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Random semilinear system of differential equations with state-dependent delay

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Abstract

In this paper we prove the existence of mild solutions for a first-order semilinear differential with state-dependent delay. The existence results are established by means of a new version of Perov's fixed point principles combined with a technique based on vector-valued matrix and convergent to zero matrix.

Keywords: retarded functional differential equations, random variable, state-dependent delay, fixed point theorem.
2010 MSC: 47H10, 47H30, 34G20, 34K20, 60H20.

1. Introduction

Random ordinary differential equations (RODEs) are ordinary differential equations (ODEs) that include a stochastic process in their vector field. They seem to have had a shadow existence to stochastic differential equations (SODEs), but have been around for as long as if not longer and have many important applications. In particular, RODEs play a fundamental role in the theory of random dynamical systems, it is more realistic to consider such equations as random operator equations. Therefore, it is more realistic to consider such equations as random operator equations which are much more difficult to handle mathematically than deterministic equations. Important contributions to the study of the mathematical aspects of such random equations have been undertaken in [9, 7, 1, 16] among others. Since sometimes we can get the random distributions of some main disturbances by historical experiences and data rather than take all random disturbances into account and assume the noise to be white noises. In a separable metric space, random

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fixed point theorems for contraction mappings were proved by Hanš [2, 3], Špaček [8], Hanš and, Špaček [4] and Mukherjee [5, 6].
In this work we prove the existence of mild solutions of the following functional differential equation with delay and random effects (random parameters) of the form:

$$\begin{cases} x'(t,\omega) &= A_1(w)x(t,\omega) \\ &+ f^1(t, x_{\rho_1(t,x_t)}(\cdot,\omega), y_{\rho_1(t,y_t)}(\cdot,\omega),\omega), \quad \text{a.e, } t \in J := [0,a] \\ y'(t,\omega) &= A_2(w)y(t,\omega) \\ &+ f^2(t, x_{\rho_2(t,x_t)}(\cdot,\omega), y_{\rho_2(t,y_t)}(\cdot,\omega),\omega) \quad \text{a.e, } t \in J := [0,a] \\ x(t,\omega) &= \phi_1(t,\omega), \ t \in (-\infty, 0] \\ y(t,\omega) &= \phi_2(t,\omega), \end{cases} \tag{1.1}$$

Here, $x(\cdot), y(\cdot)$ takes the value in the separable Hilbert space X with inner product $\langle \cdot, \cdot \rangle$ induced by the norm $\|\cdot\|$, $A_i : \Omega \times X \longrightarrow X$, $i = 1, 2$ are random operators and $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, $w \in \Omega$, $J := [0, a]$ for fixed $a > 0$ and X is a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ induced by norm $\|\cdot\|$, ϕ_1, ϕ_2 are two random maps and $f^1, f^2 : J \times \mathcal{B} \times \mathcal{B} \times \Omega \longrightarrow X$ and $\rho_1, \rho_1 : J \times \Omega \to \mathbb{R}$, $\mathcal{B}$ is a phase space to be specified later. For any function x defined on $(-\infty, a] \times \Omega$ and any $t \in J$ we denote by $x_t(., w)$ the element of $\mathcal{B} \times \Omega$ defined by $x_t(\theta, w) = x(t + \theta, w)$, $\theta \in (-\infty, 0]$. Here $x_t(., w)$ represents the history of the state from time $-\infty$, up to the present time t. We assume that the histories $x_t(., w)$ belong to the abstract phase $\mathcal{B}$. To our knowledge, the literature on the local existence of random evolution equations with delay is very limited, so the present paper can be considered as a contribution to this question. We refer the reader to [11, 17] for the properties of the first order abstract Cauchy problem and the semigroup theory.

The paper is organized as follows. In Section Íš,we introduce all the background material needed such as generalized metric spaces, some random fixed point theorems . In Section Íş, by some new random versions of Perov's fixed point theorems in a vector Banach space.

