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Research Article

Certain classes of k-uniformly functions with bounded radius rotation associated with linear operator

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Abstract

In this paper we use linear operator to define certain classes of analytic functions related to conic domains. Inclusion results, radius problems, rate of growth and other interesting properties are investigated.

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1. Introduction

Let $\mathcal{A}$ be the class functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{E} = \{z \in \mathbb{C} : |z| < 1\}$. By $\mathcal{S}$, $\mathcal{S}^*(\alpha)$, $\mathcal{C}(\alpha)$, we denote the subclasses of $\mathcal{A}$ which consist of univalent, starlike and convex functions of order α, $(0 \leq \alpha < 1)$, respectively. For details see [5].

If f and g are two analytic functions in $\mathcal{E}$, we say that f is subordinate to g, written symbolically as $f(z) \prec g(z)$, if there exists a Schwarz function w, which is analytic in $\mathcal{E}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$ for all $z \in \mathcal{E}$. Furthermore, if the function g is univalent in $\mathcal{E}$, then we have the following equivalence, see [13].

$$f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(\mathcal{E}) \subset g(\mathcal{E}).$$

The convolution (or the Hadamard product) of two functions $f(z)$ and $g(z)$ where $f(z)$is given by (1.1) and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is defined as

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n., z \in \mathcal{E} \quad .$$

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Furthermore, we recall the two interesting subclasses of $\mathcal{S}$ consisting, respectively, of functions which are k-uniformly convex and k-starlike in $\mathcal{E}$ denote it by $k-\mathcal{UCV}$ and $k-\mathcal{ST}$ introduced in [9] . So, we have

$$k-\mathcal{UCV}=\{f(z)\in\mathcal{S}\colon \mathfrak{Re}\left[1+\frac{zf''(z)}{f'(z)}\right]>k\left|\frac{zf''(z)}{f'(z)}\right|,\ z\in\mathcal{E},\ 0\le k<\infty\}.$$

$$k-\mathcal{ST}=\{f(z)\in\mathcal{S}\colon \mathfrak{Re}\left[\frac{zf'(z)}{f(z)}\right]>k\left|\frac{zf'(z)}{f(z)}-1\right|,\ z\in\mathcal{E},\ 0\le k<\infty\}.$$

It is obvious for $k=0$, the class $k-\mathcal{UCV}$ reduces to the class of convex univalent functions $\mathcal{C}$ and class of starlike functions $\mathcal{ST}$. Moreover for $k=1$ corresponds to the class of uniformly convex functions $\mathcal{UCV}$ introduced by Goodman [6] and studied extensively by Rønning [20] . The class $k-\mathcal{ST}$ was investigated in [10]. For $k\in[0,\infty)$, S. Kanas in [9, 10] defined the conic domain Ω_k as follows

$$\Omega_k=\left\{u+\iota v;u>k\sqrt{(u-1)^2+v^2}\right\}. \tag{1.2}$$

For a fixed value of k, the domain Ω_k represents the conic region bounded by ellipse for $k>1$, hyperbolic when $0\le k<1$, parabolic for the value $k=1$and the right half plane when $k=0$. Now the domain $\Omega_{k,\alpha}$, related to Ω_k is defined as:

$$\Omega_{k,\alpha}=(1-\alpha)\,\Omega_k+\alpha,$$

where

$$\alpha=\begin{cases}[0,1)\,, & \text{if}\quad k\in[0,1]\,,\\ \left[0,1-\frac{\sqrt{k^2-1}}{k}\right) & \text{if}\quad k>1.\end{cases} \tag{1.3}$$

condition (1.3) on α is imposed to ensure that the point $(0,1)$ is inside the domain $\Omega_{k,\alpha}$(see [16, 17]). Extremal functions for these conic regions denoted by $p_{k,\alpha}(z)$, are analytic in $\mathcal{E}$ and map $\mathcal{E}$ onto $\Omega_{k,\alpha}$ such that $p_{k,\alpha}(0)=1$ and $p'_{k,\alpha}(0)>1$. $p_{k,\alpha}(z)$ is given as:

$$p_{k,\alpha}(z)=\begin{cases}\frac{1+(1-2\alpha)z}{1-z}, & k=0,\\ 1+\frac{2(1-\alpha)}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & k=1,\\ 1+\frac{2(1-\alpha)}{1-k^2}\sinh^2\left[\left(\frac{2}{\pi}\arccos k\right)\operatorname{arctan h}\sqrt{z}\right], & 0<k<1,\\ 1+\frac{(1-\alpha)}{k^2-1}\sin\left[\frac{\pi}{2R(t)}\int_0^{\frac{u(z)}{\sqrt{t}}}\frac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}dx\right]+\frac{1-\gamma}{k^2-1}, & k>1,\end{cases} \tag{1.4}$$

where $u(z)=\frac{z-\sqrt{t}}{1-\sqrt{tz}}$, $t\in(0,1)$, $z\in\mathcal{E}$ and z is chosen such that $k=\cosh\frac{\pi R'(t)}{4R(t)}$, where $R(t)$ is the Legendre's complete elliptic integral of the first kind and $R'(t)$ is complementary integral of $R(t)$; see [9] and [8] . Let $\mathcal{P}(p_{k,\alpha}(z))$ denotes the class of all those functions $p(z)$ which are analytic in $\mathcal{E}$ with $p(0)=1$ such that $p(z)\prec p_{k,\alpha}(z)$ for $z\in\mathcal{E}$. Clearly it can be seen that $\mathcal{P}(p_{k,\alpha}(z))\subset\mathcal{P}$ where $\mathcal{P}$ is the class of Caratheodory functions with positive real part, see [5]. It can easily be seen that if $p\in\mathcal{P}(p_{k,\alpha}(z))$, then $p\prec p_{k,\alpha}$. Therefore it can be written as

$$\mathcal{P}(p_{k,\alpha}(z))\subset\mathcal{P}\left(\frac{k+\alpha}{1+k}\right).$$

Noor in ([15]) extended the class $\mathcal{P}(p_{k,\alpha}(z))$ by defining the following class.

