



On Mahler's S -numbers, T -numbers, and U -numbers

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Abstract

In this work, we consider some power series with algebraic coefficients from a certain algebraic number field, whose radiuses of convergence are infinite. We show that under certain conditions these series take transcendental values at non-zero algebraic number arguments, and we determine the classes to which these transcendental values belong in Mahler's classification. Then we consider these series for certain Liouville number arguments and obtain similar results.

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1. Introduction

The transcendental numbers are divided into three disjoint classes, called S -numbers, T -numbers, and U -numbers, according to the well-known classification set up by Mahler [5] in 1932. (See Bugeaud [2] and Schneider [11] for information about Mahler's classification of the transcendental numbers.) The main purpose of the present work is to give new results for obtaining S -numbers, T -numbers, and U -numbers in Mahler's classification of the transcendental numbers.

Oryan [6] considered some power series with rational coefficients, whose radiuses of convergence are infinite and showed that under certain conditions these series take transcendental values at non-zero algebraic number arguments, and he determined the classes to which these transcendental values belong in Mahler's classification. (See also Oryan [7] which is related to Oryan [6].) In the present work, in Theorem 3.1, we extend these results of Oryan [6] to the power series with algebraic coefficients from a fixed algebraic number field by making use of the result Zeren [13, Teil II, Satz 1] and the method of Zeren [15]. Then, in Theorem 3.4, we consider the series treated in Theorem 3.1 for certain Liouville number arguments by making use of the results Oryan [8, Teil II, Satz 3, Folgerung 3] and Oryan [9, Theorem 3, Corollary 3]. (See Perron [10] and Schneider [11] for information about Liouville numbers.) Our main results are stated and proved in Section 3, and the preliminary results we need to prove the main results of this work are given in Section 2.

2. Preliminary results

Let $P(x) = a_n x^n + \cdots + a_0$ be a polynomial with rational integral coefficients. The height $H(P)$ of P is defined by $H(P) = \max(|a_n|, \dots, |a_0|)$, and the size $s(P)$ of P is defined by $s(P) = |a_n| + \cdots + |a_0|$. Suppose that α is an algebraic number. Let $P(x)$ be the minimal defining polynomial of α such that its coefficients are rational integers and relatively prime, and its highest coefficient is positive. Then the height $H(\alpha)$ of α is defined by $H(\alpha) = H(P)$, the size $s(\alpha)$ of α is defined by $s(\alpha) = s(P)$, and the degree $\deg(\alpha)$ of α is defined as the degree of P . Let K be an algebraic number field, and let α be in K . Then the field height $H_K(\alpha)$ of α for K is defined as the height of the field polynomial of α for K , and the field size $s_K(\alpha)$ of α for K is defined as the size of the field polynomial of α for K .

Theorem 2.1 (LeVeque [4], Chapter 4 The Thue-Siegel-Roth Theorem). *Let K be an algebraic number field, and let α be an algebraic number. Then for each $\chi > 2$, the inequality*

$$|\alpha - \zeta| < \frac{1}{(H(\zeta))^\chi}$$

has only finitely many solutions ζ in K .

Theorem 2.2 (Baker [1]). *Let ξ be a complex number, $\chi > 2$ be a real number, and $\alpha_1, \alpha_2, \dots$ be distinct numbers in an algebraic number field K with field heights $H_K(\alpha_1), H_K(\alpha_2), \dots$ such that*

$$|\xi - \alpha_i| < \frac{1}{(H_K(\alpha_i))^\chi} \quad (i = 1, 2, \dots)$$

and

$$\limsup_{i \rightarrow \infty} \frac{\log H_K(\alpha_{i+1})}{\log H_K(\alpha_i)} < \infty.$$

Then ξ is either an S -number or a T -number.

Lemma 2.3 (LeVeque [3]). *Let α be an algebraic number of degree m , and let $\alpha^{\{1\}} = \alpha, \dots, \alpha^{\{m\}}$ be its conjugates. Then*

$$|\bar{\alpha}| \leq 2H(\alpha),$$

where $|\bar{\alpha}| = \max(|\alpha^{\{1\}}|, \dots, |\alpha^{\{m\}}|)$.

Lemma 2.4 (Zeren [14]). *Let $\alpha_1, \dots, \alpha_k$ be algebraic numbers in an algebraic number field K , and let η be any algebraic number such that*

$$\eta = \frac{f(\alpha_1, \dots, \alpha_k)}{g(\alpha_1, \dots, \alpha_k)},$$

where $f(x_1, \dots, x_k)$ and $g(x_1, \dots, x_k)$ are polynomials in x_1, \dots, x_k with rational integral coefficients. Then

$$H_K(\eta) < 2^m \left(\prod_{i=1}^k (l_i + 1) \right)^m H^m \prod_{i=1}^k (s_K(\alpha_i))^{l_i},$$

where m is the degree of K over the field \mathbb{Q} of rational numbers, l_i ($i = 1, \dots, k$) is the maximum of the degree of $f(x_1, \dots, x_k)$ in x_i and that of $g(x_1, \dots, x_k)$ in x_i , and H is the maximum of the absolute values of the coefficients of $f(x_1, \dots, x_k)$ and those of $g(x_1, \dots, x_k)$.