2. Preliminaries

In this section, we introduce some notations, recall some definitions, and preliminary facts which are used throughout this paper. Actually we will borrow it from [20, 10]. Although we could simply refer to this paper whenever we need it, we prefer to include this summary in order to make our paper as much self-contained as possible.
Let $(\Omega, \mathcal{F})$ be a measurable space. We equip the metric space X with a σ-algebra $\mathcal{B}(X)$ of Borel subsets of X so that $(X, \mathcal{B}(X))$ becomes a measurable space. A mapping $z : \Omega \to X$ is called a random variable if

$$z^{-1}(B) = \{w \in \Omega : z(w) \in B\} \in \mathcal{F},$$

for all Borel sets $B \in \mathcal{B}(X)$

Definition 2.1. Let X, Y is a real separable Hilbert space, a mapping $A : \Omega \times X \to Y$ is called a random operator if $w \longmapsto A(w, z)$ is measurable for all $z \in X$. We also denote a random operator A on X by

$$A(z)(w) = A(w, z), \ w \in \Omega, \ \ z \in X.$$

Definition 2.2. A random fixed point of A is a measurable function $z : \Omega \to X$ such that

$$z(w) = A(w, z(w)) \quad \text{for all} \quad w \in \Omega.$$

Definition 2.3. Let $A : \Omega \times X \to Y$ be a random operator.

- A is called continuous on X if $A(w, \cdot)$ is continuous for each $w \in \Omega$,

- A is is called compact if for every bounded subset C of X, $A(w, C)$ is a relatively compact subset of Y for each $w \in \Omega$.

Definition 2.4. Let $g : [, b] \times X \times \Omega \to Y$ is called random Carathéodory if the following conditions are satisfied:

(i) The map $(t, w) \longmapsto g(t, z, w)$ is jointly measurable for all $z \in X$,

(ii) The map $z \longmapsto g(t, z, w)$ is continuous for all $t \in [0, b]$ and $w \in \Omega$.

In this paper, we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato in [12] and follow the terminology used in [13]. Thus, $(\mathcal{B}, \|.\|_{\mathcal{B}})$ will be a semi norm linear space of functions mapping $(-\infty, 0]$ into X, and satisfying the following axioms :

- A_1 If $x : (-\infty, \sigma + a) \to X$, $a > 0$, is continuous on $[\sigma, \sigma + a]$ and $x_\sigma \in \mathcal{B}$ then for every t in $[\sigma, \sigma + a)$ the following conditions hold:

  (i) $x_t \in \mathcal{B}$.

  (ii) $\|x\| \leq H\|x_t\|_{\mathcal{B}}$.

  (iii) $\|x_t\|_{\mathcal{B}} \leq K(t - s) \sup\{\|x(s)\|, \ \sigma \leq s \leq t\} + M(t - \sigma)\|x_\sigma\|_{\mathcal{B}}$

  where $H \geq 0$ is a constant, $\mathbb{R}^+ \to \mathbb{R}^+$, K is continuous and M is locally bounded and H, K and M are independent of x.

- A_2 For the function x in A_1, x_t is a $\mathcal{B}$-valued continuous functions on $[\sigma, \sigma + a]$.

- A_3 The space $\mathcal{B}$ is complete

Remark 2.5. 1. (ii) is equivalent to $\|\phi(0)\| \leq H\|\phi\|_{\mathcal{B}}$ for every $\phi \in \mathcal{B}$

2. Since $\|.\|_{\mathcal{B}}$ is a seminorm, two elements $\phi, \psi \in \mathcal{B}$ can verify $\|\phi - \psi\|_{\mathcal{B}} = 0$ without necessarily $\phi(\theta) = \psi(\theta)$ for all $\theta \leq 0$.
3. From the equivalence of in the first remark, we can see that for all $\phi, \psi \in \mathcal{B}$ such that $\|\phi - \psi\|_{\mathcal{B}} = 0$. We necessarily have that $\phi(0) = \psi(0)$.

$(C2)$ If a uniformly bounded sequence $(\phi_n)_n$ in $\mathcal{B}$ converges to a function ϕ in the compact-open topology, then ϕ belongs to $\mathcal{B}$ and $\|\phi_n - \phi\|_{\mathcal{B}} \to 0$, as $n \to \infty$

Remark 2.6. Let $S(t) : \mathcal{B} \to \mathcal{B}$ be the C_0-semigroup defined by $S(t)\phi(\theta) = \phi(0)$ for $\theta \in [-t, 0]$ and $S(t)\phi(\theta) = \phi(t + \theta)$. Let $\mathcal{B}_0 = \{\phi \in \mathcal{B} : \phi(0) = 0\}$. We denote by $S_0(t)$ the restriction of $S(t)$ to $\mathcal{B}_0$.