Definition 1.1. Let $p(z)$ be analytic in $\mathcal{E}$ with $p(0)=1$. Then $p\in\mathcal{P}_m(p_{k,\alpha}(z))$ if and only if, for $m\geq 2$, $k\geq 0$ and α given by (1.3)

$$p(z)=\left(\frac{m}{4}+\frac{1}{2}\right)p_1(z)-\left(\frac{m}{4}-\frac{1}{2}\right)p_2(z),$$

where $p_1(z),p_2(z)\in\mathcal{P}(p_{k,\alpha}(z))$. Taking $k=0$ and $\alpha=0$, we obtain the class $\mathcal{P}_m$ introduced by Pinchuk in [18]. Also $\mathcal{P}_2(p_{k,\alpha}(z))=\mathcal{P}(p_{k,\alpha}(z))$.

Definition 1.2. For $\lambda,\mu>0$ and $\sigma\in\mathbb{Z}=\{\cdots-2,-1,0,1,2,\cdots\}$ P. Sharma et al. [19, 21] defined a linear operator $\mathcal{L}^{\sigma}_{\lambda,\mu}:\mathcal{A}\to\mathcal{A}$ as:

$$\mathcal{L}^{\sigma}_{\lambda,\mu}f(z)=\begin{cases} I_{\lambda,\mu}\mathcal{L}^{\sigma+1}_{\lambda,\mu}f(z) & \text{for}\quad \sigma=-1,-2,\cdots \\ D_{\lambda,\mu}\mathcal{L}^{\sigma+1}_{\lambda,\mu}f(z) & \text{for}\quad \sigma=1,2,\cdots, \\ f(z) & \text{for}\quad m=0, \end{cases} \tag{1.5}$$

where integral operator $I_{\lambda,\mu}$ is given as

$$I_{\lambda,\mu}f(z)=\frac{\lambda}{\mu}z^{1-\frac{\lambda}{\mu}}\int_0^z t^{\frac{\lambda}{\mu}-2}f(t)\,dt$$

and differential operator

$$D_{\lambda,\mu}f(z)=\frac{\mu}{\lambda}t^{2-\frac{\lambda}{\mu}}\frac{d}{dz}\left(z^{\frac{\lambda}{\mu}-1}f(z)\right).$$

Let for $b>0$, $a,c\in\mathbb{C}$, the Erdelyi-Kober type integral operator $\mathcal{J}^{a,c}_b f(z):\mathcal{A}\to\mathcal{A}$ be defined for $\mathfrak{Re}(c-b)>0$ and $\mathfrak{Re}(b)>-b$ as

$$\mathcal{J}^{a,c}_b f(z)=\frac{\Gamma(c+b)}{\Gamma(a+b)}\frac{1}{\Gamma(c-a)}\int_0^1(1-t)^{c-a-1}t^{a-1}f\left(zt^b\right)dt. \tag{1.6}$$

The linear operator $\mathcal{L}^{\sigma}_{\lambda,\mu}(a,c,b):\mathcal{A}\to\mathcal{A}$ is the composition of two linear operators defined above by

$$\mathcal{L}^{\sigma}_{\lambda,\mu}(a,c,b)f(z)=\mathcal{L}^{\sigma}_{\lambda,\mu}\left(\mathcal{J}^{a,c}_b f(z)\right)=\mathcal{J}^{a,c}_b\left(\mathcal{L}^{\sigma}_{\lambda,\mu}f(z)\right),$$

and the series is given by

$$\mathcal{L}^{\sigma}_{\lambda,\mu}(a,c,b)f(z)=z+\frac{\Gamma(c+b)}{\Gamma(a+b)}\sum_{n=2}^{\infty}\left(1+\frac{\mu(n-1)}{\lambda}\right)^{\sigma}\frac{\Gamma(a+nb)}{\Gamma(c+nb)}a_n z^n. \tag{1.7}$$

It follows from (1.7), that

$$z\left[\mathcal{L}^{\sigma}_{\lambda,\mu}(a,c,b)f(z)\right]'=\frac{\lambda}{\mu}\mathcal{L}^{\sigma+1}_{\lambda,\mu}(a,c,b)f(z)+\left(1-\frac{\lambda}{\mu}\right)\mathcal{L}^{\sigma}_{\lambda,\mu}(a,c,b)f(z), \tag{1.8}$$

and

$$z\left[\mathcal{L}^{\sigma}_{\lambda,\mu}(a,c,b)f(z)\right]'=\frac{a+b}{b}\mathcal{L}^{\sigma}_{\lambda,\mu}(a+1,c,b)f(z)-\frac{a}{b}\mathcal{L}^{\sigma}_{\lambda,\mu}(a,c,b)f(z). \tag{1.9}$$

Different analytical classes associated with the linear operator can also be found in [1], [2], [12] and [11].
Using the linear operator $\mathcal{L}^{\sigma}_{\lambda,\mu}(a,c,b)$ we will define the following classes of analytic functions.

