights make the combined system well-conditioned even when the scatterer comprises multiple heterogeneous regions [10, 11].

2.3. Construction of the Method of Moments (MoM) System

The surface is discretized using triangles and RWG basis functions $\mathbf{f}_q$ [12] are defined for every pair of triangles in S_q that share a common edge. Using N_{S_q} , the total number of edges on S_q , the unknown current $\mathbf{J}_{S_q}^+$ and $\mathbf{K}_{S_q}^+$ are expanded as

$$\left\{ \mathbf{J}_{S_q}^+(\mathbf{r}), \mathbf{K}_{S_q}^+(\mathbf{r}) \right\} = \sum_{n=1}^{N_{S_q}} \left\{ X_{q,n}^J, X_{q,n}^K \right\} \mathbf{f}_{q,n}(\mathbf{r}). \quad (8)$$

Similarly, the volume is discretized using tetrahedra and SWG basis functions $\tilde{\mathbf{f}}_u$ [13] are defined for every pair of tetrahedra in V_u that share a common face. Using N_{V_u} , the total number of triangular faces on V_u , $\tilde{\mathbf{J}}_u^v$ is expanded as

$$\tilde{\mathbf{J}}_u^v = \sum_{n=1}^{N_{V_u}} \tilde{X}_{u,n}^J \tilde{\mathbf{f}}_{u,n}(\mathbf{r}) \quad (9)$$

The MoM system obtained via the Galerkin testing can be represented as

$$\begin{bmatrix} \mathbf{Z}_{1,1} & \cdots & \mathbf{Z}_{1,p} & \cdots \\ \vdots & & \vdots & \\ \mathbf{Z}_{q,1} & \cdots & \mathbf{Z}_{q,p} & \cdots \\ \vdots & \cdots & & \end{bmatrix} \cdot \begin{bmatrix} X_1 \\ \vdots \\ X_p \\ \vdots \end{bmatrix} = \begin{bmatrix} R_1 \\ \vdots \\ R_q \\ \vdots \end{bmatrix} \quad (10)$$

Here, $\mathbf{Z}_{q,p}$ is obtained when the surface basis functions in the surface $S_q \in \partial V_u$ and the volume basis functions in V_u are tested with surface basis functions in $S_p \in \partial V_t$ and the volume basis functions in V_t . The elements of $\mathbf{Z}_{q,p}$ are given as

$$\mathbf{Z}_{q,p} = \begin{bmatrix} \left\langle \mathbf{f}_q, \frac{(\epsilon_0 - \alpha_{int,p})}{2} \mathbf{f}_p \right\rangle \delta_{qp} - \left\langle \mathbf{f}_q, \hat{n}_q \times (\epsilon_0 K_{eff}^0 + \epsilon_{eff}^t K_{eff}^t) \mathbf{f}_p \right\rangle & \left\langle \mathbf{f}_q, \hat{n}_q \times (\eta_b \epsilon_0 L_{eff}^t + \epsilon_{eff}^t \eta_{eff}^t L_{eff}^t) \mathbf{f}_p \right\rangle & \left\langle \mathbf{f}_q, -\hat{n}_q \times \epsilon_{eff}^t L_{eff}^t \tilde{\mathbf{f}}_t \right\rangle \\ \left\langle \mathbf{f}_q, \hat{n}_q \times \left(\frac{1}{\eta_0} L_{eff}^0 - \frac{1}{\eta_{eff}^t} L_{eff}^t \right) \mathbf{f}_p \right\rangle & -\left\langle \mathbf{f}_q, \mathbf{f}_p \right\rangle \delta_{qp} + \left\langle \mathbf{f}_q, \hat{n}_q \times (K_{eff}^0 - K_{eff}^t) \mathbf{f}_p \right\rangle & \left\langle \mathbf{f}_q, \hat{n}_q \times K_{eff}^t \tilde{\mathbf{f}}_t \right\rangle \\ \left\langle \tilde{\mathbf{f}}_u, -(\epsilon_0 K_{eff}^{p0} + \epsilon_{eff}^t K_{eff}^t) \mathbf{f}_p \right\rangle & \left\langle \tilde{\mathbf{f}}_u, (\eta_b \epsilon_0 L_{eff}^{p0} + \epsilon_{eff}^t \eta_{eff}^t L_{eff}^t) \mathbf{f}_p \right\rangle & \left\langle \tilde{\mathbf{f}}_u, \epsilon_{eff}^t \left(\frac{f_p}{\epsilon} - L_{eff}^t \right) \tilde{\mathbf{f}}_t \right\rangle \delta_{ut} \end{bmatrix} \quad (11)$$