- (FMS) The space $\mathcal{B}$ is said to be a fading memory space if it verifies axiom (C2) and $S_0\phi(0) \to 0$ as $t \to \infty$ for all $\phi \in \mathcal{B}_0$.

- (UFMS) The space $\mathcal{B}$ is said to be a uniformly fading memory space if it verifies (C2) and $\|S_0(t)\|_{\mathcal{B}} \to 0$ as $t \to \infty$.

We now indicate some examples of phase spaces. For other details we refer, for instance to the book by Hino*et al.* [13].

Example 2.7. Let: C_b the space of bounded continuous functions defined from $(-\infty, 0]$ to X, C_{bu} the space of bounded uniformly continuous functions defined from $(-\infty, 0]$ to X,

$$C^\infty = \left\{\phi \in C_b : \lim_{\theta \to -\infty} \phi(\theta) \text{ exist in } X\right\};$$

$$C^0 = \Big\{\phi \in C_b \,:\, \lim_{\theta\to-\infty}\phi(\theta) = 0\Big\},$$

endowed with the uniform norm

$$\|\phi\| = \sup\{|\phi|, \theta \le 0\}.$$

We have that the spaces C_{bu}, C^∞ and C^0 satisfy conditions $(A_1) - (A_3)$. However, C_b satisfies (A_1), (A_3) but (A_2) is not satisfied.

Example 2.8. Phase space $C_g^0(X)$. Let $\mathcal{B} = C_g^0(X)$ be the space consisting of continuous functions $\phi : (-\infty, 0] \to X$ such that $\lim_{\theta\to-\infty}\dfrac{\|\phi(\theta)\|}{g(\theta)} = 0$,

where $g : (-\infty, 0] \to (0, +\infty)$ is a continuous function that satisfies conditions (g_1) and (g_2) in the terminology of [13]. This means that

(g_1) The function $G(t) = \frac{g(t+\theta)}{g(\theta)}$ is locally bounded for $t \ge 0$.

(g_2) $g(\theta) \to \infty$ as $\theta \to -\infty$.

The norm in $\mathcal{B}$ is defined by

$$\|\phi\|_{\mathcal{B}} = \sup_{\theta\le 0}\frac{\|\phi(\theta)\|}{g(\theta)},\ \ \phi \in \mathcal{B}$$

The space $\mathcal{B}$ is a phase space ([13], Theorem 1.3.2). If G is bounded, then $\mathcal{B}$ verifies (FMS) and, if $G(t) \to 0$ as $t \to \infty$, then $\mathcal{B}$ verifies (UFMS) ([13], Example 7.1.7). To simplify some estimate, in this text we always assume that g is decreasing and $g(0) = 1$.

Example 2.9. For any real positive constant γ, we define the functional space C_γ by

$$C_\gamma := \Big\{\phi \in C((-\infty, 0), X) : \lim_{\theta\to-\infty} e^{\gamma\theta}\phi(\phi) \text{ exists in } X\Big\},$$

endowed with the following norm

$$\|\phi\| = \sup\{e^{\gamma\theta}|\phi(\theta)| :\ \theta \le 0\}.$$

Then in the space C_γ the axioms $(A_1) - (A_3)$ are satisfied.

3. Vector metric space and Random variable

If, $x, y \in \mathbb{R}^n$, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, by $x \le y$ we mean $x_i \le y_i$ for all $i = 1, \ldots, n$. Also $|x| = (|x_1|, \ldots, |x_n|)$, $\max(x, y) = \max(\max(x_1, y_1), \ldots, \max(x_n, y_n))$ and $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n :\ \ x_i > 0\}$. If $c \in \mathbb{R}$, then $x \le c$ means $x_i \le c$ for each $i = 1, \ldots, n$.

Definition 3.1. Let X be a nonempty set. By a vector-valued metric on X we mean a map $d : X \times X \to \mathbb{R}^n_+$ with the following properties:

- $d(u, v) \ge 0$ for all $u, v \in X$; if $d(u, v)$ then $u = v$;
- $d(u, v) = d(v, u)$ for all $u, v \in X$;
- $d(u, v) \le d(u, w) + d(w, v)$ for all $u, v, w \in X$.