Definition 1.3. Let $\lambda, \mu, b > 0,\ a, c \in \mathbb{C},\ \sigma \in \mathbb{Z},\ k \in [0, \infty), m \geq 2$ and $\mathfrak{Re}(c-a) \geq 0$ with $\mathfrak{Re}(a) > -b$. Then for α given by (1.3)

$$k - \mathcal{UR}^{m,\sigma}_{\lambda,\mu}(a,c,b,\alpha) = \left\{ f(z) \in \mathcal{A} : \frac{z\left[\pounds^{\sigma}_{\lambda,\mu}(a,c,b) f(z)\right]'}{\pounds^{\sigma}_{\lambda,\mu}(a,c,b) f(z)} \in \mathcal{P}_m(p_{k,\alpha}) \right\}.$$

Definition 1.4. Let $f(z) \in \mathcal{A}$. Then $f(z) \in k - \mathcal{UV}^{m,\sigma}_{\lambda,\mu}(a,c,b,\alpha),\ a, c \in \mathbb{C}, \lambda, \mu, b > 0$ if and only if

$$\frac{\left(z\left[\pounds^{\sigma}_{\lambda,\mu}(a,c,b) f(z)\right]'\right)'}{\left(\pounds^{\sigma}_{\lambda,\mu}(a,c,b) f(z)\right)'} \in \mathcal{P}_m(p_{k,\alpha}).$$

It can be easily seen that

$$f(z) \in k - \mathcal{UV}^{m,\sigma}_{\lambda,\mu}(a,c,b,\alpha) \Leftrightarrow zf'(z) \in k - \mathcal{UR}^{m,\sigma}_{\lambda,\mu}(a,c,b,\alpha). \tag{1.10}$$

Special cases:

i) For $m = 2,\ \sigma = \lambda = \mu = 1,\ \alpha = 0$ it reduces to the the class $k - \mathcal{UCV}$ which was introduced in [9].

ii) Taking $\sigma = \lambda = \mu = 1$, it coincides with the class of functions of k-uniform bounded boundary rotation m with order α , see details in [15].

Similarly, by giving specific values to the parameters involved in $k - \mathcal{UV}^{m,\sigma}_{\lambda,\mu}(a,c,b,\alpha)$ and $k - \mathcal{UR}^{m,\sigma}_{\lambda,\mu}(a,c,b,\alpha)$, we obtain many well-known as well as new subclasses of analytic, univalent functions, see [3-22].

We shall assume, throughout this study unless otherwise stated, that $\lambda, \mu, b > 0,\ a, c \in \mathbb{C}$, $\sigma \in \mathbb{Z},\ k \in [0, \infty), m \geq 2$ and $\mathfrak{Re}(c-a) \geq 0$ with $\mathfrak{Re}(a) > -b$.

Now to establish our main results, we need the following lemmas.

2. Preliminary lemmas

Lemma 2.1. [5] *Let $u = u_1 + iu_2,\ v = v_1 + iv_2$ and $\Psi(u,v)$ be a complex valued function satisfying the conditions:*

(i) $\psi(u,v)$ is continuous in a domain $\mathcal{D} \subset \mathbb{C}^2$,

(ii) $(1,0) \in \mathcal{D}$ and $\mathfrak{Re}\Psi(1,0) > 0$,

(iii) $\mathfrak{Re}\Psi(iu_2, v_1) \leq 0$, whenever $(iu_2, v_1) \in \mathcal{D}$ and $v_1 \leq -\frac{1}{2}(1+u_2^2)$.

If $h(z) = 1 + c_1 z + \cdots$ is a function analytic in E such that $(h(z), zh'(z)) \in \mathcal{D}$ and $\mathfrak{Re}\Psi(h(z), zh'(z)) > 0$ for $z \in \mathcal{E}$, then $\mathfrak{Re}h(z) > 0$ in $\mathcal{E}$.

Lemma 2.2. [7] *Let σ, λ with any complex numbers with $\lambda \neq 0$ and $0 \leq \gamma \leq \mathfrak{Re}(\lambda k/(k+1) + \sigma)$. If $\phi(z)$ is analytic in $\mathcal{E}$ with $\phi(0) = 1$ and satisfies*

$$\left(\phi(z) + \frac{z\phi'(z)}{\lambda\phi(z) + \sigma}\right) \prec p_{k,\gamma}(z), \tag{2.1}$$

and $\varphi_{k,\gamma}(z)$ is an analytic solution of

$$\varphi_{k,\gamma}(z) + \frac{z\varphi'_{k,\gamma}(z)}{\lambda\varphi_{k,\gamma}(z) + \sigma} = p_{k,\gamma}(z),$$

then $\varphi_{k,\gamma}(z)$ is univalent,

$$\phi(z) \prec \varphi_{k,\gamma}(z) \prec p_{k,\gamma}(z),$$

and $\varphi_{k,\gamma}(z)$ is the best dominant of (2.1), where $\varphi_{k,\gamma}(z)$ is given as

$$\varphi_{k,\gamma}(z) = \left\{ \left[\left(\int_0^1 \exp \int_t^{tz} \frac{p_{k,\gamma}(u) - 1}{u} du \right) dt \right]^{-1} + \frac{\sigma}{\lambda} \right\}.$$

Lemma 2.3. [4] *Let $f(z)$ be univalent and $0 \le r < 1$. Then there exists a number z_1 with $|z_1| = r$ such that for all, $|z| = r$, we have*

$$|z - z_1| |f(z)| \le \frac{2r^2}{1 - r^2}.$$

3. Main results

Theorem 3.1. *Let $k \in [0, \infty)$, $m \ge 2$. Then $k - \mathcal{UR}^{2,\sigma+1}_{\lambda,\mu}(a, c, b, \alpha) \subset k - \mathcal{UR}^{2,\sigma}_{\lambda,\mu}(a, c, b, \alpha)$.*

