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ON THE ASSOCIATED PRIMES OF THE d-LOCAL COHOMOLOGY MODULES

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Abstract. This paper is concerned to relationship between the sets of associated primes of the d-local cohomology modules and the ordinary local cohomology modules. Let R be a commutative Noetherian local ring, M an R-module and d, t two integers. We prove that $\mathrm{Ass}(H^t_d(M)) = \bigcup_{I\in\Phi} \mathrm{Ass}(H^t_I(M))$ whenever $H^i_d(M) = 0$ for all $i < t$ and $\Phi = \{I : I \text{ is an ideal of R with} \dim R/I \leq d\}$. We give some information about the non-vanishing of the d-local cohomology modules. To be more precise, we prove that $H^i_d(R) \neq 0$ if and only if $i = n - d$ whenever R is a Gorenstein ring of dimension n. This result leads to an example which shows that $\mathrm{Ass}(H^{n-d}_d(R))$ is not necessarily a finite set.

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1. Introduction

Throughout this paper, R denotes a commutative Noetherian ring with non-zero identity. For an ideal I of R and an R-module M, the ith local cohomology module of M with respect to I is defined as

$$H^i_I(M) \cong \varinjlim_{n\in\mathbb{N}} \mathrm{Ext}^i_R(R/I^n, M).$$

The reader can refer to [6] for the basic properties of local cohomology modules. An important problem in commutative algebra is determining when the set of associated primes of the ith local cohomology module $H^i_I(M)$ is finite. In [10], Huneke raised the following question: If M is a finitely generated R-module, then the set of associated primes of $H^i_I(M)$ is finite for all ideals I of R and all $i \geq 0$. This problem has been studied by many authors, it was shown that it is true in many situations, for examples see [4,5,8]. In particular, it is shown in [5] that if for a finitely generated R-module M and integer t, the local cohomology modules $H^i_I(M)$ are finitely generated for all $i < t$, then the set $\mathrm{Ass}(H^t_I(M))$ is finite. There are several papers devoted to the extension of the above results to more general situations, for examples see [1,9]. However there are counterexamples which show that it is not true in general, for examples see [11,16]. The purpose of this paper is to make a counterexample to above question in the context of general local cohomology modules. The theory of general local cohomology modules over commutative Noetherian rings introduced by Bijan-Zadeh in [3]. General local cohomology theory described as follows.

Let Φ be a non-empty set of ideals of R. We call Φ a system of ideals of R if, whenever $I, I' \in \Phi$, then there exists $J \in \Phi$ such that $J \subseteq II'$. Such a system of ideals gives rise to an additive, left exact functor

$$\Gamma_\Phi(M) = \{x \in M : Ix = 0 \text{ for some ideal } I \in \Phi\}$$

from the category of R-modules and R-homomorphisms to itself. $\Gamma_\Phi(-)$ is called the Φ-torsion functor. For each $i \geq 0$, the ith right derived functor of $\Gamma_\Phi(-)$ is denoted by $H^i_\Phi(-)$. For an ideal I of R, if $\Phi = \{I^n : n \in \mathbb{N}\}$, then $H^i_\Phi(-)$ coincides with the ordinary local cohomology functor $H^i_I(-)$.

Let $d \geq 0$ be an integer. We denote $\Gamma_\Phi(-)$ and $H^i_\Phi(-)$ by $\Gamma_d(-)$ and $H^i_d(-)$ respectively, for the system of ideals $\Phi = \{I : I \text{ is an ideal of R with} \dim R/I \leq d\}$. The functor $\Gamma_d(-)$ was originally defined in [2] and the modules $H^i_d(M)$ were called d-local cohomology modules associated to M were studied in [18,19]. After some preliminary results in Section 2, for an R-module M and an integer t we prove that

$$\operatorname{Ass}(H^t_d(M)) = \bigcup_{I \in \Phi} \operatorname{Ass}(\operatorname{Ext}^t_R(R/I, M)) = \bigcup_{I \in \Phi} \operatorname{Ass}(H^t_I(M)),$$