Lemma 2.5 (Zeren [14]). *Let α and β be distinct algebraic numbers in an algebraic number field K . Then*

$$|\alpha - \beta| \geq \frac{1}{\tau(m)H_K(\alpha)H_K(\beta)},$$

where m is the degree of K over \mathbb{Q} and $\tau(m) = 2^{m-1}(m+1)^2$.

Lemma 2.6 (Zeren [14]). *Let α be an algebraic number in an algebraic number field K . Then*

$$s_K(\alpha) \leq (s(\alpha))^{\frac{m}{n}},$$

where m and n are the degrees of K and α over \mathbb{Q} , respectively.

We have the following corollary obtained from Lemma 2.6.

Corollary 2.7 (Zeren [14]). *Let α be an algebraic number in an algebraic number field K . Then*

$$H_K(\alpha) \leq (2H(\alpha))^m,$$

where m is the degree of K over \mathbb{Q} .

Lemma 2.8 (Zeren [14], Folgerung, p. 83). *Let α be an algebraic number in an algebraic number field K . Then*

$$H(\alpha) < 2^m(m+1)H_K(\alpha),$$

where m is the degree of K over \mathbb{Q} .

3. The main results

Let K be an algebraic number field, and let

$$f(x) = \sum_{k=0}^{\infty} \frac{\eta_k}{a_k} x^{e_k} \tag{3.1}$$

be a power series, where η_k ($k = 0, 1, 2, \dots$) is a non-zero algebraic integer in K , $a_k > 1$ ($k = 0, 1, 2, \dots$) is a rational integer, and $\{e_k\}_{k=0}^{\infty}$ is a strictly increasing sequence of non-negative rational integers. Suppose that the following conditions

$$\sigma := \liminf_{k \rightarrow \infty} \frac{\log a_{k+1}}{\log a_k} > 1, \tag{3.2}$$

$$\theta := \limsup_{k \rightarrow \infty} \frac{\log H(\eta_k)}{\log a_k} < 1, \tag{3.3}$$

and

$$\lim_{k \rightarrow \infty} \frac{\log a_k}{e_k} = \infty \tag{3.4}$$

hold. Then the radius of convergence of the power series $f(x)$ is infinite (Zeren [15]):

By (3.2), there exist a sufficiently small real number $\varepsilon_1 > 0$ with $\sigma - \varepsilon_1 > 1$ and a positive rational integer $N_1 = N_1(\varepsilon_1)$ such that

$$\log a_{k+1} > (\sigma - \varepsilon_1) \log a_k \tag{3.5}$$

for $k \geq N_1$. It follows from (3.5) that

$$\log a_k > (\sigma - \varepsilon_1)^{k-N_1} \log a_{N_1} \tag{3.6}$$

for $k > N_1$. By (3.6),

$$\lim_{k \rightarrow \infty} a_k = \infty, \tag{3.7}$$

and by (3.5),

$$a_{k+1} > a_k^{\sigma - \varepsilon_1} > a_k \quad (k \geq N_1). \tag{3.8}$$

We deduce from (3.7) and (3.8) that

$$\lim_{k \rightarrow \infty} \frac{a_{k+1}}{a_k} = \infty. \tag{3.9}$$

By (3.3), there exist a sufficiently small real number $\varepsilon_2 > 0$ with $\theta + \varepsilon_2 < 1$ and a positive rational integer $N_2 = N_2(\varepsilon_2)$ such that

$$H(\eta_k) < a_k^{\theta + \varepsilon_2} \tag{3.10}$$

for $k \geq N_2$. By Lemma 2.3 and (3.10), we have for sufficiently large k

$$0 < \sqrt[k]{\left| \frac{\eta_k}{a_k} \right|} \leq \sqrt[k]{\frac{2H(\eta_k)}{a_k}} < \frac{\varepsilon_k \sqrt{2}}{\sqrt[k]{a_k^{1-\theta-\varepsilon_2}}}. \quad (3.11)$$

We infer from (3.4) and (3.11) that

$$\lim_{k \rightarrow \infty} \sqrt[k]{\left| \frac{\eta_k}{a_k} \right|} = 0. \quad (3.12)$$

Then the radius of convergence of the power series $f(x)$ is infinite.

Let A_k ($k = 1, 2, 3, \dots$) denote the least common multiple of the rational integers a_0, a_1, \dots, a_k . By (3.6), we get

$$a_l < a_k \left(\left(\frac{1}{\sigma - \varepsilon_1} \right)^{k-l} \right) \quad (3.13)$$

for $k > l \geq N_1$. By (3.7) and (3.13), we have for sufficiently large k

$$A_k \leq a_0 a_1 \cdots a_{N_1-1} a_{N_1} \cdots a_k \leq C_0 a_k^{(\frac{1}{\sigma - \varepsilon_1})^{k-N_1} + \dots + \frac{1}{\sigma - \varepsilon_1} + 1} \leq a_k^{\varepsilon_3 + \frac{\sigma - \varepsilon_1}{\sigma - \varepsilon_1 - 1}}, \quad (3.14)$$

where $C_0 = a_0 a_1 \cdots a_{N_1-1} > 1$ and $\varepsilon_3 > 0$ is a sufficiently small real number.

