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Computational Enumeration of Colorings of Hyperplanes of Hypercubes for all Irreducible Representations and Applications

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Abstract

We obtain the generating functions for the combinatorial enumeration of colorings of all hyperplanes of hypercubes for all irreducible representations of the hyperoctahedral groups. The computational group theoretical techniques involve the construction of generalized character cycle indices of all irreducible representations for all hyperplanes of the hypercube using the Möbius function, polynomial generators for all cycle types and for all hyperplanes. This is followed by the construction of the generating functions for colorings of all (n-q)-hyperplanes of the hypercube, for example, vertices (q=5), edges (q=4), faces (q=3), cells (q=2) and tesseracts (q=4) for a 5D-hypercube. Tables are constructed for the combinatorial numbers for coloring all hyperplanes of 5D-hypercubes for 36 irreducible representations. Applications to chirality, chemistry and biology are also pointed out.

1. Introduction

Hypercubes [1]-[29] and related combinatorics of wreath product groups [30]-[54] have been the focus of a number of research investigations owing to their importance in numerous applications in a variety of disciplines. Hypercubes are natural representations of Boolean functions, as 2n possible Boolean functions from a set of n entities that take binary values can be represented by the vertices of a hypercube. Thus hypercubes find applications in chemistry, biology, finite automata, electrical circuits, genetics, enumeration of isomers, isomerization reactions, visualization and computer graphics, chirality, protein-protein interactions, intrinsically disordered proteins, partitioning of massively large databases, and parallel computing [1]-[11], [19]-[29], [41]-[55], [56]-[59]. The automorphism groups of hypercubes which are hyperoctahedral wreath products find applications in enumerative combinatorics, isomerization reactions, chirality, nuclear spin statistics, weakly-bound non-rigid water clusters, non-rigid molecules, and in proteomics [41]-[55], [56]-[59]. The hypercubes have also been connected to Goldbach conjecture, last Fermat's theorem, Erdös discrepancy conjecture, modern multi-dimensional representation of time measures, quantum similarity measures, [1]-[5], biochemical imaging [6], multi-dimensional imaging [19],[20], [22]-[26], classification of large data, Quantitative Shape-Activity Relations (QShAR)etc. [7]-[10].

Combinatorial enumeration of colorings of different hyperplanes, especially vertices of hypercubes has been the topic of several studies for the past two centuries. In fact, subsequent to publication of his classic 1937 [15] paper on combinatorics of groups, graphs and chemical compounds, Pólya in a subsequent work [17] has pointed out the errors in previous enumeration of colorings of vertices hypercubes. As pointed out recently by Banks et al. [19],[20] in the context of computer visualization, in 1877, Clifford [12],[13] has enumerated the number of equivalence classes for 2-colorings of a 4D-hypercube vertices as 396 which was subsequently shown to be incorrect by Pólya [17] in 1940 who obtained 402 equivalence classes for 2-colorings of a 4d-hypercube. Historically Pólya's theorem was anticipated in Redfield's paper on superposition theorem [16]. Although in more recent mathematical literature, cycle indices of hypercubes and enumerations of colorings of the vertices of hypercubes have been considered [17]-[29], [34] these studies have been restricted only to the totally symmetric irreducible representations of the hyperoctahedral groups. Moreover in the most recent work on the 5D-hypercube enumeration [29] of vertex colorings there are errors, as we show here. Pólya's theorem and its variation [1]-[6], [17]-[21] have been applied extensively which generate equivalence classes for different distribution of colors called the pattern inventory and also the total number of colorings. However, several chemical and spectroscopic applications require more powerful and generalized enumeration techniques that span all the irreducible representations of the groups where Pólya's theorem becomes a special case for the totally symmetric representation. Furthermore in the

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case of hypercubes, most of the previous combinatorics is restricted to the enumeration of vertex colorings. The vertices of hypercubes are only one of several possible hypercube's hyperplanes. The present author [39]-[40] has generalized Pólya's theorem, De Bruijn's theorem [60] and Harary-Palmer power group theorem [31] to characters of all irreducible representations of a group cast into the form of generalized character cycle indices or GCCIs. Such combinatorial and graph theoretical methods have several applications to rovibronic spectroscopy, non-rigid molecules, water clusters, nuclear spin statistics, multiple-quantum NMR spectroscopy, dynamic NMR, enumeration of isomerization reactions, chirality, ESR spectroscopy, topological indices in QSAR [36]-[58], [61]-[63].

The n-dimensional hypercube's automorphism group is comprised of $2^n \times n!$ operations, and thus the order of this group increases both exponentially and factorially. For example, the automorphism group of a 6D-hypercube consists of 46,680 operations spanning 65 irreducible representation. In ordinary Pólya's theory, different conjugacy classes that give rise to the same cycle types under group action on a given set are combined into a single term, as they give rise to the same monomial for patterns, and in general with the exception of full symmetric group S_n, multiple conjugacy classes often contribute to the same cycle type. This poses a problem when one needs to consider all irreducible representation, as character values in general are based on conjugacy classes and not cycle types. Furthermore there is no one-to-one correspondence between cycle types and conjugacy classes for hyperoctahedral wreath product groups of hypercubes. Thus we need both cycle types of each conjugacy class and the character table of the group unlike the ordinary Pólya cycle index which only needs the cycle types that compose the cycle index of a group. The other computational challenge that arises for hypercube colorings is that the cycle types of induced permutation for different hyperplanes need to be obtained. In general there are n hyper planes for an nD-hypercube represented by q values ranging from 1 to n with of course q=0 being the trivial single vertex and hence is not considered. When q=n we obtain the vertices of the hypercube, q=n-1 we obtain the edges, q=n-2 yields faces, and in general q represents (n-q)-hyperplanes of an nD-hypercube. Each such hyperplane generates a set of cycle types for each conjugacy class. Thus computing the equivalence classes of the colorings of various hyperplanes requires the computation of the cycle types of different (n-q)-hyperplanes of the hypercube with q=1 through n. Previous works in the mathematical literature [17]-[29] have focused on the total number of equivalence classes rather than the inventory of patterns or a generating function that yields number of colorings for a given number of colors of various kinds. Such a distribution of patterns for various colors is quite important for a number of practical applications, and thus we focus in the present study the computational techniques to obtain such generating functions for all hyperplanes and all irreducible representations of the hypercube. Moreover none of the previous studies [17]-[29] has dealt with irreducible representations other than totally symmetric representation in their enumerations. The present author [11] has previously considered multinomial colorings of 4D-hypercube for different hyperplanes, and with chemical applications to water pentamer in mind, the present author has considered colorings of tesseracts [64] of the 5D-hypercube, and recently vertices (q=4) and tesseracts q=1 for all irreducible representations and 2-colorings of (q=2) 3-faces only for the totally symmetric irreducible representation of the 5D-hypercube [61]. The present work considers for the first time enumeration of colorings for all hyperplanes (q=1 through q=5) of the 5D-hypercube for all 36 irreducible representations.

2. Mathematical and computational techniques

In general, the automorphism group of an nD-hypercube is the wreath product $S_n[S_2]$ where S_n is the full permutation group of n objects comprising of $n!$ permutations. The order of the nD-hypercube wreath product group is $2^n \times n!$ and hence it grows in astronomical proportion as a function of n. For example, the automorphism group of a 10D-hypercube consists of $2^{10} \times 10!$ permutations that give rise to 481 conjugacy classes, and 481 irreducible representations, 10 hyperplanes, thus demonstrating the combinatorial complexity of the problem of enumerating colorings of different hyperplanes of an nD-hypercube for all irreducible representations. Coxeter [65] has discussed in depth hypercubes and various other regular polytopes and their mathematical characterizations using various projections and graph theory. An nD-hypercube is comprised of (n-q)-hyperplanes where q goes from 0 to n. The largest value of q = n represents the vertices, q=n-1 represents the edges, q=n-2 represents the faces, q=n-3 represents the cells, q=n-4 represents tesseracts, and so on. The induced permutation of the automorphism group of the nD-hypercube on each of these hyperplanes is quite different and it cannot be deduced from a simple inspection with the exception of a 2D-hypercube (square) and a 3D-hypercube (a regular cube). Thus the first step is to construct the cycle types for each conjugacy class of the hypercube's wreath product group for the induced permutations of all hyperplanes of the hypercube. We note that although for ordinary Pólya enumeration one needs only the cycle index which can be constructed by other methods as cycle types of several conjugacy classes become degenerate for wreath products, the enumerations that involve all irreducible representations require the cycle types of each conjugacy class, as there is no one-to-one correspondence between the conjugacy classes and cycle types for wreath product groups. The cycle types of q=1 or (n-1)-hyperplanes are the ones that can be readily constructed as they are natural representations of the hypercube permutations.

The techniques to construct the conjugacy class cycle types of q=1 or (n-1)-hyperplanes and the character table for all irreducible representations of the hypercube group involve matrix generating functions and we shall consider this first. We use the 5D-hypercube as not only an illustrative example but also to carry out all of the needed computations. For a 5D-hypercube the special case of q=1 enumerates the various tesseracts of the hypercube, and Fig.1 shows a graph that exemplifies the underlying relationship between the tesseracts of the 5D-hypercube. In Fig. 1 the vertices represent the tesseracts while the edges represent the underlying connectivity among the ten tesseracts of the 5D-hypercube. The cycle types of the permutations of q=1 tesseracts are isomorphic with the permutations of vertices of the automorphism group of the graph in Fig. 1.

In general, let a permutation $g \in S_n$ upon its action on the set Ω of $q=1$ hyperplanes of the hypercube generate a_1 cycles of length 1, a_2 cycles of length 2, a_3 cycles of length 3, ..., a_n cycles of length n, which can be represented by $1^{a_1}2^{a_2}3^{a_3}\dots n^{a_n}$. Alternatively, the cycle type T_g of g can be denoted as $T_g = (a_1, a_2, a_3, \dots, a_n)$. As the composing group in $S_n[S_2]$, S_2 of the wreath product has only two conjugacy classes, the conjugacy class of the wreath product $S_n[S_2]$ and he cycle types of action on q=1 hyperplanes can be expressed as a cycle type comprised of a $2 \times n$ matrix, where the first row corresponds to the action of $\{(g;\pi)\}$permutations where $\pi = e \in S_2$ and $g \in S_n$ and the second row represents the permutations $\{(g;\pi)\}$, for $\pi = (12) \in S_2$. The cycle type of any conjugacy class, $T(g;\pi)$, where $(g;\pi)$ is any representative in then a $2 \times n$ matrix is obtained using the orbit structure of $g \in S_n$ and the corresponding conjugacy class of S_2. For the particular case of $S_5[S_2]$ under consideration, the cycle type of $(g;\pi)$ for a conjugacy class of $S_5[S_2]$ is given by

$$T(g;\pi) = a_{ik} \quad (1 \le i \le 2), (1 \le k \le 5) \tag{2.1}$$

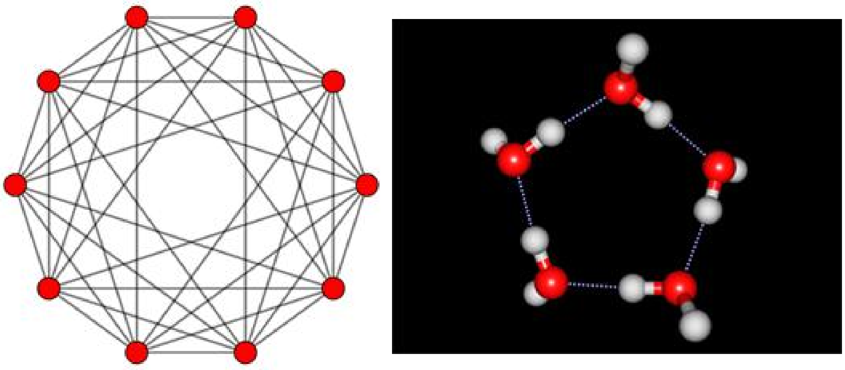

Figure 2.1: Ten tesseracts of the 5D-hypercube are represented by the vertices of the graph shown in this figure (reproduced from ref.[59]). Right: Water Pentamer. The automorphism group of this graph is also the automorphism group of the 5D-hypercube and fully non-rigid water pentamer or S5[S2] comprising of 3840 permutations that span 36 conjugacy classes.