and $X_q = [\mathbf{K}_{S_q}^+ \quad \mathbf{J}_{S_q}^+ \quad \tilde{\mathbf{J}}_u^v]^T$ is the vector holding the unknown current coefficients to be solved for, and $R_q = [\langle \mathbf{f}_q, -\varepsilon_{eff}^u \hat{n}_u \times \mathbf{E}^{inc} \rangle \quad \langle \mathbf{f}_q, -\hat{n}_u \times \mathbf{H}^{inc} \rangle \quad \langle \mathbf{f}_u, -\varepsilon_{eff}^u \mathbf{E}^{inc} \rangle]^T$ represents the tested incident field vector.

2.4. Junction Treatment

Surface basis functions formed on either side of a surface are treated as junction basis [14]. Initially, the touching surfaces are separated into two different surfaces and separate surface basis functions are formed on either side of the surface [15]. While the MoM matrix is constructed, the two separate bases are combined into a single basis with appropriate signs. If the current flow on either side of the surface is in the same direction, two bases are combined into a single basis with opposite signs. For example, in Figure 2(a), $\mathbf{f}_{S_2}$ and $\mathbf{f}_{S_3}$ have the same current direction, which are combined into a single $\mathbf{f}_{S_2}$ and $\mathbf{f}_{S_3}$ on S_3 and is replaced with $-\mathbf{f}_{S_2}$ as shown in Figure 2(b). Similarly, if the direction of current on either side of the surface is opposite, the two bases are combined with the same sign, i.e. $\mathbf{f}_{S_3}$ would be replaced with $\mathbf{f}_{S_2}$.

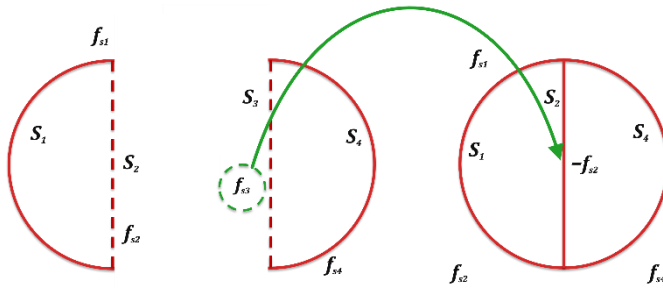


Figure 2. (a) Touching surfaces are separated into two disconnected surfaces. (b) The basis functions on the two sides of the touching surface are combined into a single basis.