Theorem 3.1. *Let $f(x)$ be a power series satisfying (3.1), (3.2), (3.3), and (3.4). Let α be a non-zero algebraic number, and suppose that*

$$t < \frac{(\sigma - 1)(1 - \theta)}{2(1 + \theta)}, \quad (3.15)$$

where t is the degree of $K(\alpha)$ over \mathbb{Q} . Then $f(\alpha)$ is a transcendental number, and we have:

a) If

$$\mu := \limsup_{k \rightarrow \infty} \frac{\log a_{k+1}}{\log a_k} < \infty, \quad (3.16)$$

then $f(\alpha)$ is either an S -number or a T -number.

b) If

$$\limsup_{k \rightarrow \infty} \frac{\log a_{k+1}}{\log a_k} = \infty, \quad (3.17)$$

then $f(\alpha)$ is a U -number of type less than or equal to t .

To prove Theorem 3.1, we make use of the methods in Zeren [13, Teil II, Satz 1, pp. 486-490] and Zeren [15].

Proof. Our proof shall appear in four steps as follows.

1) We shall define the algebraic numbers

$$\beta_n := \sum_{k=0}^n \frac{\eta_k}{a_k} \alpha^{e_k} \in K(\alpha) \quad (n = 1, 2, 3, \dots). \quad (3.18)$$

Since $\beta_n \in K(\alpha)$ ($n = 1, 2, 3, \dots$), we have $\deg(\beta_n) \leq t$ ($n = 1, 2, 3, \dots$). By multiplying both sides of (3.18) by A_n , we obtain

$$A_n \beta_n = \sum_{k=0}^n A_n \frac{\eta_k}{a_k} \alpha^{e_k} \quad (n = 1, 2, 3, \dots).$$

Then we have

$$\beta_n = \frac{g_1(\eta_0, \eta_1, \dots, \eta_m, \alpha)}{g_2(\eta_0, \eta_1, \dots, \eta_m, \alpha)},$$

where

$$g_1(x_0, x_1, \dots, x_n, y) = \sum_{k=0}^n A_n \frac{x_k}{a_k} y^{e_k} \quad \text{and} \quad g_2(x_0, x_1, \dots, x_n, y) = A_n$$

are polynomials in x_0, x_1, \dots, x_n, y with rational integral coefficients. Hence, by Lemma 2.4, we get

$$H_{K(\alpha)}(\beta_n) < 2^t (2^{n+1})^t (e_n + 1)^t A_n^t (s_{K(\alpha)}(\alpha))^{e_n} \prod_{k=0}^n s_{K(\alpha)}(\eta_k) \quad (3.19)$$

for $n = 1, 2, 3, \dots$. By (3.19), the inequality $(e_n + 1) \leq 2^{e_n}$ ($n = 1, 2, 3, \dots$), and Lemma 2.6, we have

$$H_{K(\alpha)}(\beta_n) < 2^{4e_n t} A_n^t (s_{K(\alpha)}(\alpha))^{e_n} \prod_{k=0}^n (s(\eta_k))^t \quad (n = 1, 2, 3, \dots). \quad (3.20)$$

It follows from (3.20), $s(\eta_k) \leq (\deg(\eta_k) + 1) H(\eta_k)$ ($k = 0, 1, \dots, n$), and $\deg(\eta_k) \leq t$ ($k = 0, 1, \dots, n$) that

$$H_{K(\alpha)}(\beta_n) < C_1^{e_n t} A_n^t (H(\eta_0))^t (H(\eta_1))^t \cdots (H(\eta_n))^t \quad (n = 1, 2, 3, \dots), \quad (3.21)$$

where $C_1 = 2^4(t+1)^2 s_{K(\alpha)}(\alpha) > 1$. By (3.10) and (3.21), we get

$$H_{K(\alpha)}(\beta_n) < C_2^{e_n t} A_n^t (a_0 a_1 \cdots a_n)^{(\theta + \varepsilon_2)t} \quad (3.22)$$

for $n \geq N_2$, where $C_2 = C_1 H(\eta_0) \cdots H(\eta_{N_2-1}) > 1$. By (3.4), there exists a sufficiently small real number $\varepsilon_4 > 0$ such that

$$C_2^{e_n} < a_n^{\varepsilon_4} \quad (3.23)$$

for sufficiently large n . By (3.14), (3.22), (3.23), and the inequality $A_n \leq a_0 a_1 \cdots a_n$ ($n = 1, 2, 3, \dots$), we have for sufficiently large n

$$H_{K(\alpha)}(\beta_n) < a_n^{\left(\left(\varepsilon_3 + \frac{\sigma - \varepsilon_1}{\sigma - \varepsilon_1 - 1} \right) (1 + \theta + \varepsilon_2) + \varepsilon_4 \right) t}. \quad (3.24)$$