Proof. Let $f(z) \in k - \mathcal{UR}^{2,\sigma+1}_{\lambda,\mu}(a, c, b, \alpha)$. Setting

$$\frac{z\left[\mathcal{L}^{\sigma}_{\lambda,\mu}(a, c, b) f(z)\right]'}{\mathcal{L}^{\sigma}_{\lambda,\mu}(a, c, b) f(z)} = p(z),$$

where $p(z)$ is analytic in $\mathcal{E}$ with $p(0) = 1$. Then using (1.8) ,we have

$$\frac{\mathcal{L}^{\sigma+1}_{\lambda,\mu}(a, c, b) f(z)}{\mathcal{L}^{\sigma}_{\lambda,\mu}(a, c, b) f(z)} = \frac{\mu}{\lambda}\left(p(z) + \frac{\lambda}{\mu} - 1\right)$$

Logarithmic differentiation and simple computation yields

$$\frac{z\left(\mathcal{L}^{\sigma+1}_{\lambda,\mu}(a, c, b) f(z)\right)'}{\mathcal{L}^{\sigma+1}_{\lambda,\mu}(a, c, b) f(z)} = p(z) + \frac{zp'(z)}{p(z) + \left(\frac{\lambda}{\mu} - 1\right)}.$$

Since $f(z) \in k - \mathcal{UR}^{2,\sigma+1}_{\lambda,\mu}(a, c, b, \alpha)$,we have

$$p(z) + \frac{zp'(z)}{p(z) + \left(\frac{\lambda}{\mu} - 1\right)} \prec p_{k,\alpha}(z),$$

where $p_{k,\alpha}(z)$ is defined by (1.4). Now applying Lemma 2.2 with $\rho = 1$ and $\sigma = \left(\frac{\lambda}{\mu} - 1\right)$, we have

$$p(z) \prec q_k(z) \prec p_{k,\alpha}(z),$$

where $q_k(z)$ being best dominant and is given by

$$q_k(z) = \left(\left[\int_0^1 t^{\frac{\lambda}{\mu}-1} \exp \int_z^{tz} \frac{p_{k,\alpha}(\xi) - 1}{\xi} d\xi\right]^{-1} + (1 - \frac{\lambda}{\mu})\right).$$

This shows that $p(z) \in \mathcal{P}(p_{k,\alpha})$ $(z \in \mathcal{E})$ and consequently $f \in k - \mathcal{UR}^{2,\sigma}_{\lambda,\mu}(a, c, b, \alpha)$ in $\mathcal{E}$. □

Theorem 3.2. *If $f(z) \in k - \mathcal{UV}^{2,\sigma+1}_{\lambda,\mu}(b, c, b, \alpha)$, then $f(z) \in k - \mathcal{UV}^{2,\sigma}_{\lambda,\mu}(a, c, b, \alpha)$.*

Proof. In view relation (1.10) and Theorem 3.1, we have

$$\begin{aligned} f(z) \in k - \mathcal{UV}^{2,\sigma+1}_{\lambda,\mu}(b, c, b, \alpha) &\Leftrightarrow zf'(z) \in k - \mathcal{UR}^{2,\sigma+1}_{\lambda,\mu}(b, c, b, \alpha) \\ &\Rightarrow zf'(z) \in k - \mathcal{UR}^{2,\sigma}_{\lambda,\mu}(b, c, b, \alpha) \\ &\Leftrightarrow f(z) \in k - \mathcal{UV}^{2,\sigma}_{\lambda,\mu}(b, c, b, \alpha). \end{aligned}$$

□

Using (1.9) and the same technique as in Theorem 3.1 and Theorem 3.2 we can easily prove the following results.

Theorem 3.3. *Let $f(z) \in \mathcal{A}$. Then*

$$k - \mathcal{UR}^{2,\sigma}_{\lambda,\mu}(a + 1, c, b, \alpha) \subset k - \mathcal{UR}^{2,\sigma}_{\lambda,\mu}(a, c, b, \alpha).$$

Theorem 3.4. *Let* $f(z) \in k - \mathcal{UR}_{\lambda,\mu}^{2,\sigma+1}(a,c,b,\alpha)$. *Then* $f(z) \in \mathcal{R}_{\lambda,\mu}^{2,\sigma}(a,c,b,\beta_1)$ *in* $\mathcal{E}$, *where*

$$\beta_1 = \frac{2(\mu + 2\beta_0(\lambda-\mu))}{(2(\lambda-\mu) - 2\beta_0\mu + \mu) + \sqrt{(2(\lambda-\mu) - 2\beta_0\mu + \mu) + 8\mu(\mu + 2\beta_0(\lambda-\mu))}}. \quad (3.1)$$

Proof. Proceeding as in Theorem 3.1, we have

$$p(z) + \frac{zp'(z)}{p(z) + \frac{\lambda-\mu}{\mu}} \in \mathcal{P}(p_{k,\alpha}) \subset \mathcal{P}(\beta_0), \ \beta_0 = \frac{k+\alpha}{1+k}.$$

Let us take

$$p(z) = (1-\beta_1)q(z) + \beta_1.$$

Then

$$\mu_1\left[q(z) + \frac{\omega_1 zq'(z)}{q(z)+\omega_2}\right] + \mu_2 \in \mathcal{P} \text{ in } \mathcal{E}.$$

where $\mu_1 = \frac{1-\beta_1}{1-\beta_0}$, $\mu_2 = \frac{\beta_1-\beta_0}{1-\beta_0}$, $\omega_1 = \frac{1}{1-\beta_1}$ and $\omega_2 = \frac{\lambda}{\mu(1-\beta_1)} - 1$. We now form the functional $\Psi(u;v)$ by choosing $u = q(z)$, $v = zq'(z)$ and note that the first two conditions of Lemma 2.1 are clearly satisfied. We check condition (iii) as follows.

$$\Psi(u,v) = \mu_1\left[u + \frac{\omega_1 v}{u+\omega_2}\right] + \mu_2.$$

Now

$$\mathfrak{Re}\Psi(iu,v) = \frac{\mu_1\omega_1\omega_2 v}{u^2+\omega_2^2} + \mu_2$$