We call the pair (X, d) a generalized metric space with $d(x, y) := \begin{pmatrix} d_1(x, y) \\ \cdots \\ d_n(x, y) \end{pmatrix}$.

Notice that d is a generalized metric space on X if and only if $d_i,\ i = 1, \ldots, n$ are metrics on X.

For $r = (r_1, \dots, r_n) \in \mathbb{R}^n_+$, we will denote by

$$B(x_0, r) = \{x \in X : d(x_0, x) < r\},$$

the open ball centered in x_0 with radius r and

$$\overline{B(x_0, r)} = \{x \in X : d(x_0, x) \leq r\},$$

the closed ball centered in x_0 with radius r. We mention that for generalized metric space, the notation of open subset, closed set, convergence, Cauchy sequence and completeness are similar to those in usual metric spaces.

Definition 3.2. A square matrix of real numbers is said to be convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1. In other words, this means that all the eigenvalues of M are in the open unit disc i.e. $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $det(M - \lambda I) = 0$, where I denote the unit matrix of $\mathcal{M}_{n\times n}(\mathbb{R})$.

Theorem 3.3. *[18] Let $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$. The following assertions are equivalent:*

- *M is convergent towards zero;*
- *$M^k \to 0$ as $k \to \infty$;*
- *The matrix $(I - M)$ is nonsingular and*
$$(I - M)^{-1} = I + M + M^2 + \ldots + M^k + \ldots,$$
- *The matrix $(I - M)$ is nonsingular and $(I - M)^{-1}$ has nonnegative elements.*

Some examples of matrices convergent to zero are the following:

- $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, where $a, b \in \mathbb{R}_+$ and $\max(a, b) < 1$
- $A = \begin{pmatrix} a & -c \\ 0 & b \end{pmatrix}$, where $a, b, c \in \mathbb{R}_+$ and $a + b < 1,\ c < 1$
- $A = \begin{pmatrix} a & -a \\ b & -b \end{pmatrix}$, where $a, b, c \in \mathbb{R}_+$ and $|a - b| < 1,\ a > 1, b > 0$.

Definition 3.4. Let (X, d) be a generalized metric space. An operator $N : X \to X$ is said to be contractive if there exists a convergent to zero matrix M such that

$$d(N(x), N(y)) \leq Md(x, y) \text{ for all } x, y \in X.$$

For $n = 1$ we recover the classical Banach's contraction fixed point result.

We shall use a random version of Perov type of random differential equations of first order for different aspects of the solutions under suitable conditions

Theorem 3.5. *[19] Let $(\Omega, \mathcal{F})$ be a measurable space, X be a real separable generalized Banach space and $F : \Omega \times X \to X$ be a continuous random operator, and let $M(w) \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ be a random variable matrix such that, for every $w \in \Omega$, the matrix $M(w)$ converges to 0 and*

$$d(F(w, x_1), F(w, x_1)) \leq M(w)d(x_1, x_2), \text{ for each } x_1, x_2 \in X, w \in \Omega.$$

Then there exists any random variable $x : \Omega \to X$ which is the unique random fixed point of F.

Lemma 3.6. *[19] Let X be a separable generalized metric space and $F : \Omega \times X \to X$ be a mapping such that $F(., x)$ is measurable for all $x \in X$ and $F(w, .)$ is continuous for all $w \in \Omega$. Then the map $(w, x) \to F(w, x)$ is jointly measurable.*

Proposition 3.7. *[15] Let X be a separable Banach space, and D be a dense linear subspace of X. Let $L : \Omega \times D \to X$ be a closed linear random operator such that, for each $w \in \Omega$, $L(w)$ is one to one and onto. Then the operator $R : \Omega \times X \to X$ defined by $R(w)x = L^{-1}(w)x$ is random.*

4. Main Results

Now we give our main existence result for problem (1.1). Before starting and proving this result, we give the definition of the mild random solution.