2) By (3.1) and (3.18), we get

$$|f(\alpha) - \beta_n| = \left| \sum_{k=n+1}^{\infty} \frac{\eta_k}{a_k} \alpha^{e_k} \right| \leq \sum_{k=n+1}^{\infty} \frac{|\eta_k|}{a_k} |\alpha|^{e_k} \quad (n = 1, 2, 3, \dots). \quad (3.25)$$

It follows from Lemma 2.3, (3.4), (3.10), and (3.25) that

$$|f(\alpha) - \beta_n| \leq \frac{1}{a_{n+1}^{1-\theta-\varepsilon_5}} \left(1 + \left(\frac{a_{n+1}}{a_{n+2}} \right)^{1-\theta-\varepsilon_5} + \left(\frac{a_{n+1}}{a_{n+3}} \right)^{1-\theta-\varepsilon_5} + \cdots \right) \quad (3.26)$$

for sufficiently large n , where $\varepsilon_5 > 0$ is a sufficiently small real number with $\varepsilon_2 < \varepsilon_5 < 1 - \theta$. By (3.9),

$$\left(\frac{a_{n+1}}{a_{n+2}} \right)^{1-\theta-\varepsilon_5} < \frac{1}{2} \quad (3.27)$$

holds for sufficiently large n . We infer from (3.7), (3.26), and (3.27) that

$$|f(\alpha) - \beta_n| \leq \frac{1}{a_{n+1}^{1-\theta-\varepsilon_5}} \left(1 + \frac{1}{2} + \left(\frac{1}{2} \right)^2 + \cdots \right) \leq \frac{1}{a_{n+1}^{1-\theta-\varepsilon_6}} \quad (3.28)$$

for sufficiently large n , where $\varepsilon_6 > 0$ is a sufficiently small real number with $\varepsilon_5 < \varepsilon_6 < 1 - \theta$. By (3.8), (3.24), and (3.28), we conclude that

$$|f(\alpha) - \beta_n| < \left(H_{K(\alpha)}(\beta_n) \right)^{-\frac{(\sigma - \varepsilon_1)(1 - \theta - \varepsilon_6)}{\left(\left(\varepsilon_3 + \frac{\sigma - \varepsilon_1}{\sigma - \varepsilon_1 - 1} \right) (1 + \theta + \varepsilon_2) + \varepsilon_4 \right) t}} \quad (3.29)$$

for sufficiently large n .

3) Since $\beta_n - \beta_{n-1} = \frac{\eta_n}{a_n} \alpha^{e_n} \neq 0$ ($n = 2, 3, \dots$), we have $\beta_{n-1} \neq \beta_n$ ($n = 2, 3, \dots$). It follows from step 2) that

$$|f(\alpha) - \beta_{n-1}| = \left| \sum_{k=n}^{\infty} \frac{\eta_k}{a_k} \alpha^{e_k} \right| \leq \sum_{k=n}^{\infty} \left| \frac{\eta_k}{a_k} \alpha^{e_k} \right| \leq \frac{1}{a_n^{1-\theta-\varepsilon_6}}$$

for sufficiently large n . Hence, we get for sufficiently large n

$$0 < |\beta_n - \beta_{n-1}| = \left| \frac{\eta_n}{a_n} \alpha^{e_n} \right| \leq \frac{1}{a_n^{1-\theta-\varepsilon_6}}. \quad (3.30)$$

We deduce from (3.30) and Lemma 2.5 that

$$\frac{1}{\tau(t)H_{K(\alpha)}(\beta_n)H_{K(\alpha)}(\beta_{n-1})} \leq \frac{1}{a_n^{1-\theta-\varepsilon_6}} \quad (3.31)$$

holds for sufficiently large n , where $\tau(t) = 2^{t-1}(t+1)^2$. By (3.8), (3.24), and (3.31), we see that

$$H_{K(\alpha)}(\beta_n) > C_3 a_{n-1}^{(\sigma-\varepsilon_1)(1-\theta-\varepsilon_6) - \left(\left(\varepsilon_3 + \frac{\sigma-\varepsilon_1}{\sigma-\varepsilon_1-1} \right) (1+\theta+\varepsilon_2) + \varepsilon_4 \right) t} \quad (3.32)$$

for sufficiently large n , where $C_3 = \frac{1}{\tau(t)} > 0$.

It follows from (3.24) and (3.32) that

$$\frac{H_{K(\alpha)}(\beta_{n+1})}{H_{K(\alpha)}(\beta_n)} > C_3 a_n^{(\sigma-\varepsilon_1)(1-\theta-\varepsilon_6) - 2 \left(\left(\varepsilon_3 + \frac{\sigma-\varepsilon_1}{\sigma-\varepsilon_1-1} \right) (1+\theta+\varepsilon_2) + \varepsilon_4 \right) t} \quad (3.33)$$

for sufficiently large n . By (3.15), we have $2t(1+\theta) < (\sigma-1)(1-\theta)$. Hence, by the appropriate choices of $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$, and ε_6 , the exponent of a_n in (3.33) is positive. Thus, by (3.7) and (3.33),

$$H_{K(\alpha)}(\beta_{n+1}) > H_{K(\alpha)}(\beta_n)$$

holds for sufficiently large n . Then the algebraic numbers β_n are all distinct from each other from some n onward and

$$\lim_{n \rightarrow \infty} H_{K(\alpha)}(\beta_n) = \infty. \quad (3.34)$$

Hence, by (3.34) and Corollary 2.7, we obtain

$$\lim_{n \rightarrow \infty} H(\beta_n) = \infty. \quad (3.35)$$

4) By (3.15),

$$2 < \frac{(\sigma-1)(1-\theta)}{t(1+\theta)}.$$

Thus, there exists a real number $\varepsilon > 0$ such that

$$2 + \varepsilon < \frac{(\sigma-1)(1-\theta)}{t(1+\theta)} - \varepsilon. \quad (3.36)$$

