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A New Generalization of Non-Unique Fixed Point Theorems of Ćirić for Akram-Zafar-Siddiqui Type Contraction

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Abstract

In this article, we establish some fixed point theorems of Ćirić's type for Akram-Zafar-Siddiqui type contractive mappings having non-unique fixed points. Our results generalize, extend and improve several ones in the literature.

1. Introduction

Let (X,d) be a complete metric space and $T : X \to X$ a self-mapping of X. Suppose that $F(T) = \{x \in X \mid Tx = x\}$ is the set of fixed points of T.
The following definitions shall be required in the sequel: $O(x,T) = \{x, Tx, T^2x, \cdots, T^nx, \cdots\}$ =orbit of T at x.

Definition 1.1. *Ćirić [1]: A metric space (X,d) is said to be $T-$orbitally complete if $T : X \to X$ is a selfmapping and if any Cauchy subsequence $\{T^{n_i}x\}$ in orbit $O(x,T)$, with $x \in X$, converges in X.*

Definition 1.2. *An operator $T : X \to X$ is orbitally continuous if*

$$\lim_{i\to\infty} d(T^{n_i}x, x^*) = 0 \implies \lim_{i\to\infty} d(T(T^{n_i}x), Tx^*) = 0.$$

Definition 1.2 was originally stated in the following equivalent form in Ćirić [1]:
An operator $T : X \to X$ is said to be *orbitally continuous* if $T^{n_i}x \to x^* \implies T(T^{n_i}x) \to Tx^*$ as $i \to \infty$.
Indeed, the notions in both Definition 1.1 and Definition 1.2 were first introduced by Ćirić [1] in 1971 to obtain some fixed point theorems. The definitions are also contained in Ćirić [2].
There are non-linear equations which may arise in applications and whose fixed points are not necessarily unique. Ćirić [3] established some results pertaining to this notion of non-unique fixed points. The classical Banach's fixed point theorem was established by Banach [4], using the following contractive definition: there exists $c \in [0,1)$ (fixed) such that $\forall\, x,\ y \in X$,

$$d(Tx,Ty) \le c\, d(x,y). \tag{1.1}$$

However, it is crucial to say that the mappings satisfying the contractive condition (1.1) are necessarily continuous. In order to have a wider class of contractive mappings than those satisfying (1.1), Kannan [5] generalized the Banach's fixed point theorem by employing the following contractive definition: there exists $a \in [0, \frac{1}{2})$ such that

$$d(Tx,Ty) \le a[d(x,Tx) + d(y,Ty)],\ \forall\, x,\ y \in X. \tag{1.2}$$

So, the mappings satisfying (1.2) need not be continuous and this is a very nice initiative by the author [5]. Several authors have generalized and extended Banach's fixed point theorem using similar notion as in (1.2). Interested readers may also consult Chatterjea [6], Zamfirescu [7] and a host of others in the literature.

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However, it is noteworthy to say that several contractive conditions including Banach's contractive condition (1.1) have always been concerned with establishing the existence and uniqueness of the fixed point of the mapping. Therefore, in order to include mappings whose fixed points may be not unique, Ćirić [3] introduced a new technique involving contractive conditions for such mappings, realizing the fact that there are also nonlinear equations with more than one fixed point as aforementioned. In particular, Ćirić [3] introduced, amongst others, the following two contractive conditions: For a mapping $T:X\to X$, there exists $\lambda\in(0,1)$ such that $\forall\, x,\ y\in X$,

$$\min\{d(Tx,Ty),d(x,Tx),d(y,Ty)\}-\min\{d(x,Ty),d(y,Tx)\}\le \lambda d(x,y), \tag{1.3}$$

where T is orbitally continuous; and also there exists $\lambda\in(0,1)$ such that $\forall\, x,\ y\in X$,

$$\min\{d(Tx,Ty),\max\{d(x,Tx),d(y,Ty)\}\}-\min\{d(x,Ty),d(y,Tx)\}\le \lambda d(x,y). \tag{1.4}$$

Another contractivity condition worthy of note is the following:

Definition 1.3. *(Akram et al. [8]): A selfmap $T:X\to X$ of a metric space (X,d) is said to be A-contraction if it satisfies the condition:*

$$d(Tx,Ty)\le \beta(d(x,y),d(x,Tx),d(y,Ty)),\ \forall\, x,\ y\in X, \tag{1.5}$$

and some $\beta\in A$, where A is the set of all functions $\beta:\mathbf{R}_+^3\to\mathbf{R}_+$ satisfying
(i) β is continuous on the set $\mathbf{R}_+^3$ (with respect to the Euclidean metric on $\mathbf{R}^3$);
(ii) $a\le kb$ for some $k\in[0,1)$ whenever $a\le\beta(a,b,b)$, or $a\le\beta(b,a,b)$, or, $a\le\beta(b,b,a)$, $\forall\, a,\ b\in\mathbf{R}_+$.

Akram *et al.* [8] employed the contractive condition (1.5) to prove that if X is a complete metric space, then the mapping T has a unique fixed point.
Olatinwo [9] generalized the results of Akram et al. [8] by employing the following more general contractive condition:

Definition 1.4. *(Olatinwo [9]): A selfmap $T:X\to X$ of a metric space (X,d) is said to be a generalized A-contraction* or G_A-contraction *if it satisfies the condition:*

$$d(Tx,Ty)\le \alpha(d(x,y),d(x,Tx),d(y,Ty),[d(x,Tx)]^r[d(y,Tx)]^p d(x,Ty),d(y,Tx)[d(x,Tx)]^m),$$

$\forall\, x,\ y\in X,\ r,\ p,\ m\in\mathbf{R}_+$ and some $\alpha\in G_A$, where G_A is the set of all functions $\alpha:\mathbf{R}_+^5\to\mathbf{R}_+$ satisfying
(i) α is continuous on the set $\mathbf{R}_+^5$ (with respect to the Euclidean metric on $\mathbf{R}^5$);
(ii) if any of the conditions $a\le\alpha(b,b,a,c,c)$, or, $a\le\alpha(b,b,a,b,b)$, or, $a\le\alpha(a,b,b,b,b)$ holds for some $a,\ b,\ c\in\mathbf{R}_+$, then there exists $k\in[0,1)$ such that $a\le kb$.

The contractive mappings of both Akram *et al.* [8] and Ćirić [3] are our motivation for the present article. Therefore, in this paper, we prove various and more general non-unique fixed point theorems by employing on a complete metric space for selfmappings by using Akram-Zafar-Siddiqui type contractive conditions which are hybrids of those used in [3, 8, 9]. Our results are generalizations, extensions and improvemens of the results of Ćirić [3] and those of the author [10, 11, 12]. Many unique fixed point theorems in the literature involving those of Akram *et al.* [8] are also special cases of the results of the present article. One can consult the reference section for detail on unique fixed point theorems. For excellent study of mappings having non-unique fixed points, we refer to Achari [13, 14, 15], Ćirić [2, 3, 16], Karapinar [17] and Pachpatte [18].
To prove our results, we shall employ the following more general contractive conditions than those stated in (1.3) and (1.4)
(a) For a mapping $T:X\to X$, there exists a function $\beta:\mathbf{R}_+^5\to\mathbf{R}_+$ such that $\forall\, x,\ y\in X$, we have

$$\begin{aligned}&\min\{d(Tx,Ty),d(x,Tx),d(y,Ty)\}-\min\{d(x,Ty),d(y,Tx)\}\le\\ &\beta(d(x,y),d(x,Tx),d(y,Ty),[d(x,Tx)]^r[d(y,Tx)]^p d(x,Ty),d(y,Tx)[d(x,Tx)]^m);\end{aligned} \tag{1.6}$$