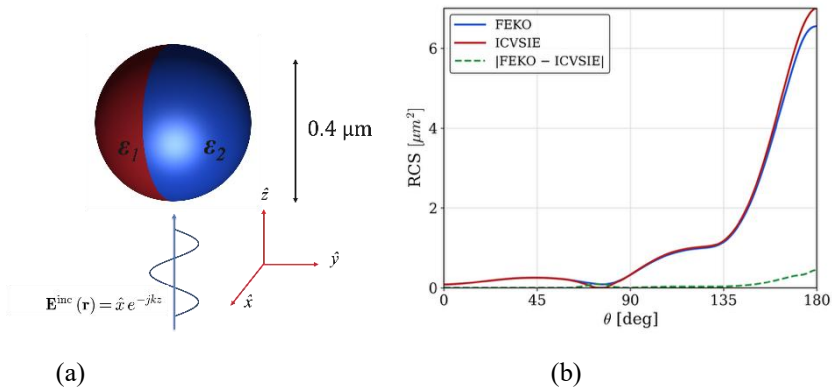
3. NUMERICAL RESULTS

This section presents numerical examples demonstrating the advantages of the proposed multi-region ICVSIE solver for EM analysis of heterogeneous scatterers with positive- and negative-permittivity regions. In all examples presented in this section, the scatterer is illuminated with an $\hat{x}$ polarized plane wave of frequency f travelling in $\hat{z}$ direction. For the iterative solution of matrix system (10), a transpose-free quasi-minimal residual (TFQMR) method is used. A Jacobi (diagonal) preconditioner is applied to both the proposed ICVSIE and the VIE systems. The TFQMR solver is terminated when the

inequality $\|\mathbf{X}^n - \mathbf{X}^{n-1}\| < \chi^{\text{TFQMR}}$ is satisfied. Here $\mathbf{X}^n$ represents the solution vector at the n th iteration, and χ^{TFQMR} is the convergence threshold, set to 10^{-6} in the examples below.

3.1. Homogeneous Hemispheres

In the first example, the scatterer consists of two hemispheres of homogeneous permittivity joined to form a sphere of radius $0.2 \mu\text{m}$ as shown in Figure 3(a). The left and right hemispheres have permittivity $\varepsilon_1 = -4\varepsilon_0$ and $\varepsilon_2 = 4\varepsilon_0$, where ε_0 is the permittivity of vacuum. The scatterer is illuminated by an incident field with $f = 700 \text{ THz}$. The proposed solver is used to analyze this problem with $\varepsilon_{\text{eff}}^1 = -3.5\varepsilon_0$ and $\varepsilon_{\text{eff}}^2 = 3.5\varepsilon_0$. The polarization currents obtained from the solver are used to compute the radar cross-section (RCS), which is compared with that obtained using the commercial software FEKO [16] in Figure 3(b). Note that FEKO uses the SIE formulation to solve this problem as the regions involved are homogeneous. In addition, the results obtained from FEKO serve as a benchmark to validate the accuracy of the proposed solver. The results in Figure 3(b) show that the RCS computed using two schemes agree well, demonstrating the accuracy of the proposed scheme. The surface and volume domains are discretized with 6316 and 34518 basis functions, respectively, and the RCS obtained from ICVSIE and FEKO differ by a relative RMS error of 5.4%.



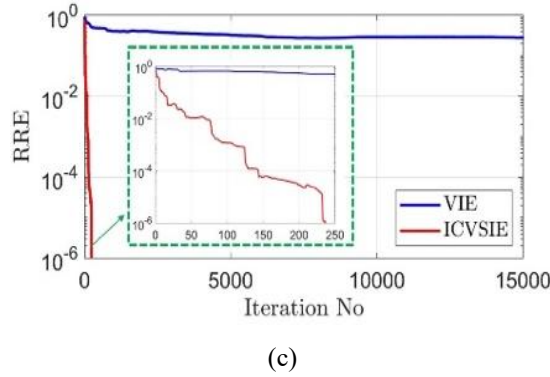


Figure 3. (a) The scattering problem, (b) comparison of the RCS obtained by FEKO and ICVSIE solvers and the absolute difference between obtained RCSs, and (c) error convergence of the ICVSIE and the VIE solvers.

To demonstrate the superiority of the proposed solver over VIE, the VIE formulation is also used to analyze the problem, and the error convergence of VIE and the proposed solver is compared in Figure 3(c). It can be seen that the solution obtained by the ICVSIE solver converges to the desired accuracy (relative residual error (RRE)) in 240 iterations, whereas the solution obtained via VIE fails to converge even after 15,000 iterations. The drastic reduction in the number of iterations required to reach the desired RRE is achieved due to the proposed solver's well-conditioned system of equations.