As $\mu_1 > 0$, $\omega_1 > 0$, so applying $v_1 = -\frac{1}{2}(1+u_2^2)$ and after simple computation we obtain

$$\begin{aligned}
\mathfrak{Re}\Psi(iu_2,v_1) &\le -\frac{\mu_1\omega_1\omega_2(1+u_2)}{2(u_2^2+\omega_2^2)} + \mu_2 \\
&= \frac{2\mu_2\omega_2^2 - \mu_1\omega_1\omega_2 + (2\mu_2 - \mu_1\omega_1\omega_2)u_2^2}{2(u_2^2+\omega_2^2)} \\
&= \frac{A + Bu_2^2}{C}, \qquad (3.2)
\end{aligned}$$

where

$$\begin{aligned}
A &= 2\mu_2\omega_2^2 - \mu_1\omega_1\omega_2, \\
B &= (2\mu_2 - \mu_1\omega_1\omega_2), \\
C &= 2\left(u_2^2 + \omega_2^2\right).
\end{aligned}$$

The right hand side of (3.2) is negative if $A \le 0$ and $B \le 0$. From $A \le 0$, we obtain β_1 as given by (3.1), and $B \le 0$ ensures that $0 \le \beta_1 < 1$. Since all the conditions of Lemma 2.1 are satisfied, it follows that $q(z) \in \mathcal{P}$, and consequently $f(z) \in \mathcal{R}_{\lambda,\mu}^{2,\sigma}(a,c,b,\beta_1)$. This completes the proof. □

Theorem 3.5. *Let* $f(z) \in k - \mathcal{UV}_{\lambda,\mu}^{2,\sigma+1}(a,c,b,\alpha)$. *Then* $f(z) \in \mathcal{V}_{\lambda,\mu}^{2,\sigma}(a,c,b,\beta_1)$ *in* $\mathcal{E}$, *where* β_1 *is defined by (3.1).*

Proof immediately follows by using Theorem 3.4 and relation (1.10).

Theorem 3.6. *Let* $f(z) \in k - \mathcal{UR}_{\lambda,\mu}^{m,\sigma}(a,c,b,\alpha)$. *Then there exist* $s_1(z)$, $s_2(z) \in k - \mathcal{US}_{\lambda,\mu}^{\sigma}(a,c,b,\alpha)$ *such that*

$$\pounds_{\lambda,\mu}^{\sigma}(a,c,b)f(z) = \frac{\left(\pounds_{\lambda,\mu}^{\sigma}(a,c,b)s_1(z)\right)^{\frac{m+2}{4}}}{\left(\pounds_{\lambda,\mu}^{\sigma}(a,c,b)s_2(z)\right)^{\frac{m-2}{4}}}, \ m \ge 2, k \in [0,\infty).$$

Proof. Let $s(z) \in k - \mathcal{US}^{\sigma}_{\lambda,\mu}(a,c,b,\alpha)$. Then

$$\frac{z\left(\pounds^{\sigma}_{\lambda,\mu}(a,c,b)\,s(z)\right)'}{\pounds^{\sigma}_{\lambda,\mu}(a,c,b)\,s(z)} \prec p_{k,\alpha}(z),$$

this implies that

$$\pounds^{\sigma}_{\lambda,\mu}(a,c,b)\,s(z) \prec z\exp\int_0^z \frac{p_{k,\alpha}(t)-1}{t}dt.$$

Let μ_m be the class of real-valued functions $\mu(t)$ of bounded variation on $[-\pi,\pi]$ satisfying the conditions

$$\int_{-\pi}^{\pi} d\mu(t) = 2 \text{ and } \int_{-\pi}^{\pi} |d\mu(t)| \leq m.$$

$$\pounds^{\sigma}_{\lambda,\mu}(a,c,b)\,f(z) \prec z\exp\int_0^z \frac{p_{k,\alpha}(t)-1}{t}d\mu(t),\ \ \mu \in \mu_m$$

We can write the real-valued function of bounded variation as

$$\mu(t) = \mu_1(t) - \mu_2(t).$$

where $\mu_1(t)$ and $\mu_2(t)$ are nonnegative increasing functions. Thus

$$\frac{\pounds^{\sigma}_{\lambda,\mu}(a,c,b)\,f(z)}{z} = \frac{\exp\int_0^z \frac{p_{k,\alpha}(t)-1}{t}d\mu_1(t),}{\exp\int_0^z \frac{p_{k,\alpha}(t)-1}{t}d\mu_2(t),} = \frac{\mathcal{N}(z)}{\mathcal{D}(z)}, \tag{3.3}$$

where

$$\int_{-\pi}^{\pi} (d\mu_1(t) - d\mu_2(t)) = 2, \text{ and } \int_{-\pi}^{\pi} |d\mu_1(t) + d\mu_2(t)| \leq m.$$

Since $\mu \in \mu_m$. These, in turn, imply that

$$\int_{-\pi}^{\pi} d\mu_1(t) \leq \frac{m+2}{2} =, \text{ and } \int_{-\pi}^{\pi} d\mu_2(t) \leq \frac{m-2}{2}. \tag{3.4}$$

From (3.3), we note that $\int_{-\pi}^{\pi} d\mu_1(t)$ and $\int_{-\pi}^{\pi} d\mu_2(t)$ are the boundary rotation of the image of $\mathcal{E}$ under the mappings

$$\omega_1 = \int_0^z \mathcal{N}(\eta)\,d\eta \text{ and } \omega_2 = \int_0^z \mathcal{D}(\eta)\,d\eta,$$

respectively. From (3.4) the functions

$$\omega_1 = (\mathcal{N}(z))^{\frac{4}{m+2}} \text{ and } \omega_2 = (\mathcal{D}(z))^{\frac{4}{m-2}},$$

are the derivatives of functions whose boundary rotations are 2. In other words, these are the derivatives of functions belonging to $k - \mathcal{UCV}(\alpha)$. Let

$$\pounds^{\sigma}_{\lambda,\mu}(a,c,b)\,s_1(z) = z(\mathcal{N}(z))^{\frac{4}{m+2}} \text{ and } \pounds^{\sigma}_{\lambda,\mu}(a,c,b)\,s_2(z) = z(\mathcal{D}(z))^{\frac{4}{m-2}}.$$