$\forall\, x,\ y\in X,\ r,\ p,\ m\in\mathbf{R}_+$, where the function β satisfies:
(i) β is continuous on the set $\mathbf{R}_+^5$ (with respect to the Euclidean metric on $\mathbf{R}^5$);
(ii) there exists some $\lambda\in[0,1)$, such that $a\le\lambda b$ whenever $a\le\beta(b,b,a,c,c)$, $\forall\, a,\ b,\ c\in\mathbf{R}_+$.
(b) For a mapping $T:X\to X$, there exists a function $\beta:\mathbf{R}_+^5\to\mathbf{R}_+$ such that $\forall\, x,\ y\in X$, we have

$$\begin{aligned}&\min\{d(Tx,Ty),\max\{d(x,Tx),d(y,Ty)\}\}-\min\{d(x,Ty),d(y,Tx)\}\le\\ &\beta(d(x,y),d(x,Tx),d(y,Ty),[d(x,Tx)]^r[d(y,Tx)]^p d(x,Ty),d(y,Tx)[d(x,Tx)]^m),\end{aligned} \tag{1.7}$$

$\forall\, x,\ y\in X,\ r,\ p,\ m\in\mathbf{R}_+$, where the function β satisfies:
(i) β is continuous on the set $\mathbf{R}_+^5$ (with respect to the Euclidean metric on $\mathbf{R}^5$);
(ii) there exists some $\lambda\in[0,1)$, such that $a\le\lambda b$ whenever $a\le\beta(b,b,a,c,c)$, or, $a\le\beta(b,b,a,b,b)$, $\forall\, a,\ b,\ c\in\mathbf{R}_+$.

Remark 1.5. *Each of the contractive conditions (1.6) and (1.7) can be reduced to several other ones in the literature. In particular, we have the following:*
(i) It is obvious that both contractive conditions (1.3) and (1.4) are special cases of contractive conditions (1.6) and (1.7) respectively if $\beta(t_1,t_2,t_3,t_4,t_5)=\lambda t_1,\ \forall\,(t_1,t_2,t_3)\in\mathbf{R}_+^5,\ \lambda\in(0,1)$.

2. Main results

Theorem 2.1. *Let (X,d) be a complete metric space and $T : X \to X$ an orbitally continuous mapping satisfying contractive condition (1.6). For $x_0 \in X$, let $\{x_n\}_{n=0}^{\infty}$ defined by $x_n = Tx_{n-1} = T^n x_0,\ n = 0,1,2,\cdots$, be the Picard iteration associated with T. Then, T has a fixed point.*

Proof. We have that $x_n = Tx_{n-1} = T^n x_0,\ x_0 \in X\ (n = 0,1,2,\cdots)$. If $d(x_q,x_{q+1}) = 0$ for some $q \geq 0$, then x_0 is the limit point of $\{T^n x_0\}$ and x_q is a fixed point of T. Suppose that $d(x_n,x_{n+1}) > 0,\ n = 0,1,2,\cdots$. Using condition (1.6) with $x = x_n,\ y = x_{n+1}$, we have

$$\begin{aligned}&\min\{d(Tx_n,Tx_{n+1}),d(x_n,Tx_n),d(x_{n+1},Tx_{n+1})\} - \min\{d(x_n,Tx_{n+1}),d(x_{n+1},Tx_n)\}\\ &\leq \beta(d(x_n,x_{n+1}),d(x_n,Tx_n),d(x_{n+1},Tx_{n+1}),[d(x_n,Tx_n)]^r[d(x_{n+1},Tx_n)]^p d(x_n,Tx_{n+1}),d(x_{n+1},Tx_n)[d(x_n,Tx_n)]^m),\end{aligned}$$

from which we obtain that

$$\min\{d(x_{n+1},x_{n+2}),d(x_n,x_{n+1})\} \leq \beta(d(x_n,x_{n+1}),d(x_n,x_{n+1}),d(x_{n+1},x_{n+2}),0,0). \tag{2.1}$$

Since $\lambda < 1$, we choose $\min\{d(x_{n+1},x_{n+2}),d(x_n,x_{n+1})\} = d(x_{n+1},x_{n+2})$ and apply Property (ii) of β so that from (2.1) we get

$$d(x_{n+1},x_{n+2}) \leq \beta(d(x_n,x_{n+1}),d(x_n,x_{n+1}),d(x_{n+1},x_{n+2}),0,0) \leq \lambda d(x_n,x_{n+1}),$$

which yields

$$d(x_{n+1},x_{n+2}) \leq \lambda d(x_n,x_{n+1}) \leq \lambda^2 d(x_{n-1},x_n) \leq \cdots \leq \lambda^{n+1} d(x_0,x_1). \tag{2.2}$$