3.2. Heterogeneous Hemispheres

Next, two heterogeneous hemispheres are combined to form a sphere. The permittivity of the left hemisphere varies randomly between $-4\epsilon_0$ and $-3\epsilon_0$, and the permittivity of the right hemisphere varies randomly between $3\epsilon_0$ and $4\epsilon_0$. As the structure is a heterogeneous scatterer, SIE cannot be used for the analysis. The proposed solver is used to characterize scattering from this structure after setting $\epsilon_{eff}^1 = -2.5\epsilon_0$ and $\epsilon_{eff}^2 = 2.5\epsilon_0$. Similar to the previous example, the RCS is computed from the polarization currents and plotted in Figure 4(a). A random relative permittivity is assigned independently to each tetrahedron of the volume mesh after discretization is performed. The discretization yields 6316 surface and 34518 volume unknowns.

Once again, the RRE convergence of the proposed solver is compared with that of the VIE formulation (Figure 4(b)). As before, the RRE of the solution obtained by ICVSIE rapidly converges in 560 iterations, whereas the RRE of the solution obtained using VIE stagnates and fails to converge even after 15,000 iterations.

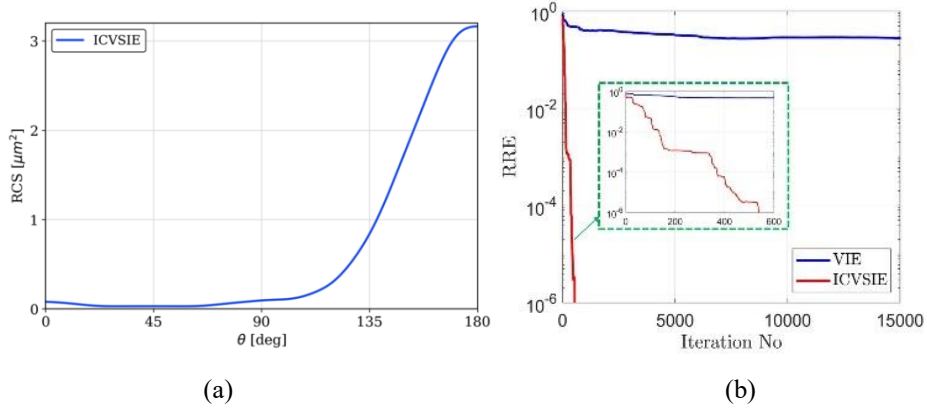


Figure 4. (a) RCS obtained from ICVSIE and (b) error convergence of the ICVSIE and the VIE solver.

3.3. Condition Number

The robustness of the proposed scheme is demonstrated by comparing the condition numbers of the MoM system matrices obtained by the proposed ICVSIE solver and the traditional VIE solver. The left and right hemispheres have relative permittivities of $\varepsilon_1 = -\varepsilon_r \varepsilon_0$ and $\varepsilon_2 = \varepsilon_r \varepsilon_0$. The magnitude of this relative permittivity ε_r is increased from 2 to 10, and the condition number of the MoM matrix of the proposed solver is computed with $\varepsilon_{eff}^1 = (-\varepsilon_r + 0.5)\varepsilon_0$ and $\varepsilon_{eff}^2 = (\varepsilon_r - 0.5)\varepsilon_0$. The corresponding condition number for a VIE system discretized with 34518 basis functions is also computed and compared, as shown in Figure 5. We see that the condition number of the system matrix in the VIE solver fluctuates between 10^5 and 10^6 , resulting in an ill-conditioned system for which the iterative solution converges very slowly or may fail to converge, whereas the condition number of the system matrix in the proposed ICVSIE solver discretized with 6316 surface and 34518 volume basis functions is around 10^2 , resulting in a well-conditioned system regardless of the permittivity of the left and right hemispheres.