Definition 4.1. A stochastic process $x, y : J \times \Omega \to X$ is said to be random mild solution of problem (1.1) if $(x(t,w)y(t,w)) = (\phi_1(t,\omega), \phi_2(t,\omega))$,$t \in (-\infty, 0]$ and the restriction of $(x(.,w), y(.,w))$ to the interval J is continuous and satisfies the following integral equation:

$$\begin{cases} x(t,w) &= S_1(w,t)\phi_1(t,\omega) + \displaystyle\int_0^t S_1(t-s) f^1(s, x_{\rho_1(s,x_s)}(\cdot,\omega), y_{\rho_1(s,y_s)}(\cdot,\omega), \omega)ds, \quad t \in J \\ y(t,w) &= S_2(w,t)\phi_2(t,\omega) + \displaystyle\int_0^t S_2(t-s) f^2(s, x_{\rho_2(s,x_s)}(\cdot,\omega), y_{\rho_2(s,y_s)}(\cdot,\omega), \omega)ds, \quad t \in J \end{cases}$$

where $\{S_1(w,t), S_2(w,t)\}$ are random C_0-Îşemigroups of bounded linear operators on X with infinitesimal generators A_1,A_2, respectively.

We will need to introduce the following hypotheses which are assumed there after:

There exist random variables $M_1, M_2 : \Omega \to (0, +\infty)$ such that.

$$\|S_i(w,t)\| \leq M(w), \ 0 \leq t \leq a \ \text{ for each } i = 1,2 \ , w \in \Omega. \tag{4.1}$$

Moreover, to abbreviate the writing, we set $K_a = \sup_{t\in[0,a]} K(t)$ and $M_a = \sup_{t\in[0,a]} M(t)$, $M = M(w)$

We will need to introduce the following hypotheses which are assumed there after:

(H_1) (1) For every $\psi_1, \psi_2 \in \mathcal{B}$ the function $f^i(., \psi_1, \psi_1) : J \to X, t \longmapsto f^i(t, \psi_1, \psi_1)$ is strongly measurable, and the function $f^i(., 0, 0)$ is integrable on I.

(2) There exists a constant $L_{f_i}, \bar{L}_{f_i} : \Omega \to \mathbb{R}^+$ such that

$$\|f^i(t, \varphi_1, \varphi_2, w) - f^i(t, \bar{\varphi}_1, \bar{\varphi}_2, w)\| \leq L_{f_i}(w)\|\varphi_1 - \bar{\varphi}_1\| + \bar{L}_{f_i}(w)\|\varphi_2 - \bar{\varphi}_2\|,$$

where

$$L_f(w) = \max\{L_{f_i}(w), \bar{L}_{f_i}(w)\}, \ i = 1,2$$

(H_2) The function $\rho_i : J \times \mathcal{B} \to [0, +\infty)$ satisfies :

(1) For every ψ, the function $t \longmapsto \rho(t, \psi)$ is continuous

(2) There exists a constant $L_\rho > 0$ such that

$$\|\rho_i(t, \psi) - \rho_i(t, \bar{\psi})\| \leq L_\rho \|\psi - \bar{\psi}\|$$

for all $(\psi, \bar{\psi}) \in \mathcal{B} \times \mathcal{B}, \ t \in [0, a]$

(H_3) For every $r > 0$, there exists a constant $L_2(r, .) : \Omega \to \mathbb{R}^+$ such that

$$\|f^i(t, x_{t_2}, y_{t_2}, w) - f^i(t, x_{t_1}, y_{t_1}, w)\| \leq L_2(r, w)|t_2 - t_1|.$$

For each $i = 1, 2, \ w \in \Omega$, $\phi_i(., w)$ is continuous and for each t, $\phi_i(t, .)$ is measurable and

$$(\sup_{s\in[0,r]} \|x(s)\|, \sup_{s\in[0,r]} \|x(s)\|) \leq (r, r).$$

Theorem 4.2. *Assume that conditions* $(H_1)-(H_3)$ *are satisfied and the matrix*

$$M_{trice} = \begin{pmatrix} \lambda_1(w) & \lambda_1(w) \\ \lambda_1(w) & \lambda_1(w) \end{pmatrix}, \lambda_1(w) \geq 0.$$

where

$$\lambda_1(w) = MK_a t(L_f(w) + L_2(r,w)L_\rho),$$

If M_{trice} *converges to zero. Then problem* (1.1) *has at least one mild random solution on* $(-\infty, a]$.

