

SLANT HELICES FOR VERSOR FIELDS ALONG A CURVE IN 4- DIMENSIONAL MINKOWSKI SPACE

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Abstract.

In this work we study investigates k - type helices obtained from versor vector fields defined along curves in four-dimensional Minkowski space. New relationships between curvature functions and helical structures were obtained by expressing the versor vector field in terms of the Frenet frame of the base curve.

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1 Introduction

The differential geometry of curves in Minkowski (or Lorentzian) spaces has been the subject of extensive research due to its significance in both mathematics and theoretical physics. The geometric structure of spacetime in general relativity is modelled by Minkowski manifolds, which provides the motivation for the investigation of curves and their invariants in such spaces. The theory of curves in four-dimensional Minkowski space E_1^4 has attracted considerable attention due to the fact that it provides a natural setting for the study of generalized helices and other special curves.

The Frenet frame constitutes a fundamental tool in the study of curves. The framework under consideration provides a moving orthonormal frame along a given curve. This framework consists of the tangent vector, the principal normal vector, and the binormal vectors. Furthermore, the geometry of the curve is described through curvature functions. In Minkowski spaces, the Frenet frame must be adapted to the causal character of vectors (spacelike, timelike, or null), which leads to several generalisations of the classical Frenet-Serret formulas.

The subject of special curves, including helices, has been a significant area of study within the field of differential geometry for some time [1-3]. In the context of three-dimensional Euclidean space, a helix is defined by the property that its tangent vector makes a constant angle with a fixed direction [4]. This concept has been subjected to various generalisations, including slant helices and higher-dimensional helices defined using Frenet frames [5, 6]. The concept of slant helices was subsequently expanded to higher-dimensional Euclidean spaces [7], wherein the principal normal vector forms a constant angle with a fixed direction in the ambient space.

In Minkowski space, several authors have investigated special curves using the Frenet apparatus [8-9]. A plethora of studies on spacelike and timelike curves

in Minkowski 4-space have yielded novel characterisations of such curves in terms of curvature functions. These studies have also given rise to the introduction of hitherto unexplored classes of special curves. In particular, Ilarslan and Nešović studied spacelike and timelike normal curves in Minkowski spacetime and provided important results related to their geometric properties [10].

Another significant development in the theory of special curves is the introduction of k -type helices [11-12]. This concept generalises classical helices by requiring that one of the Frenet frame vectors maintains a constant scalar product with a fixed direction in the ambient space. In four-dimensional Minkowski space, this definition gives rise to different types of helices, depending on whether the tangent, normal or binormal vectors satisfy the constant angle condition [13].

Further developments in this area include the study of B_2 - slant helices and other generalized helical curves in Minkowski space E_1^4 [14].

The curves mentioned above are characterized by the feature that one of the higher-order Frenet vectors, like the second binormal vector, maintains a constant angle with a specific fixed direction. This property aids in achieving a deeper understanding of the geometry of curves within higher-dimensional Minkowski spaces.

In addition to the classical Frenet frame, alternative moving frames have also been utilised in the study of special curves. For instance, the application of extended Darboux frames and parallel transport frames has been utilised in the investigation of slant helices and other generalized curves in four-dimensional spaces [15-16]. These approaches provide alternative geometric descriptions and frequently result in novel characterisations of special curves.

Recent studies have also investigated generalized helices and slant helices using different geometric frameworks, including fractional differential geometry and other generalized curvature concepts [17-18]. These developments demonstrate that the study of helices and their generalisations remains an active research area in modern differential geometry.

Despite these advances, the interaction between vector fields defined along curves and the geometry of helices in Minkowski spaces remains an area of exploration. In particular, the role of versor vector fields, i.e. vector fields of constant norm along a curve, has received relatively little attention in the context of k -type helices in Minkowski space. It is therefore hypothesised that the investigation of the Frenet frames generated by such vector fields may provide new geometric characterisations of special curves in E_1^4 . From a different perspective, (k, m) -type slant helices were defined and studies were conducted in various spaces [19-20].

Motivated by these observations, the present study investigates k -type helices obtained from versor vector fields defined along curves in four-dimensional Minkowski space. By expressing the versor vector field in terms of the Frenet frame of the base curve, new relationships between curvature functions and helical structures obtained.