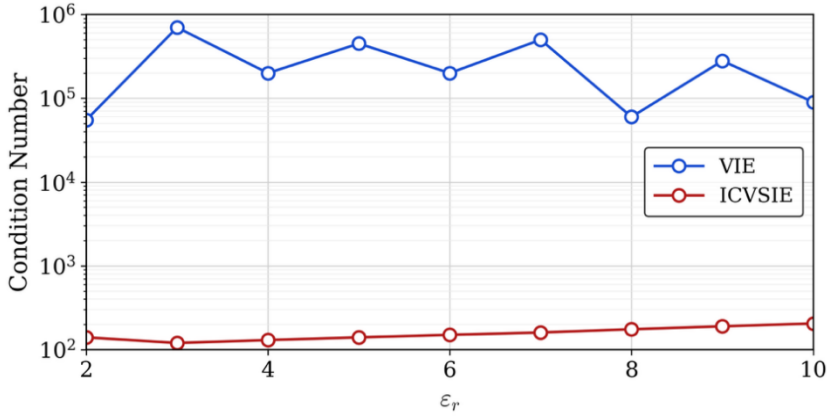


Figure 5. Comparison of condition numbers of VIE and ICVSIE systems with different relative permittivity.

3.4. Homogeneous Spheres

Finally, the scattering from two spheres of permittivity $\epsilon_1 = -0.5\epsilon_0$ and $\epsilon_2 = 10\epsilon_0$ (as shown in Figure 6(a)) is characterized using the proposed ICVSIE solver. The solver uses $\epsilon_{eff}^1 = -0.2\epsilon_0$ and $\epsilon_{eff}^2 = 9.5\epsilon_0$, and the RCS obtained from the polarization currents is compared with that obtained by a commercial finite element solver, COMSOL [17]. The discretization results in 8684 surface unknowns and 69036 volume unknowns.

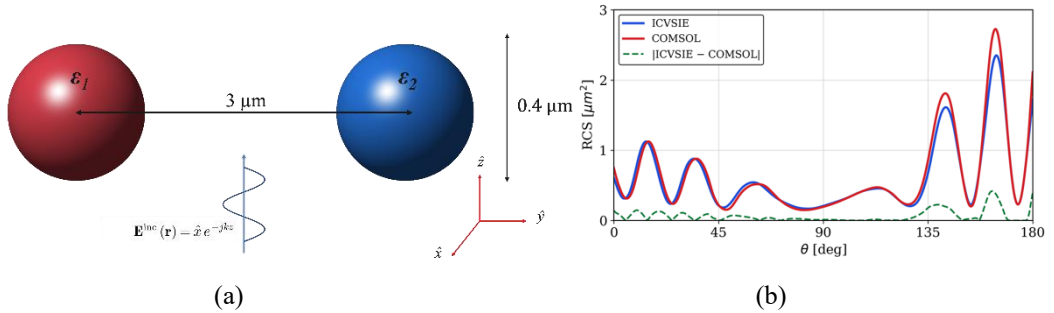


Figure 6. (a) Two-sphere scattering problem and (b) RCS comparison of the ICVSIE and COMSOL.

The results shown in Figure 6(b) demonstrate good agreement; a small mismatch at some peak values is expected as the proposed solver and COMSOL use different mesh and solution techniques. It is important to note that COMSOL fails to converge when ϵ_1 is set

to be smaller than -0.5. The RCS obtained from ICVSIE and COMSOL differ by a relative RMS error of 12.42%.

4. CONCLUSION

A multi-region ICVSIE is proposed for analyzing scattering from heterogeneous targets with positive- and negative-permittivity regions. The proposed solver obtains an accurate and robust matrix system by artificially changing the media in which the polarization currents radiate. The system combines the surface basis functions on either side of the surface with a junction basis with appropriate signs. This results in a reduced system with fewer unknowns. The exterior and interior equations are carefully combined to obtain a well-conditioned system, regardless of the scatterer's permittivity. As future work, the proposed solver would be extended to analyze epsilon-near-zero materials with sign changes.

Research and Publication Ethics Statement

The entire process of the manuscript was carried out in compliance with the research and publication ethics guidelines of the Celal Bayar University Journal of Science. An ethical committee approval and/or legal/special permission has not been required within the scope of this study.

Author's Contributions

Sadeed Bin Sayed: Implemented the solver, performed the numerical experiments and result analysis, drafted and wrote the manuscript.

Abdulkadir C. Yücel: Conceived the study and developed the theoretical formulation, designed the numerical experiments, supervised the work, interpreted and validated the results, reviewed and edited the manuscript.

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