Using (2.2) inductively in the repeated application of the triangle inequality yields, for $p \in \mathbb{N}$,

$$d(x_n,x_{n+p}) \leq \frac{\lambda^n(1-\lambda^p)}{1-\lambda} d(x_0,x_1) \to\ 0 \text{ as n} \to \infty. \tag{2.3}$$

Hence, from (2.3) we have that $\{x_n\}$ is a Cauchy sequence in X. Since (X,d) is a complete metric space, there exists $u \in X$ such that $\lim\limits_{n\to\infty} d(x_n,u) = 0$, that is, $\lim\limits_{n\to\infty} x_n = u$. Therefore, since $x_n = T^n x_0$ and T is orbitally continuous, we have

$$0 = d(\lim_{n\to\infty} T(T^n x_0),Tu) = \lim_{n\to\infty} d(T(T^n x_0),Tu) = \lim_{n\to\infty} d(Tx_n,Tu) = \lim_{n\to\infty} d(x_{n+1},Tu) = d(u,Tu).$$

Thus, proving that $Tu = u$, that is, $u \in X$ is a fixed point of T. □

Theorem 2.2. *Let (X,d) be a complete metric space and $T : X \to X$ a mapping satisfying contractive condition (1.7) For $x_0 \in X$, let $\{x_n\}_{n=0}^{\infty}$ defined by $x_n = Tx_{n-1} = T^n x_0,\ n = 0,1,2,\cdots$, be the Picard iteration associated with T. Then, T has a fixed point.*

Proof. We have that $x_n = Tx_{n-1} = T^n x_0,\ x_0 \in X\ (n = 0,1,2,\cdots)$. If $d(x_q,x_{q+1}) = 0$ for some $q \geq 0$, then x_0 is the limit point of $\{T^n x_0\}$ and x_q is a fixed point of T. Suppose that $d(x_n,x_{n+1}) > 0,\ n = 0,1,2,\cdots$. Using condition (1.7) with $x = x_n,\ y = x_{n+1}$, we have

$$\begin{aligned}&\min\{d(Tx_n,Tx_{n+1}),\max\{d(x_n,Tx_n),d(x_{n+1},Tx_{n+1})\}\} - \min\{d(x_n,Tx_{n+1}),d(x_{n+1},Tx_n)\} \leq\\ &\beta(d(x_n,x_{n+1}),d(x_n,Tx_n),d(x_{n+1},Tx_{n+1}),[d(x_n,Tx_n)]^r[d(x_{n+1},Tx_n)]^p d(x_n,Tx_{n+1}),d(x_{n+1},Tx_n)[d(x_n,Tx_n)]^m),\end{aligned}$$

which reduces to

$$\begin{aligned}&\min\{d(x_{n+1},x_{n+2}),\max\{d(x_n,x_{n+1}),d(x_{n+1},x_{n+2})\}\} \leq\\ &\beta(d(x_n,x_{n+1}),d(x_n,x_{n+1}),d(x_{n+1},x_{n+2}),0,0).\end{aligned} \tag{2.4}$$

Since

$$\min\{d(x_{n+1},x_{n+2}),\max\{d(x_n,x_{n+1}),d(x_{n+1},x_{n+2})\}\} = \max\{d(x_n,x_{n+1}),d(x_{n+1},x_{n+2})\},$$

we obtain from (2.4) that

$$\max\{d(x_{n+1},x_{n+2}),d(x_n,x_{n+1})\} \leq \beta(d(x_n,x_{n+1}),d(x_n,x_{n+1}),d(x_{n+1},x_{n+2}),0,0). \tag{2.5}$$