To illustrate, the conjugacy class $\{(1)(2)(345);(12)\}$ of $S_5[S_2]$ given by (2.2)

$$T[\{(1)(2)(345);(12)\}] = \begin{bmatrix} 20000 \\ 00100 \end{bmatrix} \tag{2.2}$$

Likewise the conjugacy class of {(1234)(5);(12)} is given by (2.3):

$$T[\{(1234)(5);(12)\}] = \begin{bmatrix} 1\,0\,0\,0\,0 \\ 0\,0\,0\,1\,0 \end{bmatrix} \tag{2.3}$$

In this manner all conjugacy classes of $S_n[S_2]$ are obtained and for the simplest example of $S_3[S_2]$ which represents the permutations of the six faces of the cube, Table 1 shows all as 2×3 matrices thus constructed for the 3D-cube. In Table 1 we have also shown the corresponding rotations or mirror planes of the cube, as the cycle types of the cube's faces can also be directly obtained by applying these operations on a regular cube and collecting the induced orbits of the permutations of the faces of the cube under the action of these operations. It can be seen from Table 1 that there is no one-to-one correspondence between the cycle types and conjugacy classes of the 3D-cube, as orbit structures of two different matrix types can be the same, for example, for matrices 3 and 5 in Table 1 have the same cycle types of $1^2 2^2$ for the six faces of the cube (q=1). However these two matrices belong to different conjugacy classes with different character values for the various irreducible representations of the octahedral (cubic) group or $S_3[S_2]$. Thus the matrices are important for the enumerations involving all irreducible representations while only the cycle types are needed for the ordinary Pólya enumeration of equivalence classes, as such enumeration becomes a special case of our formalism applied to the totally symmetric A_1 irreducible representation.
We can obtain the orders of the conjugacy classes and the cycle types for the q=1 or (n-1)hyper planes of the hypercube directly from their $2\times n$ matrices. Suppose $P(m)$ denotes the number of partitions of integer m with $P(0)=1$. Then all ordered partitions of n into pairs or compositions of n into two parts, denoted by (n_1,n_2) such that $\sum n_i = n$, yields the number of conjugacy classes of $S_n[S_2]$. That is, the total number of conjugacy classes of $S_n[S_2]$is given by

$$N_C = \sum\nolimits_{(n)} P(n_1)\,P(n_2) \tag{2.4}$$

where the sum is over all ordered pairs of partitions of n. Furthermore, the order any conjugacy class of $S_n[S_2]$ with the matrix type $T(g;\pi) = a_{ik}$ can be obtained with Eq (2.5):

$$|T(g;\pi)| = \frac{n!}{\prod_{i,k} a_{ik}!(2k)^{a_{ik}}} \tag{2.5}$$

For example, for the 6-D hyperoctahedral group, $S_6[S_2]$, the ordered partitions of 6 into 2 parts are given by

$$\{(6,0),(0,6),(5,1),(1,5),(4,2),(2,4),(3,3)\}$$

and hence the number of conjugacy classes of the $S_6[S_2]$ group is

$$2P(6)P(0)+2P(5)P(1)+2P(4)P(2)+P(3)^2 = 65 \tag{2.6}$$

The number of elements in any particular conjugacy class of $S_n[S_2]$ can also be readily computed from the corresponding matrix cycle type. For example, application of (2.5) to the conjugacy class 6 in Table 1 gives:

$$\left|\begin{pmatrix} 1\,0\,0 \\ 0\,1\,0 \end{pmatrix}\right| = \frac{3!2^3}{1!(2.1)^1 1!(2.2)^1} = 6 \tag{2.7}$$

The orders of conjugacy classes thus obtained for the cube are shown in Table 1 for each conjugacy class. The cycle types for the permutations induced on the $q=1$ or $(n-1)-$ hyperplanes are also obtained readily from the $2\times n$ matrices by mapping place values for the non-zero entries in the matrix type. That is, assign a cycle of length $(k^2)^{a_{1k}}$ for each non-zero entry column k in the first row while for the second row

the contribution is $2k$ for nonzero entries. Thus the above matrix yields the overall cycle type $1^2 2^2$ for the regular cube's 6 faces. The cycle types thus obtained for $q=1$ or tesseracts of the 5D-hypercubeand for all conjugacy classes of the cubic group, $S_3[S_2]$ group are shown in Tables 2 and 1 together with the orders of each conjugacy class.
The above process for finding the cycle types of conjugacy classes and their orders can be likewise applied to the 5D-hypercube and the results are shown in Table 2. The next step is to compute the cycle types of the induced permutations for each conjugacy class for all of the remaining (n-q)-hyperplanes. For the 5d-hypercube this corresponds to $q=2$ (cells), $q=3$ (faces), $q=4$ (edges) and $q=5$ (vertices). Although there are previous studies [17]-[29] that have discussed the techniques for obtaining the cycle indices of the hypercube including the 5D-hypercube, these previous works have been predominantly restricted to the Pólya cycle indices of the vertices of a hypercube with the exception of Lemmis [23] who has explicitly considered other cycle types for a 4D-hypercube even though Lemmis [23] does not compute or report any results for the equivalence classes even for the totally symmetric irreducible representation . The explicit expressions have also been constructed for the ordinary cycle indices of hypercubes up to six dimensions [26], [28], [29]. In the present study we outline techniques for constructing the generalized character cycle indices for all irreducible representations and all cycle types of the various (n-q)-hyperplanes of the hypercube.
The process of computing the generating functions for the cycle types of various $(n-q)-$ hyperplanes of the hypercube involve the Möbius function, a fundamental enumerative combinatorial technique that encompasses generalization of the fundamental combinatorial principle of inclusion and exclusion that has been applied to many disciplines [66], [67] including music theory [35] and isomers with nearest neighbor exclusions [63]. The Möbius functions appear in a natural way, as the construction of various cycle types for the $(n-q)-$hyperplanes is related to the divisors of the set of all hyperplanes and it relates to the simplest cycle types of $q=1$. Thus the technique involves computing the polynomial generating functions via Möbius sums. We accomplish this from the matrix types of the conjugacy classes of the $S_n[S_2]$ groups to generate all of the cycle types for all $(n-q)-$hyperplanes through polynomial generating functions. The techniques employed are similar to the ones outlined in Krishnamurthy's book [67] and the work of Lemmis [24] who has made use of the enumerative Möbius inversion technique. That is, the generating functions for all cycle types for all values of q representing $(n-q)-$hyperplanes are generated as coefficient of x^q in the polynomial generating function $Q_p(x)$ obtained using the Möbius functions shown below:

$$Q_p(x) = \frac{1}{p}\sum_{d/p}\mu(p/d)F_d(x) \tag{2.8}$$

where the sum is strictly over all divisors d of p, and $\mu(p/d)$ is the Möbius function which takes values

$$1\,,\ -1,\ -1,\ 0,\ -1,\ 1,\ -1,\ 0,\ 0,\ 1\ \ldots$$

for arguments 1 to 10; in general, the Möbius function is obtained as follows for any number:
$\mu(m)=1$ if one of m's prime factors is not a perfect square and m contains even number of prime factors,
$\mu(m)=-1$ if m satisfies the same perfect-square condition as before but m contains odd number of prime factors,
$\mu(m)=0$ if m has a perfect square as one of its factors.
$F_d(x)$ in the above Eq (2.8) is defined as a polynomial in x constructed from the matrix cycle types shown in the first column of Table 1 or Table 2. Consider the non-zero columns of the matrix cycle types of $S_n[S_2]$ (see Tables 1 and 2) . Recall that the first row of these elements are represented by a_{1k} while the second rows are denoted by a_{2k} $(k=1,n)$. Then if p is the period of the matrix type shown in the first column of Table 1 or 2, and define, $g=\gcd(k;p)$, $p'=\frac{k}{g}$, $h=\gcd(2k;p)$; $p''=\frac{2k}{h}$ and define the polynomial $F_p(x)$ in terms of these divisors of the cycle type as

$$\begin{aligned} F_p(x) &= \prod_k^{nc}(1+2x^{p'})^{ga_{1k}}(1+2x^{p''})^{\frac{ha_{2k}}{2}}, \text{ if } h \text{ does not divide } k; \\ F_p(x) &= \prod_k^{nc}(1+2x^{p'})^{ga_{1k}}, \text{ if } h \text{ divides } k, \end{aligned} \tag{2.9}$$

where the product is taken only over nc, non-zero columns of the $2\times n$ matrix cycle type shown in Tables 1 or 2. The coefficient of x^q in $Q_p(x)$ obtained from the Möbius sums of various F_d polynomials where d's are strictly divisors of p generate the various cycle types for $(n-q)-$ hyperplanes of the nD-hypercube. We shall illustrate this by one of the matrix cycle types in Table 2. Consider the 31st matrix shown in Table 2 for $S_5[S_2]$:

$$\begin{pmatrix} 0\ 1\ 1\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0 \end{pmatrix} \tag{2.10}$$

As only 2^{nd} and 3^{rd} columns contain non-zero values, hence we need to consider only these two columns. Thus the maximum period to consider is 6 and hence the possible F polynomials are F_6, F_3, F_2 and F_1 as divisors of 6 are 1, 2, 3, and 6. Applying the GCD followed by the use of Eq (2.9), we obtain each of these polynomials as

$$F_1(x) = \left(1+2x^2\right)\left(1+2x^3\right) \tag{2.11}$$

$$F_2(x) = (1+2x)^2\left(1+2x^3\right) \tag{2.12}$$

$$F_3(x) = \left(1+2x^2\right)(1+2x)^3 \tag{2.13}$$

$$F_6(x) = (1+2x)^5. \tag{2.14}$$

From the F_d polynomials thus constructed above, we obtain the Q_p polynomials using the Möbius sum, shown in Eq (2.8). Thus we obtain

$$Q_1 = F_1 = 1+2x^2+2x^3+4x^5 \tag{2.15}$$

$$Q_2 = \frac{\mu(2)F_1+\mu(1)F_2}{2} = \frac{F_2-F_1}{2} = \frac{(1+2x)^2(1+2x^3)-(1+2x^2)(1+2x^3)}{2} = 2x+x^2+4x^4+2x^5 \tag{2.16}$$

$$Q_3 = \frac{\mu(1)F_3+\mu(3)F_1}{3} = \frac{F_3-F_1}{3} = \frac{(1+2x^2)(1+2x)^3-(1+2x^2)(1+2x^3)}{3} = 2x+4x^2+6x^3+8x^4+4x^5 \tag{2.17}$$

$$Q_6 = \frac{\mu(1)F_6+\mu(2)F_3+\mu(3)F_2+\mu(6)F_1}{6} = \frac{F_6-F_3-F_2+F_1}{6} = 4x^2+10x^3+8x^4+2x^5 \tag{2.18}$$

The coefficients of $x^q s$ are tabulated below for all possible Q_p polynomials which yield the cycle types for various $(n-q)-$perplanes as shown below:

Q_p	x	x^2	x^3	x^4	x^5
Q_1		2	2		4
Q_2	2	1		4	2
Q_3	2	4	6	8	4
Q_6		4	10	8	2
`Cycle type`	$2^2 3^2$	$1^2 2^1 3^4 6^4$	$1^2 3^6 6^{10}$	$2^4 3^8 6^8$	$1^4 2^2 3^4 6^2$
`Hyperplane`	$q=1$ (tesseracts)	$q=2$ (cells)	$q=3$ (faces)	$q=2$ (edges)	$q=5$ (vertices)

The results thus obtained for all cycle types of the hyperplanes of 5D-hypercube are shown in Table 2. We believe this is the first time that these cycle types have been tabulated for all hyperplanes of the 5D-hypercube. Although previously the cycle index for the vertices of the 5D-hypercube have been reported in the literature [24]-[26], [28], [29] using different techniques, and our results agree with those results, Table 2 is exhaustive as it includes all hyperplanes, not just $q=5$ (vertices). Moreover, as outlined below we consider all irreducible representations for coloring the $(n-q)-$ hyperplanes, and not just the totally symmetric A_1 representation. In our previous studies [51],[52] we have shown how the character tables of the $S_n[S_2]$ groups can be obtained from matrix generating functions and thus we shall not repeat the techniques in detail. Instead we shall focus on the colorings of the hyperplanes using the character table of $S_5[S_2]$, and the cycle types obtained for various hyperplanes of the 5D-hypercube shown in Table 2.
The character table of $S_5[S_2]$ containing 36 irreducible representations have been constructed before and thus we employ the GCCIs of the irreducible representation with character of the group $S_5[S_2]$. In general, the GCCI for the character χ of a group G' is defined as

$$P_{G'}^{\chi} = \frac{1}{|G'|}\sum_{g\in G'}\chi(g)S_1^{b_1}S_2^{b_2}\ldots S_n^{b_n} \tag{2.19}$$

where the sum is over all permutation representations of $g\in G'$ that generate b_1 cycles of length 1, b_2 cycles of length 2, ... , b_n cycles of length n upon its action on the set Ω of the $(n-q)-$ hyperplanes of the 5D-hypercube. Upon construction of the GCCIs for each irreducible representation and each of the $(n-q)-$hyperplane's cycle types shown in Table 2, one can carry out generalized Pólya substitution in the GCCIs for each representation of $S_5[S_2]$ with a multinomial expansion. Let[n] be an ordered partition, also called the composition of n into p parts such that $n_1\geq 0,\ n_2\geq 0,\ \ldots,\ n_p\geq 0,\ \sum_{i=1}^{p} n_i = n$. A multinomial generating function in λs is obtained as

$$(\lambda_1+\lambda_2+\ldots..+\lambda_p)^n = \sum_{[n]}^{p}\begin{pmatrix} & & n & & \\ n_1 & n_2 & . & . & n_p\end{pmatrix}\lambda_1^{n_1}\lambda_2^{n_2}\ldots\ldots..\lambda_{p-1}^{n_{p-1}}\lambda_p^{n_p} \tag{2.20}$$

where $\begin{pmatrix} & & n & & \\ n_1 & n_2 & . & . & n_p\end{pmatrix}$ are multinomials given by

$$\begin{pmatrix} & & n & & \\ n_1 & n_2 & . & . & n_p\end{pmatrix} = \frac{n!}{n_1!n_2!\ldots\ldots n_{p-1}!n_p!} \tag{2.21}$$

Define two sets, the set D which contains a set of $(n-q)-$hyperplanes for a given q to be colored and the set R which contains different colors. Let wi be the weight of each color r in R . The weight of a function f from D to R is defined as

$$W(f) = \prod_{i=1}^{|R|} w(f(d_i)) \tag{2.22}$$

The generating function for each irreducible representation of the nD-hyperoctahedral group is obtained by the substitution as

$$GF^{\chi}(\lambda_1,\lambda_2\ldots.\lambda_p) = P_G^{\chi}\left\{s_k \to \left(w_1^k + w_1^k + \ldots. + w_{p-1}^k + w_p^k\right)\right\} \tag{2.23}$$

The above GFs are computed for each irreducible representation of the 5D-hyperoctahedral group. The coefficient of each term ${w_1}^{n1}{w_2}^{n2}\ldots.{w_p}^{np}$ generates the number of functions in the set R^D that transform according to the irreducible representation Γ with character χ . For the special case of the totally symmetric irreducible representation A_1 , the GF becomes the ordinary Pólya's theorem, thus enumerating the number of equivalence classes of colorings.

In the case of hyperplanes of nD-hypercubes the number of $(n-q)-$hyperplanes for a given value of q increase as $\binom{n}{q}2^q$ and thus, for example, a 10D-hypercube would have 13,440 4-hyperplanes (q=6) and 15,360 3-hyperplanes (q=7). Consequently, as the multinomial generators explode in astronomical proportions for such large sets, it is practically not possible to consider more than 2 colors in the set R or only 2-colorings for larger hypercubes are feasible. We have developed Fortran '95 codes that compute the cycle types for all hyperplanes using the Möbius method, the character tables and then finally the generating functions for 2-colorings of various $(n-q)-$hyperplanes of the hypercube. All of the arithmetic were carried out in Real*16 or quadruple precision arithmetic and thus we can rely on an accuracy of up to 32 digits, which appears to suffice for 2-colorings for all possible distribution of colors up to six-dimensional cases. However, for larger cases either only first k coefficients that contain 32 or fewer digits be considered for colorings or the codes have to be enhanced with multiple arrays to store beyond 32 digits as presently most compilers handle at most quadruple precision for real numbers. The special cases of multinomials for 2 colorings were computed in a single step for 2-colorings recursively, and stored in memory for computations of each of the monomials, sorting and collection of the coefficients for the final GF without computation of any factorials to save time. Moreover the expansion of multinomials, sorting and collection of coefficients is done only for the A_1 IR and for the remaining IRs the computed terms for each cycle type of A_1 are used. For the present case of the 5D-hypercube we were able to compute all of the possible 2-colorings for all $(n-q)-$hyperplanes as discussed in the next section within real quadruple precision or REAL*16 precision.

3. Results and discussions

As seen from Table 2, the 5D-hypercube contains 5 different hyperplanes, where $q = 1$ to 5, represent tesseracts, cells, 3-faces, edges and vertices, respectively. Owing to the simplicity of $q = 1$which yields only 10 tesseracts that can be represented by 10 vertices of a graph (Fig. 1) and as these 10 vertices also represent the protons of the fully nonrigid water pentamer $(H_2O)_5$, colorings of these ten vertices have been considered previously [64] and thus we shall not repeat the results. However for other q values with the exception of $q = 5$ (vertices) restricted to A_1 , complete enumeration results for all IRs have not been considered previously. We note that the problem of coloring the vertices of the hypercube is equivalent to generating the equivalence classes of 2^n Boolean functions of a $n-$ dimensional hypercube which is of considerable interest [24]-[26], [28], [29]. Previous exhaustive combinatorial enumerations for the 4d-hypercube for all irreducible representations have been considered by the current author recently [11].

Tables 3-6 show the unique terms for 2-colorings of $(5-q)-$ hyperplanes $q = 2-5$, respectively for the 5D-hypercube. In all these tables irreducible representations of the $S_5[S_2]$ group are denoted as A_1 to A_{36}, respectively. We note that only A_1 to A_4 are one-dimensional, $A_5 - A_8$ are 4-dimensional, $A_9 - A_{16}$ are five-dimensional, $A_{17} - A_{18}$ are 6-dimensional, $A_{19} - A_{28}$ are 10-dimensional, $A_{29} - A_{32}$ are 15-dimensional, $A_{33} - A_{36}$ are 20-dimensional IRs of the 5d-hypercube. The number of colorings that transform according to the irreducible representation $A_i\,(i = 1-36)$ are shown in Tables 3-6 for unique partition of colors. For example, the number of colorings which transform as the given irreducible representation in a row and contain 35 red colors and 5 green colors for coloring the cells $(q = 2)$ of the 5D-hyercube are shown in Table 3 in the fifth column. We use the notation $[\lambda]$ to denote the unique partitions for the colorings and in order to save space, owing to the symmetry of binomial numbers the results are shown only for $[\,n_1, n_2]$ where n_1 GE n_2 as the other case $(n_2,\, n_1)$ is equivalent to $(n_1,\, n_2)$. As can be seen from Table 3, there are 1, 1, 5, 18, 84, and 362 colorings that transform as A_1 for 40 reds, 39 reds, 38 reds, 37 reds, 36 reds, and 35 reds (remaining 40-red = greens), respectively. The number of colorings that transform as A_1 irreducible representation is simply the number of equivalence classes under the action of the 5D-hyperoctahedal group on the cells for Table 3. Thus from Table 3, there are 36,600,432 ways to color the cells of the 5D-hypercube with 20 red colors and 20 green colors.- a result that is not known up to now. In the mathematics literature, the focus has been often on the total number of equivalence classes for the vertex colorings as opposed to the detailed enumeration for each possible distribution of colors $(n_1,\, n_2)$ that we show in Table 3. The results in Tables 3-5 have not been obtained before.

As can be seen from Table 4 the number of equivalence classes for coloring faces $(q = 3)$ of the 5D-hypercube are 1, 8, 54, 633 and 7287 for 1, 2, 3, 4, 5 green colors (remaining being red colors), respectively. The fact that the number of equivalence classes for 79 red and 1 green colors for the face colorings is one implies all the faces of the hypercube are equivalent, a result that is expected. As seen from Table 4, the number of equivalence classes (A_1 colorings) for 40 red and 40 green colors is a result that is unknown up to now. The numbers for other 35 irreducible representations $(A_2 - A_{36})$ correspond to the number of functions out of 2^{80} functions in the set R^D that transform as the corresponding irreducible representation. Consequently, the numbers in each row multiplied by the dimensions of the corresponding irreducible representations for all 36 IRs and all color distributions, that is, doubling each number in Table 4 for $[\lambda]$ with the exception [40 40] we obtain 2^{80} which is the total number of functions in the set of all maps. Likewise the sum of twice all numbers for the A_1 representation with the exception that [40 40] is added only once, generates the total number of equivalence classes. This result can also be directly obtained from the cycle index for the A_1 IR by replacing every x_k by 2. That is, for the results in Table 3, total equivalence classes count is given by

$$\begin{aligned} I(faces;2) &= \tfrac{1}{3840}\left\{\begin{array}{l} 2^{80}+5\times 2^{56}+10\times 2^{44}+10\times 2^{40}+5\times 2^{40}+1\times 2^{40}+20\times 2^{50}+20\times 2^{26} \\ +60\times 2^{44}+60\times 2^{22}+60\times 2^{42}+60\times 2^{22}+20\times 2^{40}+20\times 2^{22}+80\times 2^{28} \\ +80\times 2^{14}+160\times 2^{20}+160\times 2^{14}+80\times 2^{16}+80\times 2^{14}+60\times 2^{44}+120\times 2^{24} \\ +60\times 2^{40}+120\times 2^{22}+60\times 2^{20}+60\times 2^{20}+240\times 2^{22}+240\times 2^{10}+240\times 2^{22} \\ +240\times 2^{10}+160\times 2^{18}+160\times 2^{14}+160\times 2^{10}+160\times 2^{8}+384\times 2^{16}+384\times 2^{8} \end{array}\right\} \\ &= 314,824,532,572,147,370,464 \end{aligned}$$