2 Preliminaries

Minkowski space in E_1^4 is an Euclidean space provided with the standart metric given by $g(X, Y) = -x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$, where $X = (x_1, x_2, x_3, x_4)$ and $Y = (y_1, y_2, y_3, y_4)$ be two vectors in E_1^4 . Since $g(,)$ is a indefinite metric, recall that a vector $u \in E_1^4$ can have one of three casual characters; it can be spacelike, if $g(u, u) > 0$ or $u = 0$, timelike if $g(u, u) < 0$, or null or lightlike if $g(u, u) = 0$ and $u \neq 0$. The norm of a vector $u \in E_1^4$ is given by $\|u\| = \sqrt{g(u, u)}$. Let $\alpha : I \rightarrow E_1^4$ be an arbitrary curve in E_1^4 . The curve α is said to be of a unit speed or parametrized by arc length function s if $g(\alpha', \alpha') = \pm 1$. We say that the curve α is spacelike, timelike or null (lightlike) for any s , if $\alpha'(s)$ is spacelike, timelike or null (lightlike), respectively [21].

Let ξ be a versor field on E_1^4 . Locally ξ can have one of three causal characteristics: spacelike, null or timelike. Thus, Frenet frames $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ can be constructed for these three cases [22].

This study will specifically build theories on the Frenet frame, assuming ξ is spacelike. Cases where ξ is timelike and null can be examined.

Let $\xi = \xi_1$ is spacelike, that is $g(\xi_1, \xi_1) = 1$ and C be a curve tangent to ξ_1 . In this case, let's construct the Frenet frame of (C, ξ) . If $\frac{d\xi_1}{ds} \neq 0$, then $\frac{d\xi_1}{ds}$ is perpendicular to ξ_1 , so take ξ_2 in the direction of $\frac{d\xi_1}{ds}$. Thus, $\frac{d\xi_1}{ds}$ has three different casual characters. However, for this study, let's choose $\frac{d\xi_1}{ds}$ as spacelike, i.e., $g(\frac{d\xi_1}{ds}, \frac{d\xi_1}{ds}) > 0$. ξ_2 is then the normalized vector field corresponding to $\frac{d\xi_1}{ds}$. The vector field ξ_3 is in the direction of the normal component $C^\perp$ of $\frac{d\xi_2}{ds}$ with respect to plane $\{\xi_1, \xi_2\}$ and can have all the causal charavters. Let's assume that $C^\perp$ is a spacelike unit vector, i.e., $g(C^\perp, C^\perp) > 0$. Then ξ_3 is the normalized vector field $C^\perp$ and ξ_4 is the unique timelike unit vector field perpendicular to the 3-dimensional subspace $\{\xi_1, \xi_2, \xi_3\}$, such that the orientation of $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ is the same as that of E_1^4 . The Frenet formulas are [22]

$$\begin{aligned} \frac{d\xi_1}{ds} &= k_1 \xi_2 \\ \frac{d\xi_2}{ds} &= -k_1 \xi_1 + k_2 \xi_3 \\ \frac{d\xi_3}{ds} &= -k_2 \xi_2 + k_3 \xi_4 \\ \frac{d\xi_4}{ds} &= k_3 \xi_3. \end{aligned} \tag{2.1}$$

Let ξ be an versor field on E_1^4 with Frenet frame $\{\xi_1, \xi_2, \xi_3, \xi_4\}$. ξ versor field said to be a k -type slant helix if there exists a non-zero fixed direction vector U such that $g(\xi_k, U) = c_k$, where c_k is a real constant and $k \in \{1, 2, 3, 4\}$ [12].

3.

Theorem 3.1 Let E_1^4 be a versor field in Minkowski space. Let ξ_1 be a spacelike unit vector, and let C be a curve tangent to ξ_1 such that $\frac{d\xi_1}{ds}$ and $C^\perp$ are spacelike vectors, and let $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ be given by the Frenet equations.

i) Consider a curve C with constant curvatures k_1 and k_2 , none of which are zero. If the curve C is a 1-type slant helix, then C is a 3-type slant helix

ii) Consider a curve C with constant curvatures k_1 and k_2 , none of which are zero. If the curve C is a 1-type slant helix, 3-type slant helix, and then C is a 4-type slant helix, then C has curvature $k_3 = 0$.

