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A note on some recent results of the conformable derivative

Omid T. Birgani[a], Sumit Chandok[b], Nebojša Dedović[c], Stojan Radenović[d]

[a]*School of Mathematics, Iran University of Science and Technology, Tehran, Iran.*

[b]*School of Mathematics, Thapar University, Patiala-147004, India.*

[c]*Faculty of Agriculture, University of Novi Sad, Novi Sad, Serbia.*

[d]*Department of Mathematics, College of Science, King Saud University, Riyadh 11451, Saudi Arabia.*

Abstract

In this note, we discuss, improve and complement some recent results of the conformable derivative introduced and established by Katugampola [11] and Khalil *et al.*[12]. Among other things we show that each function f defined on (a, b), $a > 0$ has a conformable derivative (CD) if and only if it has a classical first derivative. At the end of the paper, we prove the Rolle's, Cauchy, Lagrange's and Darboux's theorem in the context of Conformable Derivatives.

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1. Introduction and preliminaries

The conformable derivatives and their algebraic properties as well as its application on conformable differential linear systems subject to impulsive effects and establish qualitative behavior of the nontrivial solutions are studied in [10]. The conformable derivative is used to develop the Swartzendruber model for description of non-Darcian flow in porous media ([25]). Motivated by a proportional-derivative controller, a more precise definition of a conformable derivative is introduced and explored in [7]. Results included basic conformable derivative and integral rules, Taylor's theorem, reduction of order, variation of parameters,

complete characterization of solutions for constant coefficient and Cauchy-Euler type conformable equations, Cauchy functions, variation of constants, a self-adjoint equation, and Sturm-Liouville problems. Authors in [6] gave the physical interpretation when these derivatives are applied to physics and engineering. Quantum mechanics served as the primary backdrop for this development. Invariant conditions for conformable fractional problems of the calculus of variations under the presence of external forces in the dynamics are studied in [18] and the authors proved the fractional versions of Noether's symmetry theorem. They showed that with conformable derivatives it is possible to formulate an Action Principle for particles under frictional forces that is far simpler than the one obtained with classical fractional derivatives. Also, it is well known that fractional analysis has wide applications in several fields of science, for instance, engineering, fluid flow, electrical networks, control theory, dynamical systems, biosciences, and so on. For more interesting details see [1]-[26].

Khalil et al. [12] introduced the new definition of conformable derivatives as follows:

Definition 1.1. [12] Given a function $f : [0, \infty) \to \mathbb{R}$. Then "*conformable derivative*" of f of order α is defined by

$$T_\alpha (f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon} \tag{1.1}$$

for all $t > 0, \alpha \in (0, 1)$.

If f is $\alpha-$ differentiable in some $(0, a), a > 0$, and $\lim_{t \to 0^+} f^{(\alpha)}(t)$ exists, then define $f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t)$.

For $\alpha \in (n, n+1]$, we have the following definition of Khalil et al. [12].

Definition 1.2. [12]. Let $\alpha \in (n, n+1]$, and f be an $n-$differentiable at t, where $t > 0$. Then the conformable derivative of f of order α is defined as

$$T_\alpha (f)(t) = \lim_{\varepsilon \to 0} \frac{f^{([\alpha]-1)}(t + \varepsilon t^{[\alpha]-\alpha}) - f^{([\alpha]-1)}(t)}{\varepsilon} \tag{1.2}$$

where $[\alpha]$ is the smallest integer greater than or equal α.

Remark 1.3.

- As a consequence of Definition 1.2, one can easily show that $T_\alpha (f)(t) = t^{[\alpha]-\alpha} f^{[\alpha]}(t) = t^{n+1-\alpha} f^{(n+1)}(t)$, where $\alpha \in (n, n+1]$, and f is $(n+1)-$ differentiable at $t > 0$.

- Also, it is clear that this definition is generalization of Definition 1.1, putting $n = 0$ in (1.2) we obtain the condition (1.1), where $f^{(0)}(t) = f(t)$.

2. Main results

In this paper we consider, discuss, improve and complement some recent results of Khalil et al. [12]. We begin with the following result for the derivative of order α, i.e. T_α.