The result thus obtained agrees with the computer code that independently computed the sum of all coefficients in the generating function, thus providing independent validation of our results. Consequently, the total number of equivalence classes for the face colorings of 5D-hypercube with 2 colors is 314,824,532,572,147,370,464.
As seen from Table 5, there are also 80 edges for the 5D-hypercube, which happens to be coincidentally same as the number of faces. We have provide all 2-coloring distributions in Table 5 and as these numbers contain less than 32-33, digits all results are computed accurately within the quadruple precision arithmetic. Once again from Table 5, we infer there are 1, 8, 50, 608, 7092 colorings for 1, 2, 3, 4, 5 green colors (remaining reds) for the edge colorings of the 5D-cube.Although the first two numbers coincide with the face coloring distribution from the third number onwards all the results differ. In general, the number of face colorings is larger than the number of edge colorings for the same color distribution. Thus we obtain $27,996,670,589,987,902,014$ as the number of equivalence classes for edge colorings with 40 red colors and 40 green colors while the corresponding number for face colorings is $27,996,675,954,790,045,648$ with 40 reds and 40 greens. The total number of equivalence classes for edge colorings of the 5D-hypercube with 2 colors is 314,824,456,456,819,827,136 which can be obtained in two independent ways as demonstrated for the face colorings.
Table 6 shows the vertex colorings for all irreducible representations for the 5D-hypercube. The results for the vertex colorings of the 5D-hypercube for the A_1 IR have been obtained previously by Chen and Guo [29] using a completely different method of generating the cycle index of the group. The results obtained by Chen and Guo [29] for the equivalence classes correspond to our numbers in Table 6 for the A_1 IR. Chen and Guo [29] obtain these numbers as 1, 1, 5, 29, 47, 131, 472, 1326, 3779, 9013, 19,963, 38,073, 65,664, 98,804, 133,576, 158,658, for greens varying from 0 to 17 (remaining red). The corresponding results that we obtain in Table 6 for the same color distribution for the vertex colorings of the 5D-hypercube are 1, 1, 5, 10, 47, 131, 472, 1326, 3779, 9013, 19,963, 38,073, 65,664, 98,804, 133,576, 158,658, respectively. In addition we obtain the number of equivalence classes for 40 red and 40 green as 169,112 that Chen and Guo [29] did not report. Evidently the number of equivalence classes reported for 3 green colors by Chen and Guo [29] as 29 is not correct, and it disagrees with our result of 10 equivalence classes for the same color distribution. Furthermore the total number of equivalence classes that we obtain by adding doubles of all the numbers for A_1 in Table 6 except that [16 16] is counted once, is 1,228,158 which clearly does not agree with the results of Chen and Guo [29] although the total number directly obtained from their cycle index by replacing every x_k with 2 agrees with our result of 1,228,158. Therefore we conclude that only the number reported for 3 green colors as 29 by Chen and Guo [29] must be incorrect. Moreover, our result of 1,228,158 for the total number of equivalence classes for 2 colors agrees with the number reported by Perez-Agulia [26] but differs from the result of Aichholzer [25] who has obtained it as 1,226,525. The difference was reconciled by Perez-Agulia [26] with the explanation that vertices with 0 to 4 polytopes were treated differently by Aichholzer [25].

4. Chiral and alternating colorings, chemical and biological applications

As discussed in the previous section the numbers enumerated for the A_1 representation (totally symmetric) for the partition $[n_1,n_2]$ of colors enumerates the number of Pólya equivalence classes for the coloring of $(n-q)-$ hyperplanes with n_1 colors of one kind and n_2 colors of another kind. A geometrical or physical interpretation for the numbers enumerated for other irreducible representations in Tables is that these numbers enumerate the number of functions that transform as the IR among the set of all R^D functions from the set D to R. That is, for hypercube's binary colorings there are 2^n such functions where n is the number of $(n-q)$ $-$hyperplanes for a given q . Thus the number of irreducible representations in Tables 3 to 6 for a given color partition $[n_1 n_2]$ gives the number of possible symmetry-adapted orthogonal functions generated from the set R^D of 2^n functions. In addition to this interpretation the numbers enumerated for irreducible representations other than A_1 can yield information on different aspects of colorings such as chirality, alternation and various other applications.
Chirality arises in a coloring if the mirror image of the coloring is not superimposable on the original coloring. Objects are chiral when they have handedness such as shoes, hands, feet, gloves, etc. In such cases, the mirror images of the object cannot be converted into the original object by any proper rotations in the physical space. The term proper rotation refers to a rotation by an angle $2\pi/m$ for a natural number m around a specified axis of rotation denoted by a C_m axis of rotation. The set of such proper rotations that leave the object in the set D invariant constitute a subgroup that we call the rotational subgroup of the nD-hyperoctahedral group and it is comprised of $2^{(n-1)}x\, n!$ operations for the nD-hypercube. While such rotational operations are readily identified for a regular three-dimensional square or a cube shown in Table 1, this is less transparent for the higher dimensional hypercubes. As seen from Table 1, for each conjugacy class we can assign a rotational operation or mirror plane or a composite improper rotation by simply applying the operation on the vertices or edges or faces of the cube and gathering the various orbits generated upon the action of the operation. An improper axis of rotation, denoted is defined as the product $C_n\sigma_h$, or $\sigma_h C_n$ where the σ_h operation is a mirror plane perpendicular to the C_n axis. For a cube these operations are assigned to the various matrix conjugacy classes in Table 1 based on the permutation's orbits it generates upon its action on the vertices or edges or faces of the 3D cube. The proper rotations for an nD-hypercube can be obtained from the $2\times n$ matrix of the corresponding conjugacy class by considering the non-zero column's place values. That is, a conjugacy class with matrix $[a_{ik}]$ is a proper rotation if and only if

$$\sum_{k}^{even} a_{1k} + \sum_{k}^{odd} a_{2k}$$

is even, where the first sum is restricted to even ks while the second to odd ks. If the above sum is odd then the operation corresponding to the $2\times n$ matrix of the conjugacy class is an improper axis of rotation, where a special case of an improper axis may also be a mirror plane of symmetry or a center of inversion. This procedure can be applied to higher dimensional cubes, and thus in Table 2 we have identified each proper rotation of the 5D-hypercube by placing the label R next to the conjugacy class. If the label R is absent it means that the conjugacy class represents an improper axis of rotation. Chirality can then be determined by the definition that an object is chiral if it does not possess an improper axis of rotation. Evidently uncolored 5D-hypercube or a 3D-cube is not chiral because of the presence of improper axes of rotations. However, once the (n-q)-hyperplanes are colored some of the colorings for certain distribution of colors may become chiral. Tables 3-6 that we have constructed enumerate and identify these chiral colorings. The chiral colorings are obtained by stipulating that the functions in R^D for the coloring distribution $[n_1\ n_2]$ must transform in accord to the irreducible representation of chirality. This irreducible representation for chirality of the nD-hypercube is rigorously identified as the uni-dimensional IR that has +1 character values for all proper rotations of the nD-hyperoctahedral group and -1 for all improper rotations. By examining the character values for the uni-dimensional representations for the 5D-hypercube we identify this IR as A_2 representation, and thus in Tables 3-5 they are identified with * in these tables. Consequently, the number of chiral colorings for a given distribution of colors $[n_1\ n_2]$is enumerated by the numbers for the A_2 row in Tables 3-6 for various $(n-q)-$ hyperplanes.

As seen from Table 3, the first few numbers or the A_2 representation are $0,0,0,0,6,84,657,3750,16,898,63,366,203,095,565,964,...$ suggesting that coloring 40 cells of the 5D-hypercube do not produce any chiral colorings for 40 reds & 0 greens, 39 reds & 1 green, 38 reds & 2 green, 37 reds & 3 greens, and in order to produce a chiral coloring one needs at least 4 green colors and remaining 36 red, and there are exactly 6 such colorings which are chiral. That is, among the 84 equivalence classes of cell colorings for [36 4] partition of colors there are exactly six chiral pairs in that mirror images of a chiral coloring is not superimposable on the original coloring. In order to illustrate this further consider a regular 3D cube. Among the total of 14 equivalence classes produced for all 2-colorings of the vertices of a 3D cube, only one coloring is chiral and all remaining colorings are achiral. The chiral coloring is shown in Figure 2.

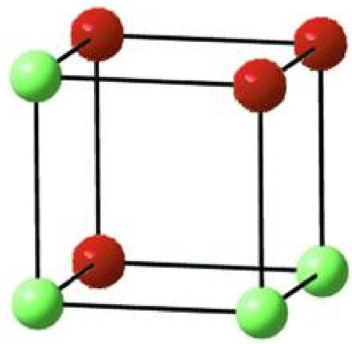

Figure 4.1: The only chiral coloring among 14 equivalence classes of 2-colorings of vertices of a cube. This is enumerated as the number of A_{1u} irreducible representations for the 2-colorings. For the 5D-hypercube the first chiral coloring appears for 4 greens and 28 reds. There are 2, 26, 148, 653, 2218, 6300, 14972, 30,730, and 54,528 such chiral colorings for 4, 5, 6, 7, 8, 9, 10, 11, and 12 green colors (remaining red), respectively for the 2-colorings of the vertices of the 5D-hypercube as enumerated by the A_2 chiral representation of the 5D-hypercube..

The numbers of chiral colorings for face-colorings of the 5D-hypercube are given by the numbers of the A_2 IR in Table 4, and it can be seen as 14, 326, 5722, 74973, 811,527, 7,477,975 and 60,113,621 for 3, 4, 5, 6, 7, 8, and 9 greens (remaining reds), respectively. The corresponding results for the edge 2-colorings are 12, 330, 5782, 75,369, 815,762, 60,219,494 and 428,191,237 for 3, 4, 5, 6, 7, 8, and 9 greens (remaining reds), respectively. Finally as can be seen from Table 6, 2-colorings of the vertices of the 5D-hypercube produce 2, 26, 148, 653, 2218, 6300, 14,972, 30,730, and 54,528 chiral colorings for 4, 5, 6, 7, 8, 9, 10, 11, and 12 green colors (remaining red), respectively. Thus in order to produce a chiral coloring of 2-coloring of the vertices of a 5D-hypercube one needs at least 4 colors of one kind and 28 colors of another kind, and there are 2 such chiral colorings for [28 4] color distribution.

The alternating irreducible representation is defined as the one that exhibits +1 character values for even permutations of q=1 (n-1)hyperplanes and -1 for the odd permutations. The set of all even permutations form the alternating subgroup of the hypercube group. The alternating representation plays an important role in the quantum chemical classification of the rovibronic total wave functions of fermions as such wave functions for fermions must transform as the alternating IR in order to comply with the Pauli Principle. For the 5D-hypercube the uni-dimensional alternating IR is the A_3 representation in Table 3-6. Thus the 2-colorings enumerated for the A_3 representation provides important information on the nuclear spin functions of rovibronic levels and nuclear spin statistical weights of fermionic particles of molecules, for example, water pentamer. We thus point out that these combinatorial enumerations aid in the analysis of experimental spectroscopic studies of weakly-bound van der waals clusters and molecular clusters of polar molecules such as ammoniated ammonia, $(H_2O)_n$, $(NH_3)_n$ [50], [64], [62] etc., as such clusters exhibit potential energy surfaces with multiple valleys separated by surmountable mountains, and consequently, these molecular clusters undergo rapid tunneling motions. Hence these tunneling motions that occur rapidly at higher room temperatures result in the splittings of the rovibronic levels to tunneling levels. Consequently, the interpretation of the rovibronic spectra of these molecular clusters requires hypercube colorings and detailed analysis for all IRs.