Proof: i) Let us assume that the curve C is a 1-type slant helix. Then

$$g(\xi_1, U) = c_1 \quad (3.1)$$

there exists a real constant c_1 such that Taking the derivative of (3.1) yields

$$g\left(\frac{d\xi_1}{ds}, U\right) = 0 \quad (3.2)$$

and using (2.1) gives

$$k_1 g(\xi_2, U) = 0 \quad (3.3)$$

Since $k_1 \neq 0$,

$$g(\xi_2, U) = 0 \quad (3.4)$$

holds. Taking the derivative of (3.4) and using (2.1) and (3.1),

$$-k_1 c_1 + k_2 g(\xi_3, U) = 0$$

or

$$g(\xi_3, U) = \frac{k_1}{k_2} c_1 \quad (3.5)$$

it follows that the curve C is a 3-type slant helix.

ii) For a 1-type slant helix and a 3-type slant helix with the C curve having $k_1 \neq 0$ and $k_2 \neq 0$, if the derivatives of both sides of (3.5) are taken and (2.1) used,

$$k_3 g(\xi_4, U) = 0$$

is obtained. Since the C curve is also a 4-type slant helix, we have $k_3 = 0$.

Theorem 3.2 Let E_1^4 be a versor field in Minkowski space. Let ξ_1 be a spacelike unit vector, and let C be a curve tangent to ξ_1 such that $\frac{d\xi_1}{ds}$ and $C^\perp$ are spacelike vectors, and let $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ be given by the Frenet equations. If the curve C is a 2-type slant helix with the constant curvature $k_1 \neq 0$, $k_2 \neq 0$, and $k_3 \neq 0$, then it is also a 4-type slant helix.

Proof: Let us assume that the curve C is a 2-type slant helix. Then

$$g(\xi_2, U) = c_2 \quad (3.6)$$

there exists a real constant c_2 such that. If we take the derivative of (3.6),

$$g\left(\frac{d\xi_2}{ds}, U\right) = 0$$

and using (2.1),

$$-k_1 g(\xi_1, U) + k_2 g(\xi_3, U) = 0. \quad (3.7)$$

And taking the derivative once more

$$-k_1 g\left(\frac{d\xi_1}{ds}, U\right) + k_2 g\left(\frac{d\xi_3}{ds}, U\right) = 0.$$

Using (2.1)

$$-k_1^2 g(\xi_2, U) - k_2^2 g(\xi_2, U) + k_2 k_3 g(\xi_4, U) = 0$$

and if (3.6) is taken into account

$$g(\xi_4, U) = \frac{(k_1^2 + k_2^2)}{k_2 k_3} c_2.$$

it follows that the curve C is a 4-type slant helix.

Theorem 3.3 Let E_1^4 be a versor field in Minkowski space. Let ξ_1 be a spacelike unit vector, and let C be a curve tangent to ξ_1 such that $\frac{d\xi_1}{ds}$ and $C^\perp$ are spacelike vectors, and let $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ be given by the Frenet equations.

i) Consider a curve C with constant curvatures k_1, k_2 and k_3 , none of which are zero. If C is a 3-type slant helix and 1-type slant helix, then C is also a 4-type slant helix.

ii) Consider a curve C with constant curvatures k_1, k_2 and k_3 , none of which are zero. If C is a 3-type slant helix and 4-type slant, then C is also a 1-type slant helix.

Proof: Let us assume that the curve C with constant curvatures k_1, k_2 and k_3 , none of which are zero is a 3-type slant helix. Then

$$g(\xi_3, U) = c_3 \quad (3.8)$$

there exists a real constant c_3 such that (3.8) holds. Taking the derivative of (3.8) and using (2.1),

$$-k_2 g(\xi_2, U) + k_3 g(\xi_4, U) = 0 \quad (3.9)$$

and taking the derivative once more

$$-k_2 g\left(\frac{d\xi_2}{ds}, U\right) + k_3 g\left(\frac{d\xi_4}{ds}, U\right) = 0$$

If (2.1) is used

$$-k_2 g(-k_1 \xi_1 + k_2 \xi_3, U) + k_3 g(-k_3 \xi_4, U) = 0$$

or

$$k_2 k_1 g(\xi_1, U) - k_2^2 g(\xi_3, U) - k_3^2 g(\xi_4, U) = 0$$

and if (3.8) is considered

$$k_2 k_1 g(\xi_1, U) - k_2^2 c_3 - k_3^2 g(\xi_4, U) = 0 \quad (3.10)$$

is obtained.