By the appropriate choices of $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$, and ε_6 , we have

$$\frac{(\sigma-\varepsilon_1)(1-\theta-\varepsilon_6)}{\left(\left(\varepsilon_3 + \frac{\sigma-\varepsilon_1}{\sigma-\varepsilon_1-1} \right) (1+\theta+\varepsilon_2) + \varepsilon_4 \right) t} > \frac{(\sigma-1)(1-\theta)}{t(1+\theta)} - \varepsilon. \quad (3.37)$$

By (3.29), (3.36), and (3.37),

$$|f(\alpha) - \beta_n| < \frac{1}{\left(H_{K(\alpha)}(\beta_n) \right)^{2+\varepsilon}} \quad (3.38)$$

holds true for infinitely many different β_n in $K(\alpha)$. Hence, by (3.35), (3.38), Theorem 2.1, and Lemma 2.8, $f(\alpha)$ must be transcendental.

Let (3.16) hold. By (3.7), (3.15), (3.16), (3.24), (3.32), and the appropriate choices of $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$, and ε_6 , we see that

$$\limsup_{n \rightarrow \infty} \frac{\log H_{K(\alpha)}(\beta_{n+1})}{\log H_{K(\alpha)}(\beta_n)} \leq \frac{\mu^2}{\frac{(\sigma-1)(1-\theta)}{t(1+\theta)} - 1} < \mu^2 < \infty. \quad (3.39)$$

We infer from (3.38), (3.39), and Theorem 2.2 that $f(\alpha)$ is either an S -number or a T -number. This completes the proof of a) in Theorem 3.1.

Let (3.17) hold. By (3.17), there exists a subsequence $\left\{ \frac{\log a_{n_i+1}}{\log a_{n_i}} \right\}_{i=1}^{\infty}$ of the sequence $\left\{ \frac{\log a_{n+1}}{\log a_n} \right\}_{n=1}^{\infty}$ such that

$$\lim_{i \rightarrow \infty} \frac{\log a_{n_i+1}}{\log a_{n_i}} = \infty. \quad (3.40)$$

It follows from (3.24) and (3.28) that

$$|f(\alpha) - \beta_{n_i}| < \left(H_{K(\alpha)}(\beta_{n_i}) \right)^{-\frac{1-\theta-\varepsilon_6}{\left(\left(\varepsilon_3 + \frac{\sigma-\varepsilon_1}{\sigma-\varepsilon_1-1} \right) (1+\theta+\varepsilon_2)+\varepsilon_4 \right) t} \frac{\log a_{n_i+1}}{\log a_{n_i}}} \quad (3.41)$$

for sufficiently large i . We deduce from (3.35), (3.40), (3.41), and Lemma 2.8 that $f(\alpha)$ is a U -number of type less than or equal to t since $\deg(\beta_{n_i}) \leq t$ ($i = 1, 2, 3, \dots$). (Here we use Koksma's classification of the transcendental numbers, set up by Koksma [?], which is equivalent to Mahler's. The reader can refer to Bugeaud [2], Schneider [11], and Wirsing [12] for detailed information about these two classifications of the transcendental numbers and their equivalence. Moreover, the reader is recommended to consult LeVeque [3] and Bugeaud [2] for detailed information about U -numbers.) This completes the proof of b) in Theorem 3.1. \square

Example 3.2. Let p be a prime number, $a > 1$ and $m > 0$ be rational integers, and α be a non-zero algebraic number. If we take $K = \mathbb{Q}(\sqrt[m]{p})$, $\eta_k = (\sqrt[m]{p})^k$ ($k = 0, 1, 2, \dots$), $a_k = a^{((3t+1)^k)}$ ($k = 0, 1, 2, \dots$), where t is the degree of $\mathbb{Q}(\sqrt[m]{p}, \alpha)$ over \mathbb{Q} , $e_k = 2^k$ ($k = 0, 1, 2, \dots$), and $x = \alpha$, then all the conditions of Theorem 3.1 are satisfied. Thus $\sum_{k=0}^{\infty} \frac{(\sqrt[m]{p})^k}{a^{((3t+1)^k)}} \alpha^{(2^k)}$ is either an S -number or a T -number.

Example 3.3. In Example 3.2, if we take $a_k = a^{((k+1)^k)}$ ($k = 0, 1, 2, \dots$) and $e_k = (k+1)^2$ ($k = 0, 1, 2, \dots$), then this yields another example for Theorem 3.1. Thus $\sum_{k=0}^{\infty} \frac{(\sqrt[m]{p})^k}{a^{((k+1)^k)}} \alpha^{((k+1)^2)}$ is a U -number of type less than or equal to t .