Finally we would like to point out applications to biology in the context of genetic regulatory network and phylogeny. The phylogenic trees are recursive in nature and they are special cases of Cayley trees and thus the automorphism groups and colorings of phylogenic trees require nested nD-hypergroups and wreath products. Likewise, in genetics it has been shown that canalization or control of one genetic trait by another trait of genetic regulatory networks is important in evolutionary processes, and such networks are represented by nD-hypercubes where the vertices of the nD-hypercube represent the 2^n possible Boolean functions for n traits. Reichhardt and Bassler [34] have shown the connection between 2-colorings of an nD-hypercube and genetic regulatory pathways, and the necessity to classify the 2-colorings of the vertices into equivalence classes in order to generate a smaller clustering subsets on the basis of equivalence classes thus enumerated for the 2-colorings of the vertices of the nD-hypercube. Thus the properties of any representative function in a class would have the same genetic expression as any other function in the equivalence class thereby reducing the amount of computations. The question of if chirality in colorings would have any implication in the probability of producing chiral traits and thus biological evolutionary implication of chirality has not been visited thus far.

5. Conclusion

Combinatorial enumeration of 2-colorings for all irreducible representations and all hyperplanes for were considered for a 5D-hypercube. The techniques involved Möbius inversion combined with generalized character cycle indices for all 36 irreducible representations of the 5D-hypercube. We also discussed applications chirality, alternation of colorings in the equivalence class. Applications to genetics and molecular spectroscopy were pointed out. As nD-hypercube colorings explode combinatorially in astronomical proportions, it remains to be seen how well the techniques will computationally scale and work for higher dimensional hypercubes.

Table 1: Conjugacy Classes, polynomials, cycle types of a regular cube or 3D-cube with group $S_3[S_2]$

CC	$\|CC\|$	O	$F_d(x)$	$q=1$ (face)	$q=2$ (edge)	$q=3$ (Vert)
$\begin{pmatrix}3&0&0\\0&0&0\end{pmatrix}$	1	E	$F_1(x)=(1+2x)^3$	1^6	1^{12}	1^8
$\begin{pmatrix}2&0&0\\1&0&0\end{pmatrix}$	3	σ_h	$F_1(x)=(1+2x)^2$ $F_2(x)=(1+2x)^3$	$1^4 2$	$1^4 2^4$	2^4
$\begin{pmatrix}1&0&0\\2&0&0\end{pmatrix}$	3	C_4^2	$F_1(x)=(1+2x)$ $F_2(x)=(1+2x)^3$	$1^2 2^2$	2^6	2^4
$\begin{pmatrix}0&0&0\\3&0&0\end{pmatrix}$	1	i	$F_1(x)=1$ $F_2(x)=(1+2x)^3$	2^3	2^6	2^4
$\begin{pmatrix}1&1&0\\0&0&0\end{pmatrix}$	6	σ_d	$F_1(x)=(1+2x)(1+2x^2)$ $F_2(x)=(1+2x)^3$	$1^2 2^2$	$1^2 2^5$	$1^4 2^2$
$\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}$	6	C_4	$F_1(x)=(1+2x)$ $F_2(x)=(1+2x)$ $F_4(x)=(1+2x)^3$	$1^2 4$	4^3	4^2
$\begin{pmatrix}0&1&0\\1&0&0\end{pmatrix}$	6	C_2	$F_1(x)=(1+2x^2)$ $F_2(x)=(1+2x)^3$	2^3	$1^2 2^5$	2^4
$\begin{pmatrix}0&0&0\\1&1&0\end{pmatrix}$	6	S_4	$F_1(x)=1$ $F_2(x)=(1+2x)$ $F_4(x)=(1+2x)^3$	$2^1 4^1$	4^3	4^2
$\begin{pmatrix}0&0&1\\0&0&0\end{pmatrix}$	8	C_3	$F_1(x)=(1+2x^3)$ $F_3(x)=(1+2x)^3$	3^2	3^4	$1^2 3^2$
$\begin{pmatrix}0&0&0\\0&0&1\end{pmatrix}$	8	S_3	$F_1(x)=1$ $F_2(x)=(1+2x^3)$ $F_3(x)=1$ $F_6(x)=(1+2x)^3$	6	6^2	$2^1\ 6$

Table 2: Conjugacy Classes of $S_5[S_2]$, their orders, F_d polynomials and cycle types generated using Möbius inversion for the 5D-hypercube's five hyperplanes*.

Conj Class C	$\lvert C\rvert$	$F_d(x)$	$q=1$ tes	$q=2$ Cel	$q=3$ fac	$q=4$ ed	$q=5$ Ver
$\begin{pmatrix}5&0&0&0&0\\0&0&0&0&0\end{pmatrix}$	1E	$F_1(x)=(1+2x)^5$	1^{10}	1^{40}	1^{80}	1^{80}	1^{32}
$\begin{pmatrix}4&0&0&0&0\\1&0&0&0&0\end{pmatrix}$	5	$F_1(x)=(1+2x)^4$ $F_2(x)=(1+2x)^5$	$1^8 2$	$1^{24}2^8$	$1^{32}2^{24}$	$1^{16}2^{32}$	2^{16}
$\begin{pmatrix}3&0&0&0&0\\2&0&0&0&0\end{pmatrix}$	10R	$F_1(x)=(1+2x)^3$ $F_2(x)=(1+2x)^5$	$1^8 2^2$	$1^{12}2^{14}$	$1^8 2^{36}$	2^{40}	2^{16}
$\begin{pmatrix}2&0&0&0&0\\3&0&0&0&0\end{pmatrix}$	10	$F_1(x)=(1+2x)^2$ $F_2(x)=(1+2x)^5$	$1^4 2^3$	$1^4 2^{18}$	2^{40}	2^{40}	2^{16}
$\begin{pmatrix}1&0&0&0&0\\4&0&0&0&0\end{pmatrix}$	5R	$F_1(x)=(1+2x)$ $F_2(x)=(1+2x)^5$	$1^2 2^4$	2^{20}	2^{40}	2^{40}	2^{16}
$\begin{pmatrix}0&0&0&0&0\\5&0&0&0&0\end{pmatrix}$	1	$F_1(x)=1$ $F_2(x)=(1+2x)^5$	2^5	2^{20}	2^{40}	2^{40}	2^{16}
$\begin{pmatrix}3&1&0&0&0\\0&0&0&0&0\end{pmatrix}$	20	$F_1(x)=(1+2x)^3(1+2x^2)$ $F_2(x)=(1+2x)^5$	$1^6 2^2$	$1^{14}2^{13}$	$1^{20}2^{30}$	$1^{24}2^{28}$	$1^{16}2^8$
$\begin{pmatrix}3&0&0&0&0\\0&1&0&0&0\end{pmatrix}$	20R	$F_1(x)=(1+2x)^3$ $F_2(x)=(1+2x)^3$ $F_4(x)=(1+2x)^5$	$1^6 4$	$1^{12}4^7$	$1^8 4^{18}$	4^{20}	4^8
$\begin{pmatrix}2&1&0&0&0\\1&0&0&0&0\end{pmatrix}$	60R	$F_1(x)=(1+2x)^2(1+2x^2)$ $F_2(x)=(1+2x)^5$	$1^4 2^3$	$1^{16}2^{17}$	$1^8 2^{36}$	$1^8 2^{36}$	2^{16}
$\begin{pmatrix}2&0&0&0&0\\1&1&0&0&0\end{pmatrix}$	60	$F_1(x)=(1+2x)^2$ $F_2(x)=(1+2x)^3$ $F_4(x)=(1+2x)^5$	$1^4 2^1 4$	$1^4 2^4 4^7$	$2^4 4^{18}$	4^{20}	4^8
$\begin{pmatrix}1&1&0&0&0\\2&0&0&0&0\end{pmatrix}$	60	$F_1(x)=(1+2x)(1+2x^2)$ $F_2(x)=(1+2x)^5$	$1^2 2^4$	$1^2 2^{19}$	$1^4 2^{38}$	2^{40}	2^{16}
$\begin{pmatrix}1&0&0&0&0\\2&1&0&0&0\end{pmatrix}$	60R	$F_1(x)=(1+2x)$ $F_2(x)=(1+2x)^3$ $F_4(x)=(1+2x)^5$	$1^4 2^2 4$	$2^6 4^7$	$2^4 4^{18}$	4^{20}	4^8
$\begin{pmatrix}0&1&0&0&0\\3&0&0&0&0\end{pmatrix}$	20R	$F_1(x)=(1+2x^2)$ $F_2(x)=(1+2x)^5$	2^5	$1^2 2^{19}$	2^{40}	2^{40}	2^{16}
$\begin{pmatrix}0&0&0&0&0\\3&1&0&0&0\end{pmatrix}$	20	$F_1(x)=1$ $F_2(x)=(1+2x)^3$ $F_4(x)=(1+2x)^5$	$2^3 4$	$2^6 4^7$	$2^4 4^{18}$	4^{20}	4^8
$\begin{pmatrix}2&0&1&0&0\\0&0&0&0&0\end{pmatrix}$	80R	$F_1(x)=(1+2x)^2(1+2x^3)$ $F_3(x)=(1+2x)^5$	$1^4 3^2$	$1^4 3^{12}$	$1^2 2^{26}$	$1^8 3^{24}$	$1^8 3^8$
$\begin{pmatrix}2&0&0&0&0\\0&0&1&0&0\end{pmatrix}$	80	$F_1(x)=(1+2x)^2$ $F_2(x)=(1+2x)^2(1+2x^3)$ $F_3(x)=(1+2x)^2$ $F_6(x)=(1+2x)^5$	$1^4 6$	$1^4 6^6$	$2 6^{13}$	$2^4 6^{12}$	$2^4 6^4$
$\begin{pmatrix}1&0&1&0&0\\1&0&0&0&0\end{pmatrix}$	160	$F_1(x)=(1+2x)(1+2x^3)$ $F_2(x)=(1+2x)^2(1+2x^3)$ $F_3(x)=(1+2x)^4$ $F_6(x)=(1+2x)^5$	$1^2 2^1 3^2$	$2^2 3^{10} 6^8$	$1^2 3^8 6^2$	$1^4 2^2 3^4 6^{10}$	$2^4 6^4$