where $\Phi = \{I : I \text{ is an ideal of R with} \dim R/I \leq d\}$ and $H^i_d(M) = 0$ for all $i < t$. In Section 3, we shall provide some results concerning the vanishing and non-vanishing of d-local cohomology modules: we shall prove that, over a local ring R, if the non-zero finitely generated R-module M has (Krull) dimension n, then there exists an integer i with $0 \leq i \leq d$ such that $H^{n-i}_d(M) \neq 0$. We shall also prove that, when R is local Gorenstein of dimension n, then $H^i_d(R) \neq 0$ if and only if $i = n - d$. Furthermore, we shall prove that $H^{n-d}_d(R)$ is a non-Artinian flat module for which $\operatorname{Ass}(H^{n-d}_d(R)) = \{\mathfrak{p} \in \operatorname{Spec}(R) : \dim R/\mathfrak{p} = d\}$. This result leads to an example which shows that the Huneke question is not true in the context of d-local cohomology modules.

2. The associated primes

It is our intention in this section to present the relationship between the sets of associated primes of the d-local cohomology modules and the ordinary local cohomology modules. So throughout this section, R will denote a ring and I is an ideal of R, d is an integer and $\Phi = \{I : I \text{ is an ideal of R with} \dim R/I \leq d\}$.

Lemma 2.1. *Let M be an R-module and t be an integer such that $H^i_d(M) = 0$ for all $i < t$. Then the following statements are true:*

(i) $H^i_I(M) \subseteq H^i_d(M)$ *for all* $I \in \Phi$ *and* $i \leq t$.

(ii) $H^i_c(M) \subseteq H^i_d(M)$ *for all* $c \leq d$ *and* $i \leq t$.

Proof. (i) Since $\mathrm{Hom}_R(R/I, \Gamma_d(M)) \cong \mathrm{Hom}_R(R/I, M)$ for all $I \in \Phi$ so, by [14, Theorem 11.38], the Grothendieck spectral sequence $E^{p,q}_2 := \mathrm{Ext}^p_R(R/I, H^q_d(M))$ converges to $E^{p+q} := \mathrm{Ext}^{p+q}_R(R/I, M)$. It follows that there is a finite filtration

$$0 = F^{q+1}E^q \subseteq F^qE^q \subseteq \cdots \subseteq F^1E^q \subseteq F^0E^q = E^q$$

of E^q such that $E^{p,q-p}_\infty \cong F^pE^q/F^{p+1}E^q$, for all $p = 0, 1, \cdots, q$. Because $E^{p,q-p}_\infty$ is a subquotient of $E^{p,q-p}_2$ so $E^{p,q}_\infty = 0$, for all $q < t$. Thus $E^{0,q}_\infty \cong E^q$, for all $q \leq t$. On the other hand, for all $q \leq t$, by the sequence $0 \longrightarrow E^{0,q}_2 \longrightarrow E^{2,q-1}_2$ and $E^{2,q-1}_2 = 0$ we have $E^{0,q}_\infty \cong E^{0,q}_2$. Thus $E^q \cong E^{0,q}_2$. Therefore, $\mathrm{Ext}^i_R(R/I, M) \cong \mathrm{Hom}_R(R/I, H^i_d(M))$ and so

$$\Gamma_I(H^i_d(M)) \cong \varinjlim_{n\in\mathbb{N}} \mathrm{Hom}_R(R/I^n, H^i_d(M)) \cong \varinjlim_{n\in\mathbb{N}} \mathrm{Ext}^i_R(R/I^n, M) \cong H^i_I(M),$$

for all $i \leq t$. The proof is therefore complete.

(ii) It is similar to that of (i). □

Corollary 2.2. *Let M be an R-module and t be an integer such that $H^i_d(M) = 0$, for all $i < t$. Then the following statements are true:*

(i) $\mathrm{Ass}(H^i_I(M)) = \mathrm{Ass}(H^i_d(M)) \cap V(I)$, *for all* $I \in \Phi$ *and* $i \leq t$.

(ii) $\mathrm{Ass}(H^i_J(M)) \subseteq \mathrm{Ass}(H^i_I(M))$ *for all* $I, J \in \Phi$ *with* $I \subseteq J$ *and* $i \leq t$.

Proof. (i) By a similar argument to that of Lemma 2.1 (i), one can shows that $\mathrm{Hom}_R(R/I, H^i_d(M)) \cong \mathrm{Hom}_R(R/I, H^i_I(M))$, for all $I \in \Phi$ and all $i \leq t$. Therefore, $\mathrm{Ass}(\mathrm{Hom}_R(R/I, H^t_d(M))) = \mathrm{Ass}(\mathrm{Hom}_R(R/I, H^t_I(M)))$. Thus $\mathrm{Ass}(H^t_I(M)) = V(I) \cap \mathrm{Ass}(H^t_d(M))$.