Theorem 3.4. Let K be an algebraic number field of degree m , and let

$$f(x) = \sum_{k=0}^{\infty} \frac{\eta_k}{a_k} x^{e_k}$$

be a power series, where η_k ($k = 0, 1, 2, \dots$) is a positive real algebraic integer in K , $a_k > 1$ ($k = 0, 1, 2, \dots$) is a rational integer, and $\{e_k\}_{k=0}^{\infty}$ is a strictly increasing sequence of non-negative rational integers. Suppose that the conditions (3.2), (3.3), and (3.4) hold. Let ξ be a positive Liouville number such that

$$\left| \xi - \frac{p_k}{q_k} \right| < \frac{1}{q_k^{e_k \omega_k}} \quad (k = 1, 2, 3, \dots), \quad (3.42)$$

where $p_k > 0$ ($k = 1, 2, 3, \dots$) and $q_k > 1$ ($k = 1, 2, 3, \dots$) are rational integers and $\{\omega_k\}_{k=1}^{\infty}$ is an infinite sequence of positive real numbers with $\lim_{k \rightarrow \infty} \omega_k = \infty$, and

$$a_k^{\delta_1} \leq q_k^{e_k} \leq a_k^{\delta_2} \quad (k = 1, 2, 3, \dots), \quad (3.43)$$

where δ_1 and δ_2 are real numbers with $0 < \delta_1 \leq \delta_2$. Moreover, assume that

$$m < \frac{(\sigma - 1)(1 - \theta)}{2(1 + \theta + \delta_2)}. \quad (3.44)$$

Then $f(\xi)$ is a transcendental number, and we have:

a) If

$$\mu := \limsup_{k \rightarrow \infty} \frac{\log a_{k+1}}{\log a_k} < \infty, \quad (3.45)$$

then $f(\xi)$ is either an S -number or a T -number.

b) If

$$\limsup_{k \rightarrow \infty} \frac{\log a_{k+1}}{\log a_k} = \infty, \quad (3.46)$$

then $f(\xi)$ is a U -number of type less than or equal to m .

Proof. 1) We shall consider the polynomials

$$f_n(x) = \sum_{k=0}^n \frac{\eta_k}{a_k} x^{e_k} \quad (n = 1, 2, 3, \dots).$$

Define the algebraic numbers

$$\beta_n = f_n\left(\frac{p_n}{q_n}\right) = \sum_{k=0}^n \frac{\eta_k}{a_k} \left(\frac{p_n}{q_n}\right)^{e_k} \in K \quad (n = 1, 2, 3, \dots). \quad (3.47)$$

Since $\beta_n \in K$ ($n = 1, 2, 3, \dots$), we have $\deg(\beta_n) \leq m$ ($n = 1, 2, 3, \dots$). By multiplying both sides of (3.47) by $A_n q_n^{e_n}$, we obtain

$$A_n q_n^{e_n} \beta_n = \sum_{k=0}^n A_n q_n^{e_n} \frac{\eta_k}{a_k} \left(\frac{p_n}{q_n}\right)^{e_k} \quad (n = 1, 2, 3, \dots).$$

Then we have

$$\beta_n = \frac{g_1(\eta_0, \eta_1, \dots, \eta_n)}{g_2(\eta_0, \eta_1, \dots, \eta_n)},$$

where

$$g_1(x_0, x_1, \dots, x_n) = \sum_{k=0}^n A_n q_n^{e_n} \frac{x_k}{a_k} \left(\frac{p_n}{q_n}\right)^{e_k} \quad \text{and} \quad g_2(x_0, x_1, \dots, x_n) = A_n q_n^{e_n}$$

are polynomials in x_0, x_1, \dots, x_n with rational integral coefficients. Hence, by Lemma 2.4, we get

$$H_K(\beta_n) < 2^m \left(2^{n+1}\right)^m H^m \prod_{k=0}^n s_K(\eta_k) \quad (n = 1, 2, 3, \dots), \quad (3.48)$$

where

$$H = \max_{k=0, \dots, n} \left(A_n q_n^{e_n}, \frac{A_n q_n^{e_n}}{a_k} \left(\frac{p_n}{q_n}\right)^{e_k} \right). \quad (3.49)$$