This means $s_1(z)$, $s_2(z) \in k - \mathcal{US}^{\sigma}_{\lambda,\mu}(a,c,b,\alpha)$. Hence

$$\pounds^{\sigma}_{\lambda,\mu}(a,c,b)\,f(z) = \frac{\left(\pounds^{\sigma}_{\lambda,\mu}(a,c,b)\,s_1(z)\right)^{\frac{m+2}{4}}}{\left(\pounds^{\sigma}_{\lambda,\mu}(a,c,b)\,s_2(z)\right)^{\frac{m-2}{4}}},\ \ m \geq 2,\ k \in [0,\infty).$$

This completes the proof. □

Theorem 3.7. *Let* $f(z) \in k - \mathcal{UV}_{\lambda,\mu}^{m,\sigma}(a,c,b,\alpha)$ *and have the form (1.1). Then*

$$a_n = O(1).n^{\alpha_1 - 2}, \quad (n \to \infty),$$

where

$$\alpha_1 = \left\{ \left(\frac{1-\alpha}{1+k}\right)\left(\frac{m}{2}+1\right) \right\},$$

and $O(1)$ *is a constant depending on* k*,* m *and* α*. The exponent is best possible when* $k = 0 = \alpha$*.*

Proof. Let

$$\mathcal{L}_{\lambda,\mu}^{\sigma}(a,c,b) f(z) = F(z) = z + \frac{\Gamma(c+b)}{\Gamma(a+b)} \sum_{n=2}^{\infty} \left(1 + \frac{\mu(n-1)}{\lambda}\right)^{\sigma} \frac{\Gamma(a+nb)}{\Gamma(c+nb)} a_n z^n.$$

Since using the result [3] $F(z) \in k - \mathcal{UV}^m(\alpha) \subset \mathcal{V}^m(\alpha_1)$, $\alpha_1 = \frac{k+\alpha}{1+k}$, there exist $F_1(z) \in \mathcal{V}^m$ such that

$$F'(z) = (F_1'(z))^{1-\alpha_1} = (F_1'(z))^{\frac{1-\alpha}{1+k}}.$$

Setting

$$\begin{aligned} G(z) &= \left(z(zF'(z))'\right)' \\ &= (zF'(z)H(z))', \quad H(z) = \frac{(zF'(z))'}{F'(z)} \in \mathcal{P}_m\left(\frac{k+\alpha}{1+k}\right) \\ &= F'(z)\left[H^2(z) + zH'(z)\right]. \end{aligned} \tag{3.5}$$

Now from Theorem (3.7) , we have

$$\begin{aligned} F(z) &= \frac{\left(\mathcal{L}_{\lambda,\mu}^{\sigma}(a,c,b)s_1(z)\right)^{\frac{m+2}{4}}}{\left(\mathcal{L}_{\lambda,\mu}^{\sigma}(a,c,b)s_2(z)\right)^{\frac{m-2}{4}}}, \text{ for } s_1(z), s_2(z) \in k - \mathcal{US}_{\lambda,\mu}^{\sigma}(a,c,b,\alpha) \\ &= \frac{\left(\mathcal{L}_{\lambda,\mu}^{\sigma}(a,c,b)g_1(z)\right)^{\left(\frac{1-\alpha}{1+k}\right)\left(\frac{m+2}{4}\right)}}{\left(\mathcal{L}_{\lambda,\mu}^{\sigma}(a,c,b)g_2(z)\right)^{\left(\frac{1-\alpha}{1+k}\right)\left(\frac{m-2}{4}\right)}}, \text{ for } g_1(z), g_2(z) \in \mathcal{C}. \end{aligned} \tag{3.6}$$

Now for $z = re^{i\theta}$,

$$n^3 |a_n| = \frac{1}{2\pi r^n} \left| \int_0^{2\pi} F(z) e^{-i\theta} d\theta \right|. \tag{3.7}$$

Using(3.6)and (3.5) in (3.7), we have

$$n^3 |a_n| = \frac{1}{2\pi r^n} \int_0^{2\pi} \left| \frac{\left(\mathcal{L}_{\lambda,\mu}^{\sigma}(a,c,b)g_1(z)\right)^{\left(\frac{1-\alpha}{1+k}\right)\left(\frac{m+2}{4}\right)}}{\left(\mathcal{L}_{\lambda,\mu}^{\sigma}(a,c,b)g_2(z)\right)^{\left(\frac{1-\alpha}{1+k}\right)\left(\frac{m-2}{4}\right)}} \right| \left|H^2(z) + zH'(z)\right| d\theta.$$

Using distortion results for the convex functions and Lemma as given in [14] with $\alpha_1 = \frac{k+\alpha}{1+k}$ and $r = 1 - \frac{1}{n}$, we obtain the required result. □

Theorem 3.8. *Let* $f(z) \in k - \mathcal{UV}_{\lambda,\mu}^{m,\sigma}(a,c,b,\alpha)$ *and let qth Hankel determinant of* $f(z)$ *for* $q \geq 1$ *and* $n \geq 1$*, be defined by . Then for* $m \geq 2$ *,*

$$\mathcal{H}_q(n) = \mathcal{O}(1)\, n, \quad (n \to \infty),$$

where $\mathcal{O}(1)$ *is a constant depending only on* n*,* m *and* α*.*

Proof. Let

$$\pounds_{\lambda,\mu}^{\sigma}\left(a,c,b\right)f\left(z\right)=F\left(z\right)=z+\frac{\Gamma\left(c+b\right)}{\Gamma\left(a+b\right)}\sum_{n=2}^{\infty}\left(1+\frac{\mu\left(n-1\right)}{\lambda}\right)^{\sigma}\frac{\Gamma\left(a+nb\right)}{\Gamma\left(c+nb\right)}a_{n}z^{n}.$$