$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}$	160R	$F_1(x) = (1+2x)$ $F_2(x) = (1+2x)^2(1+2x^3)$ $F_3(x) = (1+2x)$ $F_6(x) = (1+2x)^5$	$1^2 2^1 6$	$2^2 6^6$	$2^1 6^{13}$	$2^4 6^{12}$	$2^4 6^4$
$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{pmatrix}$	80R	$F_1(x) = (1+2x^3)$ $F_2(x) = (1+2x)^2(1+2x^3)$ $F_3(x) = (1+2x)^3$ $F_6(x) = (1+2x)^5$	$2^2 3^2$	$2^2 3^4 6^4$	$1^2 3^2 6^{12}$	$2^4 6^{12}$	$2^4 6^4$
$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \end{pmatrix}$	80	$F_1(x) = 1$ $F_2(x) = (1+2x)^2(1+2x^3)$ $F_3(x) = 1$ $F_6(x) = (1+2x)^5$	$2^2 6$	$2^2 6^6$	$2^1 6^{13}$	$2^4 6^{12}$	$2^4 6^4$
$\begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	60R	$F_1(x) = (1+2x)(1+2x^2)^2$ $F_2(x) = (1+2x)^5$	$1^2 2^4$	$1^4 2^{18}$	$1^8 2^{36}$	$1^4 2^{38}$	$1^8 2^{12}$
$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$	120	$F_1(x) = (1+2x)(1+2x^2)$ $F_2(x) = (1+2x)^3$ $F_4(x) = (1+2x)^5$	$1^2 2^2 4$	$1^2 2^5 4^7$	$1^4 2^2 4^{18}$	4^{20}	4^8
$\begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$	60	$F_1(x) = (1+2x^2)^2$ $F_2(x) = (1+2x)^5$	2^5	$1^4 2^{18}$	2^{40}	$1^4 2^{38}$	2^{16}
$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$	120R	$F_1(x) = (1+2x^2)$ $F_2(x) = (1+2x)^3$ $F_4(x) = (1+2x)^5$	$2^3 4$	$1^2 2^5 4^7$	$2^4 4^{18}$	4^{20}	4^8
$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \end{pmatrix}$	60R	$F_1(x) = (1+2x)$ $F_2(x) = (1+2x)$ $F_4(x) = (1+2x)^5$	$1^2 4^2 4$	4^{10}	4^{20}	4^{20}	4^8
$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \end{pmatrix}$	60	$F_1(x) = 1$ $F_2(x) = (1+2x)$ $F_4(x) = (1+2x)^5$	$2^1 4^2$	4^{10}	4^{20}	4^{20}	4^8
$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	240	$F_1(x) = (1+2x)(1+2x^4)$ $F_2(x) = (1+2x)(1+2x^2)^2$ $F_4(x) = (1+2x)^5$	$1^2 4^2 4$	$2^2 4^9$	$2^4 4^{18}$	$1^2 2^1 4^{19}$	$1^4 2^2 4^6$
$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$	240R	$F_1(x) = (1+2x)$ $F_2(x) = (1+2x)$ $F_4(x) = (1+2x)$ $F_8(x) = (1+2x)^5$	$1^2 8$	8^5	8^{10}	8^{10}	8^4
$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$	240R	$F_1(x) = (1+2x)(1+2x^4)$ $F_2(x) = (1+2x)(1+2x^2)^2$ $F_4(x) = (1+2x)^5$	$2^1 4^2$	$2^2 4^9$	$2^4 4^{18}$	$1^2 2^1 4^{19}$	$2^4 4^6$
$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$	240	$F_1(x) = 1$ $F_2(x) = (1+2x)$ $F_4(x) = (1+2x)$ $F_8(x) = (1+2x)^5$	$2^1 8$	8^5	8^{10}	8^{10}	8^4
$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$	160	$F_1(x) = (1+2x^2)(1+2x^3)$ $F_2(x) = (1+2x)^2(1+2x^3)$ $F_3(x) = (1+2x^2)(1+2x)^3$ $F_6(x) = (1+2x)^5$	$2^3 3^2$	$1^2 2^1 3^4 6^4$	$1^2 3^6 6^{10}$	$2^4 3^8 6^8$	$1^4 2^2 3^4 6^2$

$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$	160R	$F_1(x) = (1+2x^2)$ $F_2(x) = (1+2x)^2(1+2x^3)$ $F_3(x) = (1+2x^2)$ $F_6(x) = (1+2x)^5$	$2^2 6$	$1^2 2^1 6^6$	$2^1 6^{13}$	$2^4 6^{12}$	$2^4 6^4$
$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$	160R	$F_1(x) = (1+2x^3)$ $F_2(x) = (1+2x^3)$ $F_3(x) = (1+2x)^3$ $F_4(x) = (1+2x)^2(1+2x^3)$ $F_6(x) = (1+2x)^3$ $F_{12}(x) = (1+2x)^5$	$4^1 3^2$	$3^4 4^1 12^2$	$1^2 3^2 12^6$	$4^2 12^6$	$4^2 12^2$
$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$	160	$F_1(x) = 1$ $F_2(x) = (1+2x^3)$ $F_3(x) = 1$ $F_4(x) = (1+2x)^2(1+2x^3)$ $F_6(x) = (1+2x)^3$ $F_{12}(x) = (1+2x)^5$	$4\ 6$	$4\ 6^2 12^2$	$2\ 6 12^6$	$4^2 12^6$	$4^2 12^2$
$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	384R	$F_1(x) = (1+2x^5)$ $F_5(x) = (1+2x)^5$	5^2	5^8	5^{16}	5^{16}	$1^2 5^6$
$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	384	$F_1(x) = 1$ $F_2(x) = (1+2x^5)$ $F_5(x) = 1$ $F_{10}(x) = (1+2x)^5$	10	10^4	10^8	10^8	$2^1 10^3$

*Label R identifies proper rotations.

Table 3: 2-colorings of $q = 2$ or 3-hyerplnes (cells) of 5D-hhypercube*

$[\lambda]$	40	39 1	38 2	37 3	36 4	35 5
A_1	1	1	5	18	84	362
A_2*	0	0	0	0	6	84
$A_3\dagger$	0	0	0	1	17	130
A_4	0	0	0	3	29	218
A_5	0	0	0	14	132	912
A_6	0	1	8	41	234	1198
A_7	0	0	0	1	33	376
A_8	0	0	0	3	53	466
A_9	0	0	3	28	211	1266
A_{10}	0	1	7	43	261	1410
A_{11}	0	0	0	2	46	502
A_{12}	0	0	0	3	57	548
A_{13}	0	0	1	11	105	753
A_{14}	0	0	0	4	59	570
A_{15}	0	1	5	36	217	1247
A_{16}	0	0	0	10	130	958
A_{17}	0	0	3	34	253	1534
A_{18}	0	0	0	3	63	632
A_{19}	0	0	1	20	225	1705
A_{20}	0	0	0	19	231	1741
A_{21}	0	1	7	48	335	2060
A_{22}	0	10	2	30	266	1853
A_{23}	0	0	0	11	161	1394
A_{24}	0	0	1	16	181	1454
A_{25}	0	0	2	27	237	1684
A_{26}	0	0	1	22	217	1624
A_{27}	0	0	1	14	158	1315
A_{28}	0	0	4	44	341	2197
A_{29}	0	0	0	11	191	1808
A_{30}	0	0	0	18	232	1991
A_{31}	0	0	4	54	471	3155
A_{32}	0	1	9	80	558	3444
A_{33}	0	0	3	50	489	3556
A_{34}	0	0	6	66	562	3797
A_{35}	0	0	1	32	376	3012
A_{36}	0	0	3	40	414	3130
$[\lambda]$	34 6	33 7	32 8	31 9	30 10	29 11
A_1	1608	6549	24447	81523	243027	645920
A_2*	657	3750	16898	63366	203095	565964
$A_3\dagger$	820	4201	18036	65883	208248	575519
A_4	1196	5575	22187	76923	234085	630118
A_5	4957	22752	89932	310271	941691	2530274
A_6	5764	24690	94419	319457	959523	2561868
A_7	2788	15437	68714	255963	817470	2273349
A_8	3112	16337	70988	260991	827766	2292449
A_9	6548	29276	114337	391745	1184645	3176086
A_{10}	6951	30250	116572	396345	1193551	3191888
A_{11}	3603	19622	86732	321822	1025657	2848796
A_{12}	3766	20073	87870	324339	1030810	2858351
A_{13}	4505	22424	94334	340422	1066636	2931379
A_{14}	3902	20774	90308	331592	1048890	2898671

A_{15}	6315	28332	111060	382756	1162511	3128715
A_{16}	5519	26226	106192	372241	1142010	3091274
A_{17}	7917	35318	137717	471282	1424118	3816104
A_{18}	4423	23823	104768	387705	1233905	3424315
A_{19}	10100	49179	202674	719261	2225769	6060963
A_{20}	10246	49608	203802	721773	2231003	6070695
A_{21}	11143	51877	209058	732893	2252661	6109809
A_{22}	10559	50479	205914	726505	2240491	6088449
A_{23}	8893	45231	191440	690879	2161351	5928638
A_{24}	9081	45720	192650	693498	2166648	5938361
A_{25}	9791	47669	197270	703613	2186655	5975122
A_{26}	9603	47180	196060	700994	2181358	5965399
A_{27}	8394	43167	184598	671959	2115409	5829890
A_{28}	11821	54521	217202	754936	2304404	6219829
A_{29}	12097	63479	273932	1001661	3160917	8722835
A_{30}	12691	65129	277938	1010491	3178627	8755543
A_{31}	17340	80747	323394	1127177	3446414	9311103
A_{32}	18136	82853	328262	1137692	3466915	9348544
A_{33}	20657	99644	408572	1445748	4466210	12149350
A_{34}	21383	101463	412836	1454636	4483602	12180432
A_{35}	18488	92296	387472	1391838	4342653	11893939
A_{36}	18860	93370	389890	1397068	4353235	11913375
$[\lambda]$	28 12	27 13	26 14	25 15	24 16	20 20
A_1	1534959	3268238	6253840	10780533	16780905	36600432
A_2*	1387615	3018198	5860684	10206958	16001831	35267044
$A_3\dagger$	1404093	3044481	5899917	10261735	16073555	35382134
A_4	1508474	3227163	6193673	10698058	16674124	36432620
A_5	6051057	12935884	24815540	42849105	66771193	145850208
A_6	6103944	13018005	24935767	43014020	66984612	146185674
A_7	5566873	12098955	23481819	40882439	64078845	141182942
A_8	5599815	12151509	23560277	40991977	64222269	141413110
A_9	7585897	16203956	31069136	53629419	83551831	182450208
A_{10}	7612322	16245031	31129219	53711894	83658502	182617894
A_{11}	6970887	15143304	29381578	51144016	80152173	176564772
A_{12}	6987365	15169587	29420811	51198793	80223897	176679862
A_{13}	7122810	15401876	29787455	51737069	80956494	177940894
A_{14}	7066962	15313232	29655841	51554067	80717484	177559178
A_{15}	4793096	16040561	30802821	53232534	83001076	181479598
A_{16}	7430012	15940878	30655982	53029221	82736568	181059380
A_{17}	9111568	19458488	37303794	64384562	100300776	219002868
A_{18}	8374980	18187785	35281495	61405751	96225728	211946906
A_{19}	14630010	31481747	60673483	105113023	164178470	359867382
A_{20}	14646966	31508798	60714034	105169696	164252800	359986806
A_{21}	14712710	31612100	60865880	105379320	164525016	360417862
A_{22}	14677526	31557917	60787385	105272259	164387422	360203692
A_{23}	14381627	31054027	59993215	104112178	162809521	357499270
A_{24}	14398357	31080628	60032854	104167367	162881749	357614990
A_{25}	14460573	31179079	60178307	104369056	163144521	358033038
A_{26}	14443843	31152478	60138668	104313867	163072293	357917318
A_{27}	14189604	30714843	59442940	103290802	161673524	355499400
A_{28}	14922940	31981159	61458432	106261406	165737190	362538286
A_{29}	21247566	46014649	89080019	154819871	242359594	533011478
A_{30}	21303354	46103293	89211549	155002873	242598504	533393068
A_{31}	22352952	47922037	92114414	159290627	248473758	543597666
A_{32}	22416036	48021720	92261253	159493940	248738266	544017884
A_{33}	29307474	63039544	121460718	210385140	328565692	720070782
A_{34}	29359594	63120751	121579741	210548843	328777586	720404344
A_{35}	28825377	62206368	120131677	208425861	325881583	715416208
A_{36}	28858823	62259556	120210947	208536219	326026015	715647636

∗Identifies Chiral Representation,
†Identifies Alternating Representation

Table 4: 2-colorings of 5D-hypercube: q=3 or 2-hyperplanes(faces)