i) Let C is a curve with constant curvatures k_1 , k_2 and k_3 , none of which are zero. Assume that C is a 3-Type slant helix and also a 1-Type slant helix, then we have

$$g(\xi_1, U) = c_1$$

and taking into account the fact stated in (3.10),

$$g(\xi_4, U) = \frac{k_2^2 c_3 - k_2 k_1 c_1}{k_3^2} = \text{constant}$$

it follows that the curve C is a 4-type slant helix.

ii) Let C is a curve with constant curvatures k_1 , k_2 and k_3 , none of which are zero. Assume that C is a 3-Type slant helix and also a 4-Type slant helix, then we have

$$g(\xi_4, U) = c_4$$

and taking into account the fact stated in (3.10),

$$g(\xi_1, U) = \frac{k_3^2 c_4 + k_2^2 c_3}{k_1 k_2} = \text{constant}$$

it follows that the curve C is a 1-type slant helix.

Teorem 3.4 Let E_1^4 be a versor field in Minkowski space. Let ξ_1 be a spacelike unit vector, and let C be a curve tangent to ξ_1 such that $\frac{d\xi_1}{ds}$ and $C^\perp$ are spacelike vectors, and let $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ be given by the Frenet equations.

i) Consider a curve C with constant curvatures k_2 and k_3 , none of which are zero. If the curve C is a 4-type slant helix, then also C is a 2-type slant helix

ii) Consider a curve C with constant curvatures k_2 and k_3 , none of which are zero. If the curve C is a 4-type slant helix, 2-type slant helix, and then C is a 1-type slant helix, then C has curvature $k_1 = 0$.

Proof: i) Let us assume that the curve C is a 4-type slant helix. Then

$$g(\xi_4, U) = c_4 \quad (3.11)$$

there exists a real constant c_4 such that Taking the derivative of (3.1) yields

$$g\left(\frac{d\xi_4}{ds}, U\right) = 0 \quad (3.12)$$

and using (2.1) gives

$$-k_3 g(\xi_3, U) = 0 \quad (3.13)$$

Since $k_3 \neq 0$,

$$g(\xi_3, U) = 0 \quad (3.14)$$

holds. Taking the derivative of (3.14) and using (2.1)

$$-k_2 g(\xi_2, U) + k_3 g(\xi_4, U) = 0$$

or

$$g(\xi_2, U) = \frac{k_3}{k_2} c_4 \quad (3.15)$$

it follows that the curve C is a 2-type slant helix.

ii) For a 4-type slant helix and a 2-type slant helix with the C curve having $k_2 \neq 0$ and $k_3 \neq 0$, if the derivatives of both sides of (3.15) are taken and (2.1) used,

$$k_1 g(\xi_1, U) = 0$$

is obtained. Since the C curve is also a 1-type slant helix, we have $k_1 = 0$.

CONCLUIONS

The Frenet frames of the vector field E_1^4 were constructed in Minkowski space using a curve C and a vector field ξ [22]. Since the ξ vector field consists of spacelike, timelike, and null vectors, the $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ vector Frenet frame gives rise to different structures. When constructing these frames, the situation became even more interesting when the spacelike, timelike, and null characteristics of the vectors ξ_1, ξ_2, ξ_3 , and ξ_4 were also taken into account. Upon closer examination, the following situations arise.

A) When ξ_1 is a spacelike vector and $\frac{d\xi_1}{ds}$ is a spacelike vector, there are three distinct Frenet versor frames depending on whether $C^\perp$ is spacelike, timelike, or null,

When ξ_1 is a spacelike vector and $\frac{d\xi_1}{ds}$ is a timelike vector, there is a Frenet versor frame,

When ξ_1 is a spacelike vector and $\frac{d\xi_1}{ds}$ is a null vector, there is a Frenet versor frame,

B) a Frenet vector frame when ξ_1 is a timelike vector,

C) a Frenet vector frame when ξ_1 is a null vector.

In total, this results in seven Frenet frames that are similar to one another but differ in the structure of their causal vectors. This actually provides a tremendous new perspective on the field of geometry, enabling the derivation of interesting results.