Hence, using a result due to [3], it follows that, for $F(z)\in k-\mathcal{UV}^{m}(\alpha)\subset\mathcal{V}^{m}(\alpha_1)$, $\alpha_1=\frac{k+\alpha}{1+k}$, there exist $F_1(z)\in\mathcal{V}^{m}$ such that

$$F'(z)=\left(F_1'(z)\right)^{1-\alpha_1}=\left(F_1'(z)\right)^{\frac{1-\alpha}{1+k}}$$
$$F'(z)=\frac{\left(\frac{s_1(z)}{z}\right)^{\left(\frac{m+2}{4}\right)\left(\frac{1-\alpha}{1+k}\right)}}{\left(\frac{s_2(z)}{z}\right)^{\left(\frac{m-2}{4}\right)\left(\frac{1-\alpha}{1+k}\right)}},\ \text{for } s_1,s_2\in\mathcal{S}^*$$

Setting

$$\begin{aligned}G(z)&=\left(z\left(zF'(z)\right)'\right)'\\&=\left(zF'(z)H(z)\right)',\quad H(z)=\frac{\left(zF'(z)\right)'}{F'(z)}\in P_m\left(\frac{k+\alpha}{1+k}\right)\\&=F'(z)\left[H^2(z)+zH'(z)\right].\qquad(3.8)\\&=\frac{\left(\frac{s_1(z)}{z}\right)^{\left(\frac{m+2}{4}\right)\left(\frac{1-\alpha}{1+k}\right)}}{\left(\frac{s_2(z)}{z}\right)^{\left(\frac{m-2}{4}\right)\left(\frac{1-\alpha}{1+k}\right)}}\left[H^2(z)+zH'(z)\right],\ \text{for } s_1,s_2\in\mathcal{S}^*\qquad(3.9)\end{aligned}$$

Now for $z=re^{i\theta}$, $j\geq 1$ and z_1 any complex number, we consider

$$\begin{aligned}\left|\triangle_j\left(n,z_1,G(z)\right)\right|&=\left|\frac{1}{2\pi r^{n+j}}\int_0^{2\pi}\left(z-z_1\right)^jG(z)\,e^{-(n+j)\theta}d\theta\right|\\&\leq\frac{1}{2\pi r^{n+j}}\int_0^{2\pi}\left|z-z_1\right|^j\left|s_1(z)\right|^j\frac{\left|s_1(z)\right|^{\left(\frac{m+2}{4}\right)\left(\frac{1-\alpha}{1+k}\right)-j}}{\left|s_2(z)\right|^{\left(\frac{m-2}{4}\right)\left(\frac{1-\alpha}{1+k}\right)}}\left|H^2(z)+zH'(z)\right|d\theta\end{aligned}$$

Using Lemma 2.6, Lemma 2.7 and well-known distortion result for starlike functions in (4.6), we obtain for $\left(\frac{m+2}{4}\right)\left(\frac{1-\alpha}{1+k}\right)>j$

$$\begin{aligned}\left|\triangle_j\left(n,z_1,G(z)\right)\right|\leq&\ \frac{1}{2\pi r^{n+j}}\left(\frac{2r^2}{1-r^2}\right)^j\left(\frac{4}{r}\right)^{\left(\frac{m-2}{4}\right)\left(\frac{1-\alpha}{1+k}\right)}\\&\times\int_0^{2\pi}\left|s_1(z)\right|^{\left(\frac{m+2}{4}\right)\left(\frac{1-\alpha}{1+k}\right)-j}\left|H^2(z)+zH'(z)\right|d\theta.\end{aligned}$$

using distortion bounds for starlike function we have

$$\begin{aligned}
|\triangle_j(n,z_1,G(z))| &\le \frac{1}{2\pi r^{n+j}}\left(\frac{2r^2}{1-r^2}\right)^j\left(\frac{4}{r}\right)^{\left(\frac{m-2}{4}\right)\left(\frac{1-\alpha}{1+k}\right)} \\
&\quad\times\int_0^{2\pi}\left(\frac{r}{(1-r)^2}\right)^{\left(\frac{m+2}{4}\right)\left(\frac{1-\alpha}{1+k}\right)-j}\left|H^2(z)+zH'(z)\right|d\theta \\
&\le \frac{1}{r^{n-j-\left(\frac{1-\alpha}{1+k}\right)}}\left(\frac{1}{1-r}\right)^{\left(\frac{m+2}{2}\right)\left(\frac{1-\alpha}{1+k}\right)-j}(2)^{\left(\frac{m-2}{2}\right)\left(\frac{1-\alpha}{1+k}\right)-j}\left[\frac{1}{2\pi}\int_0^{2\pi}\left|H^2(z)+zH'(z)\right|d\theta\right] \\
&\le \frac{1}{r^{n-2j-\left(\frac{1-\alpha}{1+k}\right)}}\left(\frac{1}{1-r}\right)^{\left(\frac{m+2}{2}\right)\left(\frac{1-\alpha}{1+k}\right)-j}(2)^{\left(\frac{m-2}{2}\right)\left(\frac{1-\alpha}{1+k}\right)-j}\left[\frac{1-\left(m^2(1-\alpha)^2-1\right)r^2}{1-r^2}\right]
\end{aligned}$$

which can be written as

$$\begin{aligned}
|\triangle_j(n,z_1,G(z))| &\le C(m,k,\alpha,j)\frac{1}{r^{n-2j-\left(\frac{1-\alpha}{1+k}\right)}}\left(\frac{1}{1-r}\right)^{\left(\frac{m+2}{2}\right)\left(\frac{1-\alpha}{1+k}\right)-j-1} \\
&= \mathcal{O}(1)\left(\frac{1}{1-r}\right)^{\left(\frac{m+2}{2}\right)\left(\frac{1-\alpha}{1+k}\right)-j-1}
\end{aligned}$$