Proof. We can choose a constant $p(w), q(w) > 0$ such that

$$M(H\|\phi_1\|_{\mathcal{B}} + L_f(w)M_a a(\|\phi_1\|_{\mathcal{B}} + |\phi_2\|_{\mathcal{B}}) + ML_f(w)K_a a(p(w)+q(w)) \\ + M\int_0^t \|f^1(s,0,0,\omega)\|ds \leq p(w) \tag{4.2}$$

and

$$M(H\|\phi_2\|_{\mathcal{B}} + L_f(w)M_a a(\|\phi_1\|_{\mathcal{B}} + |\phi_2\|_{\mathcal{B}}) + ML_f(w)K_a a(p(w)+q(w)) \\ + M\int_0^t \|f^2(s,0,0,\omega)\|ds \leq q(w). \tag{4.3}$$

Let $Y = \{x, y \in C(J,X) : (x(0,w), y(0,w)) = (\phi_1(0,w), \phi_2(0,w)) = (0,0)\}$ endowed with the uniform convergence topology. Consider the operator $N : \Omega \times Y \times Y \to Y \times Y$ be the random operator defined by

$$(x,y) \longmapsto (N_1(w,x,y), N_2(w,x,y)),$$

where

$$N_1(x(t,w), y(t,w), w) = S_1(t,w)\phi_1(0,w) \\ + \int_0^t S_1(t-s,w)f^1(s, x_{\rho_1(s,x_s)}(s,\omega), y_{\rho_1(s,y_s)}(s,\omega), \omega)ds, \quad t \in J$$

and

$$N_2(x(t,w), y(t,w), w) = S_2(t,w)\phi_2(0,w) \\ + \int_0^t S_2(t-s,w)f^2(s, x_{\rho_1(s,x_s)}(s,\omega), y_{\rho_1(s,y_s)}(s,\omega), \omega)ds, \quad t \in J.$$

First we show that N is a random operator on $Y \times Y$. Since f^1 and f^2 are Caratheodory functions, then $w \longmapsto f^1(t,x,y,w)$ and $w \longmapsto f^2(t,x,y,w)$ are measurable maps in view of Lemma 3.6. By the Crandall-Liggett formula, we have

$$S_i(w,t) = \lim_{n\to\infty}\left(I - \frac{t}{n}A_i(w)\right)^{-n} x, \qquad i = 1,2.$$

From Proposition 3.7, we know that $w \to \left(I - \frac{t}{n}A_i(w)\right)^{-n} x$ are measurable operators, thus $w \to S_i(w,t)$ are measurable. Using the continuity properties of the semigroups $S_1(w,.), S_2(w,.)$, we get

$$w \to S_i(t,w)\phi_i(w) \quad \text{and} \quad (s,w) \to S_i(t-s,w)f^i(s, x_{\rho_1(s,x_s)}, y_{\rho_1(s,y_s)}, \omega)$$

are measurable. Further, the integral is a limit of a finite sum of measurable functions; therefore, the maps

$$w \longmapsto N_1(x(t,w), y(t,w), w), \qquad w \longmapsto N_2(x(t,w), y(t,w), w)$$

are measurable. As a result, N is a random operator on $Y \times Y \times \Omega$ into $Y \times Y$.Let $B_p, B_q : \Omega \to 2^Y$ be defined by:

$$B_p(w) \times B_q(w) = \left\{(x,y) \in Y \times Y : \left\| \begin{pmatrix} x(t,w) \\ y(t,w) \end{pmatrix} \right\| \leq \begin{pmatrix} p(w) \\ q(w) \end{pmatrix} \right\}$$

where

$$\left\| \begin{pmatrix} x(t,w) \\ y(t,w) \end{pmatrix} \right\| = \begin{pmatrix} \|x(t,w)\| \\ \|y(t,w)\| \end{pmatrix}.$$

The set $B_p(w) \times B_q(w)$ bounded, closed, convex and solid for all $w \in \Omega$. Then $B_p(w) \times B_q(w)$ is measurable. Let $w \in \Omega$ be fixed.