Again, since $\lambda < 1$, we choose $\max\{d(x_{n+1},x_{n+2}),d(x_n,x_{n+1})\} = d(x_{n+1},x_{n+2})$, so that from (2.5) we obtain

$$d(x_{n+1},x_{n+2}) \leq \beta(d(x_n,x_{n+1}),d(x_n,x_{n+1}),d(x_{n+1},x_{n+2}),0,0) \leq \lambda d(x_n,x_{n+1}),$$

which inductively leads again (as in the proof of Theorem 2.1) to

$$d(x_n,x_{n+1}) \leq \lambda^n d(x_0,x_1).$$

For $p \in \mathbb{N}$, we therefore, have again as in the proof of Theorem 2.1 that $d(x_n,x_{n+p}) \to 0$ as $n \to \infty$.
Hence, we have that $\{x_n\}$ is a Cauchy sequence in X. Since (X,d) is complete, there exists $u \in X$ such that $\lim\limits_{n\to\infty} x_n = u$.
Using (1.7) again with $x = x_n,\ y = u$ we obtain

$$\begin{aligned}&\min\{d(Tx_n,Tu),\max\{d(x_n,Tx_n),d(u,Tu)\}\} - \min\{d(x_n,Tu),d(u,Tx_n)\} \leq\\ &\beta(d(x_n,u),d(x_n,Tx_n),d(u,Tu),[d(x_n,Tx_n)]^r[d(u,Tx_n)]^p d(x_n,Tu),d(u,Tx_n)[d(x_n,Tx_n)]^m),\end{aligned}$$

which reduces to

$$\min\{d(x_{n+1},Tu),\max\{d(x_n,x_{n+1}),d(u,Tu)\}\}-\min\{d(x_n,Tu),d(u,x_{n+1})\}\leq$$
$$\beta(d(x_n,u),d(x_n,x_{n+1}),d(u,Tu),[d(x_n,x_{n+1})]^r[d(u,x_{n+1})]^p d(x_n,Tu),d(u,x_{n+1})[d(x_n,x_{n+1})]^m). \tag{2.6}$$

As $n\to\infty$, we obtain from (2.6) that

$$\min\{d(u,Tu),d(u,Tu)\}\leq\beta(0,0,d(u,Tu),0,0). \tag{2.7}$$

Using Property(ii) of β in (2.7) yields

$$d(u,Tu)\leq\beta(0,0,d(u,Tu),0,0)\leq\lambda.0=0,$$

from which it follows that $d(u,Tu)\leq 0$.
Therefore, due to nonnegativity of the metric, we obtain $d(Tu,u)=0 \iff Tu=u$. Thus, T has a fixed point $u\in X$. □

The next two results are Maia type (see [19]) which extend both Theorem 2.1 and Theorem 2.2

Theorem 2.3. *Let X be a non-empty set, d and ρ two metrics on X and $T: X\to X$ a mapping. For $x_0\in X$, let $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1}=Tx_n,\ n=0,1,2,\cdots$, be the Picard iteration associated with T. Suppose that*
(i) there exists $M>0$ such that $\rho(Tx,Ty)\leq Md(x,y),\ \forall\, x,\ y\in X$;
(ii) (X,ρ) is a complete metric space;
(iii) $T: (X,\rho)\to(X,\rho)$ is orbitally continuous;
(iv) $T: (X,d)\to(X,d)$ is a mapping satisfying (Δ).
Then, $T: (X,\rho)\to(X,\rho)$ has a fixed point.

Proof. By condition (iv), we obtain as in Theorem 2.1 that, for $p\in\mathbb{N}, d(x_n,x_{n+p})\to\ 0$ as n $\to\infty$. That is, $\{x_n\}$ is a Cauchy sequence in (X,d).
We now show that $\{x_n\}$ is a Cauchy sequence in (X,ρ) as follows: By condition (i), we have, for $p\in\mathbb{N}$,

$$\rho(x_n,x_{n+p})=\rho(Tx_{n-1},Tx_{n+p-1})\leq Md(x_{n-1},x_{n+p-1})\to\ 0 \text{ as } n\to\infty,$$

that is, $\rho(x_n,x_{n+p})\to\ 0$ as n $\to\infty$. Thus, $\{x_n\}$ is a Cauchy sequence in (X,ρ) too.
By condition (ii), (X,ρ) is a complete metric space implies that there exists $u\in X$ such that $\lim_{n\to\infty}\rho(x_n,u)=0$, that is, $\lim_{n\to\infty}x_n=u$.
By condition (iii), since $x_n=T^nx_0$ and $T: (X,\rho)\to(X,\rho)$ is orbitally continuous, we have

$$0=\rho(\lim_{n\to\infty}T(T^nx_0),Tu)=\lim_{n\to\infty}\rho(T(T^nx_0),Tu)=\lim_{n\to\infty}\rho(Tx_n,Tu)=\lim_{n\to\infty}\rho(x_{n+1},Tu)=\rho(u,Tu).$$