In this study, the necessary and sufficient conditions for the Frenet frames of the versor to be slant helices when ξ_1 is a spacelike vector, $\frac{d\xi_1}{ds}$ is a spacelike vector, and $C^\perp$ is spacelike, have been investigated. The remaining six distinct cases will be addressed from the same perspective in a master's thesis titled Frenet frames of Versor vector fields along a curve in 4-dimensional Lorentz space.

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3 References

- [1] Chouaieb, N., Goriely, A., & Maddocks, J. H. (2006). Helices. *Proceedings of the National Academy of Sciences*, 103(25), 9398-9403.
- [2] Sy, S. (2001). General helices and other topics in the differential geometry of curves (Master's thesis, Michigan Technological University).
- [3] Uzunoğlu, B., Gök, İ., & Yaylı, Y. (2016). A new approach on curves of constant precession. *Applied Mathematics and Computation*, 275, 317-323.
- [4] M. Barros, (1997). General helices and a theorem of Lancret, *Proc. Am.Math. Soc.* 125, 1503–1509.
- [5] Kula, L., Ekmekci, N., Yaylı, Y., & İlarslan, K. (2010). Characterizations of slant helices in Euclidean 3-space. *Turkish Journal of Mathematics*, 34(2), 261-274.
- [6] Camci, C., İlarslan, K., Kula, L., & Hacisalihoglu, H. (2009). Harmonic curvatures and generalized helices in E^n . *Chaos Solitons & Fractals*, 40(5).
- [7] Gök, I., Camci, C., & Hacisalihoglu, H. (2009). V_n -slant helices in Euclidean n -space E^n . *Mathematical communications*, 14(2), 317-329.
- [8] Abdel-Aziz, H. S., Serry, H., & Saad, M. K. (2022). Evolution equations of pseudo spherical images for timelike curves in Minkowski 3-space. *Mathematics and Statistics*, 10(4), 884-893.
- [9] Hanif, M., Hou, Z. H., & Nisar, K. S. (2018). On special kinds of involute and evolute curves in 4-dimensional Minkowski space. *Symmetry*, 10(8), 317.
- [10] İlarslan, K., & Nešović, E. (2009). Spacelike and timelike normal curves in Minkowski space-time. *Publications de l'Institut Mathématique*, (105), 111-118.
- [11] Zhou, A., Yao, K., & Pei, D. (2024). k -type hyperbolic framed slant helices in hyperbolic 3-space. *Filomat*, 38(11), 3839-3850.
- [12] Ali, A. T., López, R., & Turgut, M. (2012). k -type partially null and pseudo null slant helices in Minkowski 4-space. *Mathematical Communications*, 17(1), 93-103.
- [13] Nešović, E., Koç Öztürk, E. B., & Öztürk, U. (2015). k -type null slant helices in Minkowski space-time. *Mathematical Communications*, 20(1), 83-95.
- [14] Ali, A. T., & López, R. (2008). Timelike B_2 -slant helices in Minkowski space E_1^4 . *arXiv preprint arXiv:0810.1460*.
- [15] Hu, N., Zhang, T., & Jiang, Y. (2022). Darboux Vector in Four-Dimensional Space-Time. *Advances in Mathematical Physics*, 2022(1), 9044567.
- [16] Babadağ, F. (2016). On similar partner curves in Bishop frames with variable transformations. *International Journal of New Technology and Research*, 2(4), 59-64.

- [17] Has, A., & Yilmaz, B. (2024). C_α -helices and C_α -slant helices in fractional differential geometry. *Arabian Journal of Mathematics*, 13(2), 291-301.
- [18] Yıldırım Y. M. and Bektaş M.(2026). k -type Slant Helices for Conformable Curve in Euclidean 4-Space,(submitted to the journal)
- [19] Khan, S. H., Jamali, M., & Aslam, M. (2023). Characterization of bi-Null Slant (BNS) Helices of (k, m) -type in R_1^3 and R_2^5 . *Facta Universitatis, Series: Mathematics and Informatics*, 731-740.
- [20] Yilmaz, M. Y., & Bektaş, M. (2019). Slant helices of (k, m) -type in E^4 . *Acta Universitatis Sapientiae, Mathematica*, 10(2), 395-401.
- [21] O'Neill, B. (1983). *Semi-Riemannian Geometry*. Academic Press .
- [22] Heroiu, B. (2011). Versor fields along a curve in a four dimensional Lorentz space. *Journal of Advanced Mathematical Studies*, 4(1), 49-57.