Step 1.- We show initially that $N(B_p(w) \times B_q(w)) \subseteq B_p(w) \times B_q(w)$. In fact, for $(x,y) \in B_p(w) \times B_q(w)$, using (4.2) and (4.3), we can estimate we can estimate

$$\begin{aligned}
&\|N_1(x(t,w),y(t,w),w)\| \\
&\quad \leq MH\|\phi_1\|_{\mathcal{B}} + M\int_0^t \|f^1(s,x_{\rho_1(s,x_s)},y_{\rho_1(s,y_s)},\omega)\|ds \\
&\leq MH\|\phi_1\|_{\mathcal{B}} + M\int_0^t \|f^1(s,x_{\rho_1(s,x_s)},y_{\rho_1(s,y_s)},\omega) - f^1(s,0,0,\omega)\|ds \\
&\quad + M\int_0^t \|f^1(s,0,0,\omega)\|ds \\
&\leq MH\|\phi_1\|_{\mathcal{B}} + ML_{f_1}(w)\int_0^t \|x_{\rho_1(s,x_s)}\|_{\mathcal{B}}ds + L_{f_2}(w)\int_0^t \|y_{\rho_1(s,y_s)}\|_{\mathcal{B}}ds \\
&\quad \leq MH\|\phi_1\|_{\mathcal{B}} + ML_{f_1}(w)\int_0^t \Big(K_a \sup_{0\leq\tau\leq\rho_1(s,x_s)} \|x(\tau)\| + M_a\|\phi_1\|_{\mathcal{B}}\Big)ds \\
&+ML_{f_2}(w)\int_0^t \Big(K_a \sup_{0\leq\tau\leq\rho_1(s,y_s)} \|y(\tau)\| + M_a\|\phi_2\|_{\mathcal{B}}\Big)ds + M\int_0^t \|f^1(s,0,0,\omega)\|ds \\
&\quad \leq MH\|\phi_1\|_{\mathcal{B}} + ML_f(w)K_a a p(w) + ML_f(w)M_a a\|\phi_1\|_{\mathcal{B}} \\
&+ML_f(w)K_a a q(w) + ML_f(w)aM_a\|\phi_2\|_{\mathcal{B}} + M\int_0^t \|f^1(s,0,0,\omega)\|ds. \\
&\quad \leq p(w).
\end{aligned}$$

Similarly we have

$$\begin{aligned}
\|N_2(x(t,w),y(t,w),w)\| &\leq MH\|\phi_2\|_{\mathcal{B}} + ML_f(w)K_a a p(w) + ML_f(w)M_a a\|\phi_2\|_{\mathcal{B}} \\
&\quad +ML_f(w)K_a a q(w) + ML_f(w)aM_a\|\phi_2\|_{\mathcal{B}} + M\int_0^t \|f^2(s,0,0,\omega)\|ds \\
&\quad \leq q(w).
\end{aligned}$$

Step 2.- we show that N is Lipschitz continuous. Let $(x,,y),(\bar{x},\bar{y}) \in B_p(w) \times B_q(w)$. Using conditions