(ii) It is obvious by (i). □

We say that an R-module M is d-torsion if $\Gamma_d(M) = M$, and it is d-torsion free if $\Gamma_d(M) = 0$. Note, if M is d-torsion, then $\mathrm{Supp}(M) \subseteq \Phi$. Also, if M is finitely generated of $\dim M = n$, then M is d-torsion if and only if $n \leq d$. We are now in a position to prove that the main result of this section.

Theorem 2.3. *Let M be an R-module and t be an integer such that $H_d^i(M)=0$, for all $i<t$. Then*

$$\operatorname{Ass}(H_d^t(M)) = \bigcup_{\mathfrak{p}\in V(\Phi)} \operatorname{Ass}(\operatorname{Ext}_R^t(R/\mathfrak{p},M)) = \bigcup_{\mathfrak{p}\in V(\Phi)} \operatorname{Ass}(H_{\mathfrak{p}}^t(M)),$$

where $V(\Phi)=\operatorname{Spec}(R)\cap\Phi$.

Proof. First of all, we show that

$$\operatorname{Ass}(H_d^t(M)) = \bigcup_{\mathfrak{p}\in V(\Phi)} \operatorname{Ass}(\operatorname{Hom}_R(R/\mathfrak{p},H_d^t(M))).$$

For an ideal I of R, it is clear that $0:_{H_d^t(M)} I \cong \operatorname{Hom}_R(R/I,H_d^t(M))$ thus $\operatorname{Ass}(H_d^t(M)) \supseteq \bigcup_{\mathfrak{p}\in V(\Phi)} \operatorname{Ass}(\operatorname{Hom}_R(R/\mathfrak{p},H_d^t(M)))$. Let $\mathfrak{p}\in\operatorname{Ass}(H_d^t(M))$. Then there exists a non-zero element m of $H_d^t(M)$ such that $\mathfrak{p}=\operatorname{Ann}(m)$ so that $\mathfrak{p}\in V(\Phi)$ by the pervious paragraph, because $H_d^t(M)$ is a d-torsion R-module. Thus $m\in 0:_{H_d^t(M)}\mathfrak{p}\cong\operatorname{Hom}_R(R/\mathfrak{p},H_d^t(M))$ and therefore $\bigcup_{\mathfrak{p}\in V(\Phi)}\operatorname{Ass}(\operatorname{Hom}_R(R/\mathfrak{p},H_d^t(M)))$ $\supseteq\operatorname{Ass}(H_d^t(M))$. The result follows now by the proof of Lemma 2.1, since

$$\operatorname{Hom}_R(R/\mathfrak{p},H_d^t(M))\cong\operatorname{Ext}_R^t(R/\mathfrak{p},M)\cong\operatorname{Hom}_R(R/\mathfrak{p},H_{\mathfrak{p}}^t(M)).$$

□

Theorem 2.4. *Let M be an R-module and t be an integer such that $H_d^i(M)=0$, for all $i<t$. Then $\operatorname{Ass}(H_d^t(M))\subseteq\{\mathfrak{p}:\dim R/\mathfrak{p}=d\}$ if and only if $H_c^t(M)=0$ for all integers $c<d$.*

Proof. Assume that $\operatorname{Ass}(H_d^t(M))\nsubseteq\{\mathfrak{p}:\dim R/\mathfrak{p}=d\}$. Thus there exists a prime ideal $\mathfrak{p}$ in $\operatorname{Ass}(H_d^t(M))$ such that $\dim R/\mathfrak{p}=c<d$. Now, the exact sequence $0\longrightarrow\Gamma_c(R/\mathfrak{p})\longrightarrow\Gamma_c(H_d^t(M))$ and $\Gamma_c(R/\mathfrak{p})=R/\mathfrak{p}$, $\Gamma_c(H_d^t(M))\cong H_c^t(M)$ show that $\mathfrak{p}\in\operatorname{Ass}(H_c^t(M))$. Hence, $H_c^t(M)\neq 0$. The converse is true by Lemma 2.1 (ii). □

3. The non-vanishing theorems

In this section, we shall provide some results concerning the vanishing and non-vanishing of d-local cohomology modules. Throughout R is a local ring with maximal ideal $\mathfrak{m}$ and d is a non negative integer.