Now choosing $z_1=\frac{n}{n+1}e^{i\theta_n}\,(n\to\infty)\,, r=1-\frac{1}{n}$ we have

$$|\triangle_j(n,z_1,G(z))| = \mathcal{O}(1)\,n^{\left(\frac{m+2}{2}\right)\left(\frac{1-\alpha}{1+k}\right)-j-1}.$$

where $\mathcal{O}(1)$ represents the constant depending on α, m and n. □

Using Lemma 2.3 and similar technique of Theorem 3.7, we can easily prove the following result.

Theorem 3.9. *Let $f(z)\in k-\mathcal{UR}^{m,\sigma}_{\lambda,\mu}(a,c,b,\alpha)$ having the series(1.1). Then for $m\ge 2$*

$$||a_{n+1}|-|a_n||\le C(\alpha,m,n)\,n^{\alpha_1-2},$$

where $C(\alpha,m,n)$ is a constant depending on α, m and n .

Theorem 3.10. *Let $f(z)\in k-\mathcal{UV}^{m,\sigma}_{\lambda,\mu}(a,c,b,\alpha)$ and $g(z)\in k-\mathcal{UR}^{m,\sigma}_{\lambda,\mu}(a,c,b,\alpha)$. Then*

$$\left(\pounds^{\sigma}_{\lambda,\mu}(a,c,b)F(z)\right)=\int_0^z\left(\frac{\pounds^{\sigma}_{\lambda,\mu}(a,c,b)g(t)}{t}\right)^{\eta}\left(\frac{\pounds^{\sigma}_{\lambda,\mu}(a,c,b)h(t)}{t}\right)^{\kappa}dt, \tag{3.10}$$

in the class $k-\mathcal{UV}^{m,\sigma}_{\lambda,\mu}(a,c,b,\delta)$, where

$$\delta=1-(\eta+\kappa)(1-\alpha). \tag{3.11}$$

Proof. From (3.10), we can write

$$\frac{\left(z\left(\pounds^{\sigma}_{\lambda,\mu}(a,c,b)F(z)\right)'\right)'}{\left(\pounds^{\sigma}_{\lambda,\mu}(a,c,b)F(z)\right)'}=\eta\frac{z\left(\pounds^{\sigma}_{\lambda,\mu}(a,c,b)g(z)\right)'}{\pounds^{\sigma}_{\lambda,\mu}(a,c,b)g(z)}+\kappa\frac{z\left(\pounds^{\sigma}_{\lambda,\mu}(a,c,b)h(z)\right)'}{\pounds^{\sigma}_{\lambda,\mu}(a,c,b)h(z)}+1-(\eta+\kappa)\,. \tag{3.12}$$

Since $f(z)\in k-\mathcal{UV}^{m,\sigma}_{\lambda,\mu}(a,c,b,\alpha)\subset k-\mathcal{UR}^{m,\sigma}_{\lambda,\mu}(a,c,b,\alpha)$ and $g(z)\in k-\mathcal{UR}^{m,\sigma}_{\lambda,\mu}(a,c,b,\alpha)$, we have $\pounds^{\sigma}_{\lambda,\mu}(a,c,b)f(z)\,,\pounds^{\sigma}_{\lambda,\mu}(a,c,b)g(z)\in k-\mathcal{UR}^{m}(\alpha)$ and therefore we obtain

$\dfrac{z\left(\pounds^{\sigma}_{\lambda,\mu}(a,c,b)\,g(z)\right)'}{\pounds^{\sigma}_{\lambda,\mu}(a,c,b)\,g(z)}, \dfrac{z\left(\pounds^{\sigma}_{\lambda,\mu}(a,c,b)\,h(z)\right)'}{\pounds^{\sigma}_{\lambda,\mu}(a,c,b)\,h(z)} \in \mathcal{P}_m(p_{k,\alpha})$. Let

$$\frac{z\left(\pounds^{\sigma}_{\lambda,\mu}(a,c,b)\,g(z)\right)'}{\pounds^{\sigma}_{\lambda,\mu}(a,c,b)\,g(z)} = (1-\alpha)\,q_1(z)+\alpha,\ \ q_1(z)\in\mathcal{P}_m(p_k)$$

$$\frac{z\left(\pounds^{\sigma}_{\lambda,\mu}(a,c,b)\,h(z)\right)'}{\pounds^{\sigma}_{\lambda,\mu}(a,c,b)\,h(z)} = (1-\alpha)\,q_2(z)+\alpha,\ \ q_1(z)\in\mathcal{P}_m(p_k).$$

from (3.12), we have

$$\frac{1}{1-\delta}\left[\frac{\left(z\left(\pounds^{\sigma}_{\lambda,\mu}(a,c,b)\,F(z)\right)'\right)'}{\left(\pounds^{\sigma}_{\lambda,\mu}(a,c,b)\,F(z)\right)'}-\delta\right] = \frac{\eta}{(\eta+\kappa)}q_1(z)+\frac{\kappa}{(\eta+\kappa)}q_2(z), \tag{3.13}$$

where δ is given by (3.11). Now by using the well-known fact that the class $\mathcal{P}_m(p_k)$ is a convex set together with (3.13), we obtain the required result. □

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