$(H_1) - (H_3)$, we have

$$\begin{aligned}
&\|N_1(x(t,w),y(t,w),w) - N_1(\bar{x}(t,w),\bar{y}(t,w),w)\| \\
&\qquad \le M\int_0^t \|f^1(s,x_{\rho_1(s,x_s)},y_{\rho_1(s,y_s)},\omega) - f^1(s,\bar{x}_{\rho_1(s,\bar{x}_s)},\bar{y}_{\rho_1(s,\bar{y}_s)},\omega)\|ds \\
&\le M\int_0^t \|f^1(s,x_{\rho_1(s,x_s)},y_{\rho_1(s,y_s)},\omega) - f^1(s,\bar{x}_{\rho_1(s,x_s)},\bar{y}_{\rho_1(s,y_s)},\omega)\|ds \\
&\qquad +M\int_0^t \|f^1(s,\bar{x}_{\rho_1(s,x_s)},\bar{y}_{\rho_1(s,y_s)},\omega) - f^1(s,\bar{x}_{\rho_1(s,\bar{x}_s)},\bar{y}_{\rho_1(s,\bar{y}_s)},\omega)\|ds \\
&\le ML_f(w)\int_0^t \|x_{\rho_1(s,x_s)} - \bar{x}_{\rho_1(s,x_s)}\|_{\mathcal{B}}ds + ML_f(w)\int_0^t \|y_{\rho_1(s,y_s)} - \bar{y}_{\rho_1(s,y_s)}\|_{\mathcal{B}}ds \\
&\qquad +ML_2(r,w)\int_0^t |\rho_1(s,x_s) - \rho_1(s,\bar{x}_s)|ds + ML_2(r,w)\int_0^t |\rho_1(s,y_s) - \rho_1(s,\bar{y}_s)|ds \\
&\le ML_f(w)K_a\int_0^t \sup_{0\le\tau\le\rho_1(s,x_s)} \|x_(\tau) - \bar{x}(\tau)\|ds + ML_f(w)K_a\int_0^t \sup_{0\le\tau\le\rho_1(s,y_s)} \|y_(\tau) - \bar{y}(\tau)\|ds \\
&\qquad +ML_2(r,w)L_\rho\int_0^t \|x_s - \bar{x}_s\|_{\mathcal{B}}ds + ML_2(r,w)L_\rho\int_0^t \|y_s - \bar{y}_s\|_{\mathcal{B}}ds \\
&\le ML_{f_1}(w)K_a\int_0^t \sup_{0\le\tau\le s} \|x_(\tau) - \bar{x}(\tau)\|ds + ML_f(w)K_a\int_0^t \sup_{0\le\tau\le s} \|y_(\tau) - \bar{y}(\tau)\|ds \\
&\qquad +ML_2(r,w)L_\rho\int_0^t \|x_s - \bar{x}_s\|_{\mathcal{B}}ds + ML_2(r,w)L_\rho\int_0^t \|y_s - \bar{y}_s\|_{\mathcal{B}}ds \\
&\le ML_f(w)K_at\sup_{0\le s\le t}\|x(s) - \bar{x}(s)\|ds + ML_f(w)K_at\sup_{0\le s\le t}\|y(s) - \bar{y}(s)\| \\
&\qquad +ML_2(r,w)L_\rho K_at\sup_{0\le s\le t}\|y(s) - \bar{y}(s)\| + ML_2(r,w)L_\rho K_at\sup_{0\le s\le t}\|y(s) - \bar{y}(s)\| \\
&\le \lambda_1(w)\sup_{0\le s\le t}\|x(s) - \bar{x}(s)\| + \lambda_1(w)\sup_{0\le s\le t}\|y(s) - \bar{y}(s)\|.
\end{aligned}$$

and

$$\begin{aligned}
\|N_2(x(t,w),y(t,w),w) - N_2(\bar{x}(t,w),\bar{y}(t,w),w)\| &\le \lambda_1(w)\sup_{0\le s\le t}\|x(s,w) - \bar{x}(s,w)\| \\
&\qquad +\lambda_1(w)\sup_{0\le s\le t}\|y(s) - \bar{y}(s)\|\Big),
\end{aligned}$$

for all $0 \le t \le a$. Consequently,

$$\begin{aligned}
\|N(x,y,w) - N(\overline{x},\overline{y},w)\|_\infty &= \begin{pmatrix}\|N_1((x,y,w) - N_1(\overline{x},\overline{y},w)\|_\infty \\ \|N_2(x,y,w) - N_2(\overline{x},\overline{y},w)\|_\infty\end{pmatrix} \\
&\le \lambda_1(w)\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}\|x(\cdot,w) - \overline{x}(\cdot,w)\|_\infty \\ \|y(\cdot,w) - \overline{y}(\cdot,w)\|_\infty\end{pmatrix}.
\end{aligned}$$

Therefore

$$\|N(x,y,w) - N(\overline{x},\overline{y},w)\|_\infty \le M_{trice}\begin{pmatrix}\|x(\cdot,w) - \overline{x}(\cdot,w)\|_\infty \\ \|y(\cdot,w) - \overline{y}(\cdot,w)\|_\infty\end{pmatrix}, \text{ for all}, (x,y), (\overline{x},\overline{y}) \in B_p(w)\times B_q(w).$$

It is clear that the radius spectral $\rho(M_{trice}) < 1$. By Lemma 3.3, $M_{trix}(w)$ converges to zero. From Theorem 3.5 there exists a unique random solution of problem (1.1). We denote by $(x(t,w),y(t,w))$ the mild solution of (1.1). □

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