Proposition 2.1. *If $f : (0, +\infty) \to \mathbb{R}, \alpha \in (0, 1]$ and f is a differentiable, then $T_\alpha (f)(t) = t^{1-\alpha} \frac{df(t)}{dt}$. Also, if $\alpha \in (n, n+1], n \in \mathbb{N}$, then $T_\alpha (f)(t) = t^{n-\alpha} \frac{d^n f}{dt^n}(t)$.*

Proof. Indeed, equation (1.1), that is, (1.2) implies that

$$
\begin{aligned}
T_\alpha (f)(t) &= \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon t^{1-\alpha}} t^{1-\alpha} \\
&= t^{1-\alpha} \frac{df(t)}{dt},
\end{aligned}
$$

that is,

$$
\begin{aligned}
T_\alpha\left(f\right)\left(t\right) &= \lim_{\varepsilon\to 0} \frac{f^{([\alpha]-1)}\left(t+\varepsilon t^{[\alpha]-\alpha}\right) - f^{([\alpha]-1)}\left(t\right)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \frac{f^{([\alpha]-1)}\left(t+\varepsilon t^{[\alpha]-\alpha}\right) - f^{([\alpha]-1)}\left(t\right)}{\varepsilon t^{[\alpha]-\alpha}} t^{[\alpha]-\alpha} \\
&= t^{n-\alpha} f^{(n)}\left(t\right),
\end{aligned}
$$

because $[\alpha] = n$. □

For $\alpha = 1$, that is, $\alpha = n$ we obtain that the conformable derivative (FD) coincides with the classical first derivative $\frac{df(t)}{dt}$, that is, with $\frac{d^n f(t)}{dt^n}$. However, the zero order derivative $T_0\left(f\right)\left(t\right) = f^{(0)}\left(t\right)$ of a function $f\left(t\right)$ does not return the function $f\left(t\right)$, because $T_0\left(f\right)\left(t\right) = t^{1-0}\frac{df}{dt}\left(t\right) = t\frac{df}{dt}\left(t\right) \neq f\left(t\right)$ in general case.

Using Proposition 2.1, we have the following result:

Proposition 2.2. *Suppose that either $\alpha \in (0,1]$ or $\alpha \in (n, n+1]$, $n \in \mathbb{N}$ and f, g are $\alpha-$differentiable at a point $t > 0$. Then*

(1) $T_\alpha\left(a_1 f + a_2 g\right) = a_1 T_\alpha\left(f\right) + a_2 T_\alpha\left(g\right)$ *in both cases, for all $a_1, a_2 \in \mathbb{R}$.*

(2) $T_\alpha\left(t^p\right) = pt^{p-\alpha}$ *or* $T_\alpha\left(t^p\right) = p\left(p-1\right)...\left(p-n+1\right)t^{p-n}$ *for all $p \in \mathbb{R}$.*

(3) $T_\alpha\left(\lambda\right) = 0$ *in both cases, for all constant functions $f\left(t\right) = \lambda$.*

(4) $T_\alpha\left(fg\right) = fT_\alpha\left(g\right) + gT_\alpha\left(f\right)$ *only in the first case, that is, $\alpha \in (0,1]$.*

(5) $T_\alpha\left(\frac{f}{g}\right) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$ *only in the first case.*

Proof. Let $\alpha \in (0,1]$.

(**1**) We have that

$$
\begin{aligned}
T_\alpha\left(a_1 f + a_2 g\right)\left(t\right) &= t^{1-\alpha}\left(a_1 f + a_2 g\right)'\left(t\right) \\
&= t^{1-\alpha}\left(a_1 f'\left(t\right) + a_2 g'\left(t\right)\right) \\
&= a_1 t^{1-\alpha} f'\left(t\right) + a_2 t^{1-\alpha} g'\left(t\right) \\
&= a_1 T_\alpha\left(f\right)\left(t\right) + a_2 T_\alpha\left(g\right)\left(t\right) \\
&= \left(a_1 T_\alpha\left(f\right) + a_2 T_\alpha\left(g\right)\right)\left(t\right),
\end{aligned}
$$

or $T_\alpha\left(a_1 f + a_2 g\right) = a_1 T_\alpha\left(f\right) + a_2 T_\alpha\left(g\right)$, that is, operator T_α is a linear.

(**2**) Again we have

$$
\begin{aligned}
T_\alpha\left(t^p\right) &= t^{1-\alpha}\left(t^p\right)' \\
&= pt^{p-\alpha}.
\end{aligned}
$$

(**3**) Since $f'\left(t\right) = \lambda' = 0$ the result follows.

(**4**) In this case, we also obtain by Proposition 2.1,

$$
\begin{aligned}
T_\alpha\left(fg\right)\left(t\right) &= t^{1-\alpha}\left(fg\right)'\left(t\right) \\
&= t^{1-\alpha}\left(f'g + fg'\right)\left(t\right) \\
&= t^{1-\alpha}\left(f'g\right)\left(t\right) + t^{1-\alpha}\left(fg'\right)\left(t\right) \\
&= t^{1-\alpha} f'\left(t\right) g\left(t\right) + t^{1-\alpha} f\left(t\right) g'\left(t\right) \\
&= T_\alpha\left(f\right)\left(t\right) g\left(t\right) + f\left(t\right) T_\alpha\left(g\right)\left(t\right) \\
&= \left(T_\alpha\left(f\right) g + fT_\alpha\left(g\right)\right)\left(t\right),
\end{aligned}
$$

or $T_\alpha\left(fg\right) = T_\alpha\left(f\right) g + fT_\alpha\left(g\right)$. Otherwise, it is worth noticing that Leibniz rule (even generalized: see [21], page 5) does not hold for the conformable derivative.