We are now in a position to prove that the non-vanishing theorems in the d-local cohomology modules.

Theorem 3.1. *Let $(R,\mathfrak{m})$ be a local ring and let M be a non-zero finitely generated R-module of dimension n. Then there is at least one j with $0\leq j\leq d$ for which $H_d^{n-j}(M)\neq 0$.*

Proof. Since $\Gamma_{\mathfrak{m}}(\Gamma_d(M)) \cong \Gamma_{\mathfrak{m}}(M)$, so there is the Grothendieck spectral sequence

$$E_2^{i,j} := H^i_{\mathfrak{m}}(H^j_d(M)) \Rightarrow H^{i+j}_{\mathfrak{m}}(M) = E^{i+j}.$$

We have $\mathrm{Supp}(H^j_d(M)) \subseteq \Phi$ and so $\dim H^j_d(M) \leq d$. Thus by [6, Theorem 6.1.2], $E_2^{i,j} = 0$ for all $i > d$. There is a filtration $0 \subseteq F^n E^n \subseteq \cdots \subseteq F^1 E^n \subseteq E^n$ with $E_\infty^{i,n-i} \cong F^i E^n / F^{i+1} E^n$. Then $F^{d+1} E^n = \cdots = F^n E^n = 0$. If $E_\infty^{j,n-j} = 0$, for all j with $0 \leq j \leq d$, then $F^d E^n = \cdots = F^0 E^n = E^n = 0$ contrary to [6, Theorem 7.3.2]. So suppose that $E_\infty^{j,n-j} \neq 0$ for some j with $0 \leq j \leq d$. Thus $E_2^{j,n-j} \neq 0$ and then $H^{n-j}_d(M) \neq 0$. The proof is therefore complete. □

Theorem 3.2. *Let R be a complete ring with respect to the $\mathfrak{m}$-adic topology. Then the following statements are true:*

(i) *If M is a finitely generated R-module of dimension n and $0 < d \leq n$, then $H^n_d(M) = 0$.*

(ii) *If $\dim R = n$ and $0 < d < n$, then either $H^{n-1}_d(R) = 0$ or $H^{n-1}_d(R)$ is not finitely generated. In particular, $H^{n-1}_1(R)$ is not finitely generated.*

Proof. (i) Let $\mathfrak{p} \in \mathrm{Ass}(M)$ and $\dim R/\mathfrak{p} = n$. In this case Φ contains an ideal I of R with $\dim R/I = 1$ and $\mathfrak{p} \subseteq I$. So that $\dim R/(I + \mathfrak{p}) = \dim R/I = 1$ thus in view of [13, Theorem 2.4], the result follows.

(ii) Let $H^{n-1}_d(R) \neq 0$. Then $H^{n-1}_d(R) \cong H^{n-1}_d(R/\Gamma_d(R))$. So that in view of [6, Lemma 2.1.1] there exists a non-zero divisor $x \in \mathfrak{m}$. The exact sequence $0 \to R \xrightarrow{x} R \to R/xR \to 0$ induces an exact sequence $\cdots \to H^{n-1}_d(R) \xrightarrow{x} H^{n-1}_d(R) \to H^{n-1}_d(R/xR)$. By assumption R/xR is a local complete ring of dimension $n-1$, so that $H^{n-1}_d(R/xR) = 0$ by (i). Thus $H^{n-1}_d(R) = xH^{n-1}_d(R)$ which implies $H^{n-1}_d(R)$ is not finitely generated by Nakayma's Lemma.

In view of Theorem 3.1, there exists j with $0 \leq j \leq 1$ such that $H^{n-j}_1(R) \neq 0$ on the other hand $H^n_1(R) = 0$ by (i). Hence, $H^{n-1}_1(R) \neq 0$ and so is not finitely generated. □

Corollary 3.3. *If R is a complete ring with respect to the $\mathfrak{m}$-adic topology and M is a non-zero finitely generated R-module of dimension $n \geq 1$, then there is at least one j with $1 \leq j \leq d$ for which $H^{n-j}_d(M) \neq 0$.*

Proof. It is obvious by Theorems 3.1 and 3.2. □

Theorem 3.4. *Let R be Gorenstein of dimension n and let $0 \leq d \leq n$. Then the following statements are true:*