$[\lambda]$	80 0	79 1	78 2	77 3	76 4	75 5
A_1	1	1	8	54	633	7287
A_2*	0	0	0	14	326	5722
$A_3\dagger$	0	0	1	2	408	699
A_4	0	0	0	19	418	661
A_5	0	0	1	86	1724	25905
A_6	0	1	14	154	2138	27755
A_7	0	0	0	48	1329	22923
A_8	0	0	2	71	1491	23876
A_9	0	0	8	136	2349	33188
A_{10}	0	1	14	171	2552	34114
A_{11}	0	0	1	73	1735	29121
A_{12}	0	0	2	85	1817	29598
A_{13}	0	0	6	110	2060	30896
A_{14}	0	0	1	73	1771	29392
A_{15}	0	1	10	168	2435	33702
A_{16}	0	0	3	106	2090	31741
A_{17}	0	0	7	167	2811	40020
A_{18}	0	0	1	79	2086	34886
A_{19}	0	0	7	201	4067	62428
A_{20}	0	0	6	213	4117	62905
A_{21}	0	1	18	275	4557	64866
A_{22}	0	0	6	220	4201	6500
A_{23}	0	0	4	173	3807	60718
A_{24}	0	0	4	165	3833	60755
A_{25}	0	1	11	245	4232	6148
A_{26}	0	0	8	210	4090	62222
A_{27}	0	0	7	180	3825	60285
A_{28}	0	1	13	271	4519	65440
A_{29}	0	0	4	233	5451	89243
A_{30}	0	0	7	270	5728	90747
A_{31}	0	0	14	354	6550	96726
A_{32}	0	1	21	416	6895	98687
A_{33}	0	0	13	421	8268	125928
A_{34}	0	1	24	487	8672	127770
A_{35}	0	0	12	381	7893	122938
A_{36}	0	0	15	406	8057	123899
$[\lambda]$	74 6	73 7	72 8	71 9	70 10	69 11
A_1	83555	849445	7641565	60729304	429970617	2732388768
A_2*	74973	811527	7477975	60113621	427758604	2725189869
$A_3\dagger$	77230	821376	7515124	60245702	428179564	2726468083
A_4	79347	833673	7583400	60540511	429376647	2730690404
A_5	319235	3344486	30366992	242293889	1717899937	10924039594
A_6	327603	3376017	30483176	242671455	1719087495	10927436302
A_7	301055	3250060	29935770	240529874	1711337285	10901617831
A_8	305566	3269746	30010065	240794016	1712179165	10904174239
A_9	402754	4193869	38008521	303022994	2147870156	13656428163
A_{10}	406924	4209641	38066550	303211787	2148463784	13658126527
A_{11}	378269	4071401	37450878	300775427	2139516551	13628085765
A_{12}	380526	4081250	37488027	300907508	2139937511	13629363979

A_{13}	388244	4115896	37642667	301493236	2142078579	13636348861
A_{14}	380718	4085029	37522094	301081473	2140734613	13632382149
A_{15}	403325	4194147	37983313	302887223	2147152220	13653690659
A_{16}	394699	4157315	37849447	302418655	2145690306	13649303434
A_{17}	483936	5037385	45625203	363695529	2577640232	16388396724
A_{18}	454418	4886903	44952736	360964567	2567578338	16354134118
A_{19}	782696	8276912	75522353	604085534	4288538040	27288670143
A_{20}	784537	8286761	75555698	604217615	4288931526	27289948357
A_{21}	794398	8323593	75700964	604686183	4290475691	27294335582
A_{22}	787906	8301941	75618251	604440771	4289681978	27292217431
A_{23}	772614	8226772	75295921	603175242	4285167659	27277249393
A_{24}	773790	8230741	75319810	603250704	4285470842	27278107820
A_{25}	783478	8273411	75466888	603775798	4287050362	27282914469
A_{26}	780121	8257639	75416431	603587005	4286511454	27281216105
A_{27}	768923	8200847	75164722	602574387	4282812548	27268730688
A_{28}	797985	8351384	75832721	605305556	4292841882	27302993771
A_{29}	1147363	12280002	112662231	903599298	6423347475	40900693107
A_{30}	1154851	12310869	112782678	904011061	6424691099	40904659819
A_{31}	1191584	12502770	113668866	907667442	6438414240	40951876998
A_{32}	1200210	12539602	113802732	908136010	6439876154	40956264223
A_{33}	1570592	16578830	151140594	1208526176	8578219760	54580887445
A_{34}	1578916	16610319	151256643	1208903649	8579406919	54584283790
A_{35}	1552712	16484362	150712329	1206762068	8571678755	54558465319
A_{36}	1557242	16504090	150786672	1207026303	8572520806	54561022090

$[\lambda]$	44 36	43 37	42 38	41 39	40 40
A_1	18847863525339251552	22413675116856521554	25362842575806673932	27313830262039356344	27996675954790045648
A_2*	18847852585019852784	22413662952649979772	25362829447471548304	27313816527678832042	27996662005552559820
$A_3\dagger$	18847853190803004294	22413663622117296124	25362830164050160934	27313817276349083728	27996662763315380740
A_4	18847862859627984748	22413674384398836626	25362841789323193438	27313829443562144630	27996675123446791678
A_5	75391452039570414338	89654698207061418894	101451367868400802692	109255318522918146262	111986701245805189142
A_6	75391453370992774196	89654699671976785110	101451369441367576670	109255320159872567790	111986702908491503538
A_7	75391410895382419988	89654652417075817924	101451318447499755284	109255266789578133400	111986648717878780904
A_8	75391412106948721622	89654653756010447008	101451319880656979158	109255268286918634892	111986650233404418984
A_9	94239315564909056836	112068373323916723632	126814210444206867570	136569148784956847548	139983377200593924674
A_{10}	94239316230620152144	112068374056374408560	126814211230690163308	136569149603434059262	139983378031936988900
A_{11}	94239264086184823660	112068316039191912380	126814148611549315596	136569084065926569990	139983311481192867368
A_{12}	94239264691967975170	112068316708659228732	126814149328127928226	136569084814596821676	139983312238955688288
A_{13}	94239275611010301062	112068328846778722822	126814162431677508060	136569098520092612896	139983326162116863420
A_{14}	94239273757571746680	112068326806566039966	126814160239766081968	136569096238736595918	139983323844407910574
A_{15}	94239306532959553232	112068363268005984602	126814199571879555474	136569137407356432010	139983365636539659632
A_{16}	94239304629041322386	112068361164802814868	126814197321003585678	136569135056193326978	139983363256737576052
A_{17}	113087179025595952234	134482048377782845824	152177052944638511992	163882978977189213078	167980053076059861824
A_{18}	113087117226508284532	134481979598318989036	152176978716635025204	163882901272468852522	167979974182415265728
A_{19}	188478589300930046414	224136700109485934928	253628370658848001576	273138245000424414272	279966701018560496814
A_{20}	188478589901989013044	224136700778953251280	253628371369956483346	273138245749094665958	279966701770579703858
A_{21}	188478591820079755620	224136702882156421014	253628373637242754192	273138248100257770990	279966704167612580446
A_{22}	188478591089716591170	224136702086708485832	253628372775384675292	273138247211973727294	279966703256945725386
A_{23}	188478568698399675330	224136677182998016880	253628345912898114958	273138219098850995800	279966674717232831548
A_{24}	188478569253703298004	224136677789475082978	253628346570512276262	273138219777714415504	279966675412902686308
A_{25}	188478571208100940998	224136679955668384664	253628348880352605006	273138222198684224492	279966677854797606408
A_{26}	188478570551838143964	224136679223210699736	253628348104809448672	273138221380207012778	279966677034941689648
A_{27}	188478549368580840744	224136655653342348792	253628322671442383030	273138194758827908970	279966650006522174306
A_{28}	188478611161999668620	224136724432806385474	253628396892881934154	273138272463548459144	279966728893274635996
A_{29}	282717823061500661778	336204982396918169290	380442482835833897378	409707290927757703692	419949973771606452858
A_{30}	282717824914939044664	336204984437130852146	380442485027745138714	409707293209113720670	419949976089315216212
$A_{31}1$	282717915740561360068	336205085534618800980	380442594154920976128	409707407449934795308	419950092087919132428
A_{32}	282717917644479590914	336205087637821970714	380442596405796945924	409707409801097900340	419950094467721216008
A_{33}	376957180390646038772	448273402196193222712	507256743434232078056	546276492212397496308	559933404275504931684
A_{34}	376957181722068167618	448273403661108470626	507256745007198636492	546276493849351789810	559933405938190990028
A_{35}	376957139250237212918	448273356406207503440	507256694017706957254	546276440479057355420	559933351752173214880
A_{36}	376957140461803631240	448273357745142250826	507256695450864273506	546276441976397984938	559933353267698982600

*Identifies Chiral Representation

†Identifies Alternating Representation

”Terms corresponding to partitions [68 12] through [45 35] are not displayed.”

Table 5: 2-colorings of 5D-hypercube for q=4 or 1-hyperplanes (edges) of 5D-hypercube

$[\lambda]$	80 0	79 1	78 2	77 3	76 4	75 5
A_1	1	1	8	50	608	7092
A_2*	0	0	0	12	330	5782
$A_3\dagger$	0	0	2	30	488	6690
A_4	0	0	0	10	319	5730
A_5	0	0	0	55	1426	23866
A_6	0	1	13	132	1990	26563
A_7	0	0	1	64	1465	23992
A_8	0	0	5	98	1781	25800
A_9	0	0	3	97	2010	30903
A_{10}	0	0	10	136	2289	32246
A_{11}	0	0	2	90	1940	30638
A_{12}	0	0	4	108	2098	31546
A_{13}	0	1	10	148	2345	32892
A_{14}	0	0	2	74	1808	29722
A_{15}	0	1	10	162	2385	33253
A_{16}	0	0	1	67	1795	29648
A_{17}	0	0	5	127	2489	37615
A_{18}	0	0	4	120	2428	37328
A_{19}	0	0	4	171	3786	60625
A_{20}	0	0	6	191	3952	61607
A_{21}	0	1	19	284	4598	65138
A_{22}	0	0	6	225	4204	63415
A_{23}	0	0	3	157	3735	60286
A_{24}	0	0	5	175	3893	61194
A_{25}	0	1	14	270	4483	64799
A_{26}	0	1	12	252	4325	63891
A_{27}	0	0	9	212	4121	62523
A_{28}	0	0	8	219	4148	62810
A_{29}	0	0	6	267	5820	91884
A_{30}	0	0	13	340	6347	95035
A_{31}	0	0	9	286	5943	92458
A_{32}	0	1	18	381	6533	96063
A_{33}	0	0	10	394	7978	124004
A_{34}	0	1	23	471	8536	126701
A_{35}	0	0	13	403	8041	124130
A_{36}	0	0	17	437	8357	125938
$[\lambda]$	74 6	73 7	72 8	71 9	70 10	69 11
A_1	82379	843038	7611823	60601324	429479585	2730645204
A_2*	75639	815762	7501366	60219494	428191237	2726763270
$A_3\dagger$	80615	837606	7592170	60547288	429312879	2730230168
A_4	75477	815283	7500045	60216779	428185149	2726758252
A_5	307123	3284074	30095715	241209472	1713913625	10910627650
A_6	320894	3339553	30319122	241978353	1716502254	10918401261
A_7	307440	3284670	30095732	241204688	1713884368	10910515598
A_8	317386	3328346	30277316	241860238	1716127616	10917449272
A_9	389378	4126847	37706909	301809609	2143390868	13641268215
A_{10}	396261	4154583	37818587	302193983	2144685133	13645154996
A_{11}	387957	4122042	37687342	301750856	2143195045	13640741264
A_{12}	392933	4143886	37778146	302078650	2144316687	13644208162
A_{13}	399267	4171384	37884523	302461562	2145574281	13648094476
A_{14}	383245	4099953	37598806	301421467	2142091443	13637273850
A_{15}	400355	4177159	37900610	302525641	2145738361	13648631429
A_{16}	383252	4099836	37601654	301428966	2142137746	13637390920
A_{17}	470161	4965306	45302969	362369722	2572751209	16371625455
A_{18}	468572	4959648	45279512	362298144	2572507924	16370971432