(**5**) Similarly as (4).

Now, assume that $\alpha \in (n, n+1]$.

We prove only (1):

$$
\begin{aligned}
T_\alpha \left(a_1 f + a_2 g\right)(t) &= t^{n-\alpha} \left(a_1 f + a_2 g\right)^{(n)}(t) \\
&= t^{n-\alpha} \left(a_1 f^{(n)} + a_2 g^{(n)}\right)(t) \\
&= a_1 t^{n-\alpha} f^{(n)}(t) + a_2 t^{n-\alpha} g^{(n)}(t) \\
&= a_1 T_\alpha(f)(t) + a_2 T_\alpha(g)(t) \\
&= \left(a_1 T_\alpha(f) + a_2 T_\alpha(g)\right)(t),
\end{aligned}
$$

or $T_\alpha \left(a_1 f + a_2 g\right) = a_1 T_\alpha(f) + a_2 T_\alpha(g)$, that is, operator T_α is a linear in the case $\alpha \in (n, n+1]$. $\square$

Proposition 2.3. *The index law, that is, $T_\alpha T_\beta(f) = T_{\alpha+\beta}(f)$ for any α, β does not hold in general.*

Indeed, if $f(t) = t^2, \alpha = \frac{1}{2}, \beta = \frac{1}{3}$ then $T_{\frac{1}{2}} T_{\frac{1}{3}}(f) = \frac{10}{3} t^{\frac{7}{6}}$, while $T_{\frac{1}{2}+\frac{1}{3}}(f) = T_{\frac{5}{6}}(f) = 2t^{\frac{7}{6}}$. Hence, $T_{\frac{1}{2}} T_{\frac{1}{3}}(f) \neq T_{\frac{1}{2}+\frac{1}{3}}(f)$. See explanation in ([21], page 6).

The following result follows from Proposition 2.1.

Theorem 2.4. *Let $f : (0, +\infty) \to \mathbb{R}$ be a given function. Then the following assertions are equivalent:*
(a) f is a differentiable;
(b) f is a $\alpha-$differentiable for some $\alpha \in (0, 1)$.

Proof. (a) implies (b) because according to Proposition 2.1, we have

$$
T_\alpha(f)(t) = f^{(\alpha)}(t) = t^{1-\alpha} \frac{df(t)}{dt}.
$$

Conversely, if (b) holds, then $\frac{df(t)}{dt} = t^{\alpha-1} f^{(\alpha)}(t)$, that is, (a) holds. Hence the proof of Theorem 2.4 is finished. $\square$

The following result is an immediate consequence of Proposition 2.1 and Theorem 2.4.

Theorem 2.5. *(Rolle's Theorem for Conformable Differentiable Functions). Let $a > 0$ and $f : [a, b] \to \mathbb{R}$ be a given function that satisfies*
(i) f is continuous on $[a, b]$.
(ii) f is $\alpha-$differentiable on (a, b) for some $\alpha \in (0, 1)$,
(iii) $f(a) = f(b)$.
Then, there exists $c \in (a, b)$, such that $f^{(\alpha)}(c) = 0$.

Proof. As f is continuous $[a,b]$ and f is $\alpha-$differentiable for some $\alpha \in (0, 1)$, so f is differentiable on (a, b). Then by classical Rolle's theorem there exists $c \in (a, b)$ such that $\frac{df}{dt}(c) = f'(c) = 0$. Now, using Proposition 2.1 $\left(T_\alpha(f)(t) = f^{(\alpha)}(t) = t^{1-\alpha} \frac{df(t)}{dt}\right)$, we obtain that

$$
T_\alpha(f)(c) = f^{(\alpha)}(c) = 0.
$$

Hence the proof of Rolle's Theorem for Conformable Differentiable functions is complete. $\square$

The subsequent two results are immediate consequences of Proposition 2.1 and Theorem 2.4-2.5.

Theorem 2.6. *(Cauchy Theorem for Conformable Differentiable Functions). Let $a > 0$ and $f, g : [a, b] \to \mathbb{R}$ be given functions that satisfy*
(i) f, g are continuous on $[a, b]$,
(ii) f, g are $\alpha-$differentiable for some $\alpha \in (0, 1)$ and $g^{(\alpha)}(x) \neq 0$ for all $x \in (a, b)$,
Then, there exists $c \in (a, b)$, such that $\frac{f^{(\alpha)}(c)}{g^{(\alpha)}(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$.