(i) *$H^i_d(R) \neq 0$ if and only if $i = n - d$.*

(ii) $\mathrm{Ass}(H_d^{n-d}(R)) = \{\mathfrak{p} \in \mathrm{Spec}(R) : \dim R/\mathfrak{p} = d\}$.
(iii) $H_d^{n-d}(R)$ *is not an Artinian module, for* $d > 0$.
(iv) $H_d^{n-d}(R)$ *has injective dimension* d.
(v) $H_d^{n-d}(R)$ *is a flat module.*

Proof. (i) Let $0 \to R \to E^0 \to E^1 \to \cdots \to E^{n-1} \to E^n \to 0$ be a minimal injective resolution of R. Then by [12, Theorems 18.1 and 18.8] we have $E^i = \bigoplus_{\dim R/\mathfrak{p}=n-i} E(R/\mathfrak{p})$. If $i \geq n-d$, then $\dim R/\mathfrak{p} = n-i \leq n-(n-d) = d$, and so $\mathfrak{p} \in \Phi$. Thus $\Gamma_d(E(R/\mathfrak{p})) = E(R/\mathfrak{p})$ and therefore $\Gamma_d(E^i) = E^i$. If $i < n-d$, then $\dim R/\mathfrak{p} = n-i > d$ and so $\mathfrak{p} \notin \Phi$. Thus $\Gamma_d(E(R/\mathfrak{p})) = 0$ and then $\Gamma_d(E^i) = 0$. It follows that $H_d^i(R) = 0$, whenever $i \neq n-d$. On the other hand, by Theorem 3.1, there is one j with $0 \leq j \leq d$ for which $H_d^{n-j}(R) \neq 0$. Hence, $H_d^{n-d}(R) \neq 0$.

(ii) This is immediate from (i) that

$$\mathrm{Ass}(H_d^{n-d}(R)) \subseteq \mathrm{Ass}(\bigoplus_{\dim R/\mathfrak{p}=d} E(R/\mathfrak{p})) = \{\mathfrak{p} \in \mathrm{Spec}(R) : \dim R/\mathfrak{p} = d\}.$$

Let $\mathfrak{p} \in \mathrm{Spec}(R)$ and $\dim R/\mathfrak{p} = d$. By [2, Lemma], $(H_d^{n-d}(R))_\mathfrak{p} \cong H^{n-d}_{d-\dim R/\mathfrak{p}}(R_\mathfrak{p}) = H_0^{n-d}(R_\mathfrak{p}) \cong H^{n-d}_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p})$ since every Gorenstein local ring is catenary and biequidimensional, see [12, Theorem 17.3]. Moreover, $H^{n-d}_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p}) \neq 0$, by [6, Theorem 7.3.2]. Thus $\mathfrak{p}$ is a minimal element of $\mathrm{Supp}(H_d^{n-d}(R))$ and then $\mathfrak{p} \in \mathrm{Ass}(H_d^{n-d}(R))$.

(iii) By (ii), $\mathrm{Ass}(H_d^{n-d}(R)) \nsubseteq \mathrm{Max}(R)$ so $H_d^{n-d}(R)$ is not Artinian.

(iv) It is obvious by the proof of (i).

(v) See [17, Theorem 2.1]. □

Example 3.5. *Let K be a field and let $R = K[X_1, \cdots, X_n]$ be the ring of polynomials over K in the indeterminates $X_1, \cdots, X_n$. Then for $\mathfrak{m} = (X_1, \cdots, X_n)$, $R_\mathfrak{m}$ is a local Goenstein ring of dimension n. So by Theorem 3.4 we have*

$$\mathrm{Ass}(H_1^{n-1}(R_\mathfrak{m})) = \{\mathfrak{p}R_\mathfrak{m} \in \mathrm{Spec}(R_\mathfrak{m}) : \dim(R/\mathfrak{p})_\mathfrak{m} = 1\}.$$

Now, $\mathrm{Ass}(H_1^{n-1}(R_\mathfrak{m}))$ *has infinite members by* [15, Exercise 15.3].