A_{19}	772214	8226800	75301905	603231076	4285454906	27278542065
A_{20}	777520	8249976	75397983	603574410	4286630098	27282141053
A_{21}	795119	8325967	75699273	604655545	4290232009	27293249472
A_{22}	786640	8293652	75571959	604229960	4288818500	27289074727
A_{23}	771209	8221878	75288996	603179822	4285332791	27278132184
A_{24}	776185	8243722	75379800	603507616	4286454433	27281599082
A_{25}	793288	8321045	75678756	604604291	4290055048	27292839591
A_{26}	788312	8299201	75587952	604276497	4288933406	27289372693
A_{27}	782308	8270850	75482201	603880760	4287661270	27285359307
A_{28}	783403	8276508	75501136	603952338	4287871653	27286013330
A_{29}	1163976	12366243	113065434	905261187	6429633640	40922345364
A_{30}	1179979	12437655	113351061	906301111	6433116307	40933165819
A_{31}	1166655	12376344	113102790	905381304	6430009399	40923404250
A_{32}	1183758	12453667	113401746	906477979	6433610014	40934644759
A_{33}	1558766	16520230	150873340	1207459954	8574271258	54567612412
A_{34}	1572537	16575709	151096684	1208228835	8576859887	54575386023
A_{35}	1559415	16520826	150876380	1207455170	8574263945	54567500360
A_{36}	1569361	16564502	151057964	1208110720	8576507193	54574434034

$[\lambda]$	44 36	43 37	42 38	41 39	40 40
A_1	18847859334620010456	22413670446997972838	25362837531743140240	27313824978896887460	27996670589987902014
A_2*	18847856749898064896	22413667593567098448	25362834461344949584	27313821778724903160	27996667338560535196
$A_{3\dagger}$	18847859257885780852	22413670364841246912	25362837441865001528	27313824887601341100	27996670495254082980
A_4	18847856766704295498	22413667612733468770	25362834481318343144	27313821800213507326	27996667359714049916
A_5	75391429606157432796	89654673254005600786	101451340942485322288	109255290345061834468	111986672634212247242
A_6	75391434741988768948	89654678922534504952	101451347043334813206	109255296702428492698	111986679094759845390
A_7	75391429507567646082	89654673145529299732	101451340825886267878	109255290223762071580	111986672510921365600
A_8	75391434523543070266	89654678688077585068	101451346786926363114	109255296441514937800	111986678824308450416
A_9	94239288940765094724	112068343700990309476	126814178474214859414	136569115323944723314	139983343224185808696
A_{10}	94239291508680725702	112068346535254721166	126814181524639564132	136569118502628011070	139983346454459568290
A_{11}	94239288765441089748	112068343510357292528	126814178267737678352	136569115111349424020	139983343006161121260
A_{12}	94239291273428805704	112068346281631440992	126814181248257730296	136569118220225861960	139983346162854669044
A_{13}	94239293852494038614	112068349135075557906	126814184312105381812	136569121420411825260	139983349407403993730
A_{14}	94239286278051269304	112068340758262768502	126814175311580690928	136569112023975578906	139983339875230284004
A_{15}	94239293986646838254	112068349287375751864	126814184469883533380	136569121590029833798	139983349573931817500
A_{16}	94239286361724523754	112068340847572699234	126814175410394391170	136569112123786737628	139983339979665122900
A_{17}	113087148246821895790	134482014116808664130	152177015972758652110	163882940268379932380	167980013779270147172
A_{18}	113087148023326870600	134482013875198539440	152177015709602679880	163882939998950765120	167980013501415204240
A_{19}	188478575214092180140	224136684459253077978	253628353780325450740	273138227347920302220	279966683093672480502
A_{20}	188478577753444741688	224136687262337963862	253628356797550900292	273138230492142003960	279966686289042830634
A_{21}	188478585356450552428	224136695670330279072	253628365831274846342	273138239923039836330	279966695856119919142
A_{22}	188478582759971824040	224136692804886249198	253628362747650794080	273138236709894870180	279966692590942719194
A_{23}	188478575127165613502	224136684357929991762	253628353678132069522	273138227235136161648	279966682985826244160
A_{24}	188478577635153329458	224136687129204140226	253628356658652121466	273138230344012599588	279966686142519791944
A_{25}	188478585260075643958	224136695569007192856	253628365718141263676	273138239810255695758	279966695736786486544
A_{26}	188478582752087928002	224136692797733044392	253628362737621211732	273138236701379257818	279966692580092938760
A_{27}	188478580130520633006	224136689893311819736	253628359623658890590	273138233444359426552	279966689282605624774
A_{28}	188478580348346687096	224136690134921944426	253628359880250742400	273138233713788593812	279966689553568287440
A_{29}	282717866380008860880	336205030620395062432	380442534902040326352	409707345433873419442	419950029122932529996
A_{30}	282717873954451546210	336205038997207759458	380442543902564924858	409707354830309573418	419950038655106147470
A_{31}	282717866710071210850	336205030982494643660	380442535290645133570	409707345837575331440	419950029533233410340
A_{32}	282717874334993525350	336205039422297696290	380442544350134275780	409707355303818427610	419950039127500104940
A_{33}	376957157974051675740	448273377264126084652	507256716527962663540	546276464057801193400	559933375684600884504
A_{34}	376957163109882955912	448273382932654988818	507256722628812154458	546276470415167851630	559933382145148421080
A_{35}	376957157879241203778	448273377155649783598	507256716415739690158	546276463936501430512	559933375565904857280
A_{36}	376957162895216627962	448273382698198068934	507256722376779785394	546276470154254296732	559933381879291942096

*Identifies Chiral Representation

†Identifies Alternating Representation

”Terms corresponding to partitions [68 12] through [45 35] are not displayed.”

Table 6: Two-Colorings of Vertices or q=5-hyperplanes of 5D-hypercube.

$[\lambda]$	32 0	31 1	30 2	29 3	28 4	27 5	26 6	25 7	24 8
A_1	1	1	5	10	47	131	472	1326	3779
A_2*	0	0	0	0	2	26	148	653	2218
$A_3\dagger$	0	1	2	10	33	131	421	1326	3616
A_4	0	0	0	0	1	26	144	653	2210
A_5	0	0	0	0	8	120	664	2870	9511
A_6	0	0	4	13	82	310	1281	4174	12576
A_7	0	0	0	0	13	120	690	2870	9600
A_8	0	0	2	13	67	310	1215	4174	12360
A_9	0	0	0	4	39	228	1092	4135	13189
A_{10}	0	0	2	11	77	324	1399	4789	14718
A_{11}	0	0	0	4	35	228	1073	4135	13128
A_{12}	0	0	1	11	64	324	1339	4789	14514
A_{13}	0	1	5	23	105	441	1657	5500	16038
A_{14}	0	0	0	0	17	146	852	3523	11868
A_{15}	0	1	4	23	100	441	1636	5500	15976
A_{16}	0	0	0	0	15	146	838	3523	11818
A_{17}	0	0	0	3	42	276	1335	5068	16098
A_{18}	0	0	0	3	45	276	1342	5068	16126
A_{19}	0	0	0	4	52	374	1922	7658	24982
A_{20}	0	0	0	3	56	396	2021	7938	25690
A_{21}	0	1	7	34	176	765	3034	10289	30678
A_{22}	0	0	0	16	100	586	2498	9242	28298
A_{23}	0	0	0	4	50	374	1911	7658	24946
A_{24}	0	0	0	3	58	396	2032	7938	25726
A_{25}	0	1	5	34	164	765	2975	10289	30490
A_{26}	0	0	2	16	112	586	2557	9242	28486
A_{27}	0	0	2	15	106	552	2447	8924	27754
A_{28}	0	0	1	15	99	552	2412	8924	27642
A_{29}	0	0	0	7	91	624	3091	12073	38804
A_{30}	0	0	2	27	171	910	3875	14031	42938
A_{31}	0	0	0	7	93	624	3105	12073	38854
A_{32}	0	0	3	27	176	910	3896	14031	43000
A_{33}	0	0	0	18	148	948	4398	16862	53220
A_{34}	0	0	4	31	220	1138	5015	18166	56276
A_{35}	0	0	1	18	157	948	4444	16862	53368
A_{36}	0	0	3	31	211	1138	4969	18166	56128

$[\lambda]$	23 9	22 10	21 11	20 12	19 13	18 14	17 15	16 16
A_1	9013	19963	38073	65664	98804	133576	158658	169112
A_2*	6300	14972	30730	54528	84854	115772	139549	148312
$A_3\dagger$	9013	19591	38073	64985	98804	132622	158658	168028
A_4	6300	14955	30730	54502	84854	115733	139549	148272
A_5	26577	62443	127170	224457	348060	473805	570371	605924
A_6	31935	72346	141756	246631	375831	509313	608445	647402
A_7	26577	62656	127170	224857	348060	474370	570371	606564
A_8	31935	71835	141756	245691	375831	507976	608445	645892
A_9	35457	82216	165022	289831	446538	607012	728648	774616
A_{10}	38137	87161	172314	300905	460423	624750	747682	795338
A_{11}	35457	82075	165022	289569	446538	606644	728648	774200
A_{12}	38137	86673	172314	299996	460423	623459	747682	793876
A_{13}	40948	91573	179829	310939	474635	640973	767103	814338
A_{14}	32877	77754	157900	279619	432914	590482	709920	755258
A_{15}	40948	91426	179829	310676	474635	640598	767103	813920
A_{16}	32877	77628	157900	279385	432914	590142	709920	754876
A_{17}	43199	99880	200138	350931	540233	733809	880619	935962
A_{18}	43199	99934	200138	351041	540233	733952	880619	936136
A_{19}	68334	159792	322922	569118	879452	1197022	1438568	1529340
A_{20}	69776	162501	327308	575734	888293	1208086	1450990	1542436
A_{21}	79085	178556	352143	611502	935058	1265251	1514785	1609132
A_{22}	75134	171312	341894	595902	916064	1240734	1489064	1580692
A_{23}	68334	159703	322922	568954	879452	1196786	1438568	1529076
A_{24}	69776	162590	327308	575898	888293	1208322	1450990	1542700
A_{25}	79085	178099	352143	610672	935058	1264057	1514785	1607796
A_{26}	75134	171769	341894	596732	916064	1241928	1489064	1582028
A_{27}	73594	169021	337336	590062	906961	1230818	1476330	1568876
A_{28}	73594	168748	337336	589565	906961	1230103	1476330	1568076
A_{29}	105233	244539	492330	865233	1334831	1814626	2179638	2316518
A_{30}	113271	258295	514208	896465	1376487	1865012	2236746	2375486
A_{31}	105233	244665	492330	865467	1334831	1814966	2179638	2316900
A_{32}	113271	258442	514208	896728	1376487	1865387	2236746	2375904
A_{33}	143370	330976	664644	1164802	1795254	2437474	2927320	3109712
A_{34}	148728	340879	679230	1186958	1823025	2472982	2965394	3151168
A_{35}	143370	331338	664644	1165463	1795254	2438425	2927320	3110776
A_{36}	148728	340517	679230	1186297	1823025	2472031	2965394	3150104

Identifies Chiral Representation

†Identifies Alternating Representation

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