By (3.42), we have

$$\frac{p_n}{q_n} < \xi + 1 \quad (n = 1, 2, 3, \dots). \quad (3.50)$$

By (3.48), (3.49), (3.50), and Lemma 2.6, we obtain

$$H_K(\beta_n) < C_1^{e_n m} A_n^m q_n^{e_n m} \prod_{k=0}^n (s(\eta_k))^m \quad (n = 1, 2, 3, \dots), \quad (3.51)$$

where $C_1 = 2^3(\xi + 1) > 1$. It follows from (3.10), (3.43), (3.51), $s(\eta_k) \leq (\deg(\eta_k) + 1)H(\eta_k)$ ($k = 0, 1, \dots, n$), and $\deg(\eta_k) \leq m$ ($k = 0, 1, \dots, n$) that

$$H_K(\beta_n) < C_2^{e_n m} a_n^{\delta_2 m} A_n^m (a_0 a_1 \dots a_n)^{(\theta + \varepsilon_2)m} \quad (3.52)$$

for $n \geq N_2$, where $C_2 = C_1(m+1)^2 H(\eta_0) \cdots H(\eta_{N_2-1}) > 1$. By (3.4), (3.14), (3.52), and the inequality $A_n \leq a_0 a_1 \cdots a_n$ ($n = 1, 2, 3, \dots$), we conclude that

$$H_K(\beta_n) < a_n^{\left(\left(\varepsilon_3 + \frac{\sigma - \varepsilon_1}{\sigma - \varepsilon_1 - 1} \right) (1 + \theta + \varepsilon_2) + \varepsilon_4 + \frac{\sigma - \varepsilon_1}{\sigma - \varepsilon_1 - 1} \delta_2 \right) m} \quad (3.53)$$

for sufficiently large n , where $\varepsilon_4 > 0$ is a sufficiently small real number.

2) We have

$$|f(\xi) - \beta_n| \leq |f(\xi) - f_n(\xi)| + |f_n(\xi) - \beta_n| \quad (n = 1, 2, 3, \dots). \quad (3.54)$$

Similarly as in step 2) in the proof of Theorem 3.1, we get

$$|f(\xi) - f_n(\xi)| = \sum_{k=n+1}^{\infty} \frac{\eta_k}{a_k} \xi^{e_k} \leq \frac{1}{2a_{n+1}^{1-\theta-\varepsilon_5}} < \frac{1}{2a_n^{(\sigma-\varepsilon_1)(1-\theta-\varepsilon_5)}} \quad (3.55)$$

for sufficiently large n , where $\varepsilon_5 > 0$ is a sufficiently small real number with $\varepsilon_5 < 1 - \theta$.

By (3.12), there exists a real number $C_3 > 1$ such that

$$\frac{\eta_k}{a_k} < C_3 \quad (k = 0, 1, 2, \dots). \quad (3.56)$$

By (3.42), (3.50), and (3.56), we get

$$|f_n(\xi) - \beta_n| \leq (e_n + 1)^2 C_3 \frac{1}{q_n^{e_n \omega_n}} (\xi + 1)^{e_n} \quad (n = 1, 2, 3, \dots). \quad (3.57)$$

Since $\lim_{n \rightarrow \infty} \sqrt[e_n]{(e_n + 1)^2} = 1$, there exists a real number $C_4 > 1$ such that

$$(e_n + 1)^2 < C_4^{e_n} \quad (3.58)$$

for sufficiently large n . By (3.4), (3.43), (3.57), (3.58), and $\lim_{n \rightarrow \infty} \omega_n = \infty$, we have for sufficiently large n

$$|f_n(\xi) - \beta_n| < \frac{C_5^{e_n}}{q_n^{e_n \omega_n}} < \frac{a_n^{\varepsilon_6}}{2a_n^{\delta_1 \omega_n}} < \frac{1}{2a_n^{(\sigma-\varepsilon_1)(1-\theta-\varepsilon_5)}}, \quad (3.59)$$

where $C_5 = C_4 C_3 (\xi + 1) > 1$ and $\varepsilon_6 > 0$ is a sufficiently small real number.

It follows from (3.54), (3.55), and (3.59) that

$$|f(\xi) - \beta_n| < \frac{1}{a_n^{(\sigma-\varepsilon_1)(1-\theta-\varepsilon_5)}} \quad (3.60)$$

for sufficiently large n . By (3.53) and (3.60), we get for sufficiently large n

$$|f(\xi) - \beta_n| < (H_K(\beta_n))^{-\frac{(\sigma-\varepsilon_1)(1-\theta-\varepsilon_5)}{\left(\left(\varepsilon_3 + \frac{\sigma - \varepsilon_1}{\sigma - \varepsilon_1 - 1} \right) (1 + \theta + \varepsilon_2) + \varepsilon_4 + \frac{\sigma - \varepsilon_1}{\sigma - \varepsilon_1 - 1} \delta_2 \right) m}}.$$

3) We can assume that the sequence $\left\{ \frac{p_n}{q_n} \right\}_{n=1}^{\infty}$ is strictly increasing or strictly decreasing by working with an appropriate subsequence of $\left\{ \frac{p_n}{q_n} \right\}_{n=1}^{\infty}$ if necessary. Hence, $\beta_n - \beta_{n-1} \neq 0$ ($n = 2, 3, \dots$). We have

$$0 < |\beta_n - \beta_{n-1}| \leq |f(\xi) - \beta_n| + |f(\xi) - \beta_{n-1}| \quad (n = 2, 3, \dots). \quad (3.61)$$

By (3.8), (3.60), and (3.61), we get for sufficiently large n

$$0 < |\beta_n - \beta_{n-1}| < \frac{2}{a_{n-1}^{(\sigma-\varepsilon_1)(1-\theta-\varepsilon_5)}}. \quad (3.62)$$