Proof. Consider the function

$$F\left(x\right) = f\left(x\right) - f\left(a\right) - \frac{f\left(b\right) - f\left(a\right)}{g\left(b\right) - g\left(a\right)}\left(g\left(x\right) - g\left(a\right)\right).$$

It is clear that the function $F\left(x\right)$ satisfies the conditions of Rolle's theorem for Conformable Differentiable functions equivalent to classical differentiable functions on $\left(a, b\right)$. Hence, according to Theorem 2.5, there exists $c \in \left(a, b\right)$, such that $F'\left(c\right) = 0$, that is,

$$\frac{f'\left(c\right)}{g'\left(c\right)} = \frac{f\left(b\right) - f\left(a\right)}{g\left(b\right) - g\left(a\right)}.$$

Since $\frac{f'(c)}{g'(c)} = \frac{c^{\alpha-1}f^{(\alpha)}(c)}{c^{\alpha-1}g^{(\alpha)}(c)} = \frac{f^{(\alpha)}(c)}{g^{(\alpha)}(c)}$. Hence the result is obtained. $\qquad\square$

Theorem 2.7. *(Mean Value Theorem for Conformable Differentiable Functions) Let $a > 0$ and $f : [a, b] \to \mathbb{R}$ be a given function that satisfies*
   (i) f is continuous on $[a, b]$,
   (ii) f is $\alpha-$differentiable on $\left(a, b\right)$ for some $\alpha \in \left(0, 1\right)$.
   Then, there exists $c \in \left(a, b\right)$, such that $f^{(\alpha)}\left(c\right) = c^{1-\alpha} \cdot \frac{f(b)-f(a)}{b-a}$.

Proof. Putting $g\left(x\right) = x$ in Theorem 2.6, the result follows. $\qquad\square$

Finally, we have the following result:

Theorem 2.8. *(Darboux's Theorem for Conformable Differentiable Functions) Let $b > a > 0$ and $f : \left(0, +\infty\right) \to \mathbb{R}$ be a given function that satisfies*
   (i) f is $\alpha-$differentiable for some $\alpha \in \left(0, 1\right)$.
   (ii) $f^{(\alpha)}\left(a\right) \cdot f^{(\alpha)}\left(b\right) < 0$.
   Then, there exists $c \in \left(a, b\right)$, such that $f^{(\alpha)}\left(c\right) = 0$.

Proof. According to Theorem 2.4, the function f is differentiable on $\left(0, +\infty\right)$. Further, from (ii) as well as by Proposition 2.1, we have that

$$a^{1-\alpha}b^{1-\alpha}f'\left(a\right)f'\left(b\right) < 0,$$

that is, $f'\left(a\right)f'\left(b\right) < 0$. Now, using classical Darboux's theorem for differentiable functions, we obtain that there exists $c \in \left(a, b\right)$ such that $f'\left(c\right) = 0$, that is, $f^{(\alpha)}\left(c\right) = 0$. Hence the proof is complete. $\qquad\square$

3. Applications

Khalil et al. [12] consider the following examples of differential equations with the conformable derivatives. But our approach for the same is different and we have used the Proposition 2.1.

$$y^{\left(\frac{1}{2}\right)} + y = x^2 + 2x^{\frac{3}{2}}, y\left(0\right) = 0; \ y^{(\alpha)} + y = 0, 0 < \alpha \leq 1;$$

$$y^{\left(\frac{1}{2}\right)} + \sqrt{x}y = xe^{-x} \text{ and } y^{\left(\frac{1}{2}\right)} = \frac{x^{\frac{3}{2}} + y\sqrt{x}}{2x + 3y}.$$

However, using Proposition 2.1 our approach is the following:

$$x^{1-\frac{1}{2}}y' + y = x^2 + 2x^{\frac{3}{2}} \text{if and only if } y' + \frac{1}{\sqrt{x}}y = x^{\frac{3}{2}} + 2x, x > 0,$$

$$x^{1-\alpha}y' + y = 0 \text{ if and only if } y' + x^{\alpha-1}y = 0, x > 0,$$

$$x^{1-\frac{1}{2}}y' + \sqrt{x}y = xe^{-x} \text{ if and only if } y' + y = \sqrt{x}e^{-x}, x > 0$$

and

$$x^{1-\frac{1}{2}}y' = \frac{x^{\frac{3}{2}} + y\sqrt{x}}{2x + 3y} \text{ if and only if } y' = \frac{x + y}{2x + 3y}.$$

Hence, for $x > 0$ all equations with conformable derivatives are equivalent to usual (very well known types) differential equations. Difficulties are possible in the point $x = 0$. With these remarks we improve complete all results from the paper [12].

Remark 3.1. For other similar results see recent paper [11]. It is not hard to check that Definition 2.1 of [11] is equivalent to the Definition 2.1 of [12].

Further, it is clear that (6) of Theorem 2.3 in [11] is wrong. Also, for the new approach of some things from the conformable calculus see recent paper [4].

4. Competing interests

The author declares that they have no competing interests.

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