Therefore, $\rho(u,Tu)=0 \iff Tu=u$. So, T has a fixed point u. □

Theorem 2.4. *Let X be a non-empty set, d and ρ two metrics on X and $T: X\to X$ a mapping. For $x_0\in X$, let $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1}=Tx_n,\ n=0,1,2,\cdots$, be the Picard iteration associated with T. Suppose that*
(i) there exists $M>0$ such that $\rho(Tx,Ty)\leq Md(x,y),\ \forall\, x,\ y\in X$;
(ii) (X,ρ) is a complete metric space;
(iii) $T: (X,\rho)\to(X,\rho)$ is continuous;
(iv) $T: (X,d)\to(X,d)$ is a mapping satisfying $(\Delta\star)$.
Then, $T: (X,\rho)\to(X,\rho)$ has a fixed point.

Proof. By condition (iv), we obtain as in Theorem 2.2 that $\{x_n\}$ is a Cauchy sequence in (X,d).
By condition (i), we have as in Theorem 2.3 that $\{x_n\}$ is a Cauchy sequence in (X,ρ) too.
By condition (ii), (X,ρ) is a complete metric space implies that there exists $u\in X$ such that $\lim_{n\to\infty}\rho(x_n,u)=0$, that is, $\lim_{n\to\infty}x_n=u$.
By condition (iii), since $T: (X,\rho)\to(X,\rho)$ is continuous, we have

$$0=\lim_{n\to\infty}\rho(x_{n+1},u)=\lim_{n\to\infty}\rho(Tx_n,u)=\rho(T(\lim_{n\to\infty}x_n),u)=\rho(Tu,u).$$

Therefore, $\rho(u,Tu)=0 \iff Tu=u$. So, T has a fixed point u. □

Remark 2.5. *Our results generalize and extend several classical results in the literature, involving unique and nonunique fixed points. In particular, both Theorem 2.1 and Theorem 2.2 are generalizations and extensions of the corresponding results of Ćirić [3, 2]. Both Theorem 2.3 and Theorem 2.4 extend both Theorem 2.1 and Theorem 2.2 respectively as well as the corresponding results of Ćirić [3, 2]. Both Theorem 2.3 and Theorem 2.4 also generalize the result of Maia [19]. Indeed, the results of our present paper generalize the corresponding results of Olatinwo [10, 11, 12], but independent of the corresponding results of the author [20]. We also observe that the unique fixed point theorems of Akram et al. [8] are special cases of the results contained in this paper.*

Remark 2.6. *We also employ this medium to announce that while proving the existence of the fixed point of T, the term "$d(T\lim_{n\to\infty}(T^nx_0),Tu)$" that appeared was a typographical misprint in Theorem 2.1 and Theorem 2.3 of [10] as well as in Theorem 2.1 and Theorem 2.4 of [20]. Since T is orbitally continuous in those Theorems (rather than being continuous), the misprint should change to "$d(\lim_{n\to\infty}T(T^nx_0),Tu)$" (which is now correctly expressed in the present article). Our interested readers can also see the correct term "$d(\lim_{n\to\infty}T(T^nx_0),Tu)$" in the articles [11, 12] (which invariably becomes "$\lim_{n\to\infty}d(T(T^nx_0),Tu)$" since metric is continuous).*

3. Conclusion

So far, the results obtained in the present article are the most general results in non-unique fixed point theory.

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This paper is dedicated to the late Professor Ljubomir Ćirić (one of my great mentors) who was born on Tuesday, 13th August, 1935 and whose demise occurred on Saturday, 23rd July, 2016. He was a pioneering expert of Fixed Point Theory in Serbia.

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