We deduce from (3.62) and Lemma 2.5 that

$$\frac{1}{\tau(m) H_K(\beta_n) H_K(\beta_{n-1})} < \frac{2}{a_{n-1}^{(\sigma-\varepsilon_1)(1-\theta-\varepsilon_5)}} \quad (3.63)$$

holds for sufficiently large n , where $\tau(m) = 2^{m-1}(m+1)^2$. By (3.53) and (3.63),

$$H_K(\beta_n) > C_6 a_{n-1}^{(\sigma-\varepsilon_1)(1-\theta-\varepsilon_5) - \left(\left(\varepsilon_3 + \frac{\sigma-\varepsilon_1}{\sigma-\varepsilon_1-1} \right) (1+\theta+\varepsilon_2) + \varepsilon_4 + \frac{\sigma-\varepsilon_1}{\sigma-\varepsilon_1-1} \delta_2 \right) m} \quad (3.64)$$

for sufficiently large n , where $C_6 = \frac{1}{2\tau(m)} > 0$. Hence, similarly as in step 3) in the proof of Theorem 3.1, we see that the algebraic numbers β_n are all distinct from each other from some n onward and

$$\lim_{n \rightarrow \infty} H(\beta_n) = \infty. \quad (3.65)$$

4) Similarly as in step 4) in the proof of Theorem 3.1, there exists a real number $\varepsilon > 0$ such that

$$|f(\xi) - \beta_n| < \frac{1}{(H_K(\beta_n))^{2+\varepsilon}} \quad (3.66)$$

holds true for infinitely many different β_n in K . Hence, by (3.65), (3.66), Theorem 2.1, and Lemma 2.8, $f(\xi)$ must be transcendental.

Let (3.45) hold. By (3.7), (3.44), (3.45), (3.53), (3.64), and the appropriate choices of $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$, and ε_5 , we see that

$$\limsup_{n \rightarrow \infty} \frac{\log H_K(\beta_{n+1})}{\log H_K(\beta_n)} \leq \frac{\mu^2}{\frac{(\sigma-1)(1-\theta)}{m(1+\theta+\delta_2)} - 1} < \mu^2 < \infty. \quad (3.67)$$

We infer from (3.66), (3.67), and Theorem 2.2 that $f(\xi)$ is either an S -number or a T -number. This completes the proof of a) in Theorem 3.4.

Let (3.46) hold. By (3.46), there exists a subsequence $\left\{ \frac{\log a_{n_i+1}}{\log a_{n_i}} \right\}_{i=1}^{\infty}$ of the sequence $\left\{ \frac{\log a_{n+1}}{\log a_n} \right\}_{n=1}^{\infty}$ such that

$$\lim_{i \rightarrow \infty} \frac{\log a_{n_i+1}}{\log a_{n_i}} = \infty. \quad (3.68)$$

By (3.53), (3.55), and (3.68), we have

$$|f(\xi) - f_{n_i}(\xi)| \leq \frac{1}{2a_{n_i+1}^{1-\theta-\varepsilon_5}} < \frac{1}{2(H_K(\beta_{n_i}))^{r_i}} \quad (3.69)$$

for sufficiently large i , where

$$r_i = \frac{\log a_{n_i+1}}{\log a_{n_i}} \frac{1-\theta-\varepsilon_5}{\left(\left(\varepsilon_3 + \frac{\sigma-\varepsilon_1}{\sigma-\varepsilon_1-1} \right) (1+\theta+\varepsilon_2) + \varepsilon_4 + \frac{\sigma-\varepsilon_1}{\sigma-\varepsilon_1-1} \delta_2 \right) m}$$

with $\lim_{i \rightarrow \infty} r_i = \infty$. It follows from (3.53) and (3.59) that

$$|f_{n_i}(\xi) - \beta_{n_i}| < \frac{1}{2d_{n_i}^{\delta_1 \omega_{n_i} - \varepsilon_6}} < \frac{1}{2(H_K(\beta_{n_i}))^{s_i}} \quad (3.70)$$

for sufficiently large i , where

$$s_i = \frac{\delta_1 \omega_{n_i} - \varepsilon_6}{\left(\left(\varepsilon_3 + \frac{\sigma-\varepsilon_1}{\sigma-\varepsilon_1-1} \right) (1+\theta+\varepsilon_2) + \varepsilon_4 + \frac{\sigma-\varepsilon_1}{\sigma-\varepsilon_1-1} \delta_2 \right) m}$$

with $\lim_{i \rightarrow \infty} s_i = \infty$. By (3.54), (3.69), and (3.70), we get for sufficiently large i

$$|f(\xi) - \beta_{n_i}| < \frac{1}{(H_K(\beta_{n_i}))^{t_i}}, \quad (3.71)$$

where $t_i = \min(r_i, s_i)$ with $\lim_{i \rightarrow \infty} t_i = \infty$. We deduce from (3.65), (3.71), and Lemma 2.8 that $f(\xi)$ is a U -number of type less than or equal to m since $\deg(\beta_{n_i}) \leq m$ ($i = 1, 2, 3, \dots$). This completes the proof of b) in Theorem 3.4. \square

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