Theorem 3.6. *Let R be Gorenstein of dimension n, I an ideal of R and let $0 \leq d \leq n$. Then the following statements are true:*

(i) *If $I \notin \Phi$, then for all $i \geq 0$, $H_I^i(H_d^{n-d}(R)) \cong H_\Psi^{n-d+i}(R)$, where $\Psi = \{I^n + J : n \geq 1,\ J \in \Phi\}$.*
(ii) *For all $I \in \Phi$ and all $i \geq 0$, $H_I^i(H_d^{n-d}(R)) \cong H_I^{n-d+i}(R)$.*

Proof. (i) Let $I \notin \Phi$ and $\Psi = \{I^n + J : n \geq 1, J \in \Phi\}$. Then $\Gamma_I(\Gamma_d(R)) = \Gamma_\Psi(R)$. Thus there is a Grothendieck spectral sequence

$$E_2^{i,j} := H_I^i(H_d^j(R)) \Longrightarrow H_\Psi^{i+j}(R) = E^{i+j}.$$

Since $E_\infty^{i,n-d-i}$ is a subquotient of $E_2^{i,n-d-i}$ so by Theorem 3.4(i) we have $E_\infty^{i,n-d-i} = 0$, for all $i \geq 1$. Thus the filtration

$$0 = F^{n-d}E^{n-d} \subseteq F^{n-d-1}E^{n-d} \subseteq \cdots \subseteq F^1E^{n-d} \subseteq F^0E^{n-d} = E^{n-d}$$

implies that $F^{n-d}E^{n-d} = \cdots = F^1E^{n-d} = 0$ and $E_\infty^{0,n-d} \cong E^{n-d}$. By the sequence $0 \longrightarrow E_2^{0,n-d} \longrightarrow E_2^{2,n-d-1}$ and $E_2^{2,n-d-1} = 0$ we have $E_\infty^{0,n-d} \cong E_2^{0,n-d}$. Therefore, $\Gamma_I(H_d^{n-d}(R)) \cong H_\Psi^{n-d}(R)$.

Let $0 \to R \to E^0 \to E^1 \to \cdots \to E^{n-1} \to E^n \to 0$ be a minimal injective resolution of R. Then $0 \to H_d^{n-d}(R) \to E^{n-d} \to \cdots \to E^{n-1} \to E^n \to 0$ is an injective resolution of $H_d^{n-d}(R)$. For all $i \geq n-d$ we have

$$\Gamma_I(E^i) \cong \oplus_{\dim R/\mathfrak{p}=n-i}\Gamma_I(E(R/\mathfrak{p})) \cong \oplus_{\dim R/\mathfrak{p}=n-i}\Gamma_\Psi(E(R/\mathfrak{p})) \cong \Gamma_\Psi(E^i).$$

Hence, $H_I^i(H_d^{n-d}(R)) \cong H_\Psi^{n-d+i}(R)$, for all $i > 0$.

(ii) The proof is similar to that of (i). □

Corollary 3.7. *Let R be Gorenstein of dimension n and let $0 \leq d \leq n$. Then $\dim H_d^{n-d}(R) = \operatorname{depth} H_d^{n-d}(R) = d$ and $H_d^{n-d}(R)$ is not a finitely generated R-module.*

Proof. It follows by the proof of Theorems 3.2(ii), 3.4, 3.6 and [7, Exercise 9.1.12(c)]. □

Lemma 3.8. *Let $\{\mathfrak{p}_\lambda : \mathfrak{p}_\lambda \notin \Phi\}_{\lambda\in\Lambda}$ be a family of prime ideals of R. Then for any d-torsion R-module M, $\operatorname{Hom}_R(M, \oplus_{\lambda\in\Lambda}E(R/\mathfrak{p}_\lambda)) = 0$.*

Proof. Suppose that $\operatorname{Hom}_R(M, E(R/\mathfrak{p})) \neq 0$, for some $\mathfrak{p} \notin \Phi$. Thus there is a nonzero element $f \in \operatorname{Hom}_R(M, E(R/\mathfrak{p}))$ so $f(x) \neq 0$ for some $x \in M$. By assumption there is an $I \in \Phi$ such that $Ix = 0$ and so $If(x) = 0$. Hence, $I \subseteq \mathfrak{p}$. Otherwise for each $a \in I\backslash\mathfrak{p}$ we have the automorphism $E(R/\mathfrak{p}) \xrightarrow{a} E(R/\mathfrak{p})$, contrary to $If(x) = 0$. Therefore, $\operatorname{Hom}_R(M, E(R/\mathfrak{p})) = 0$ and so $\operatorname{Hom}_R(M, \oplus_{\lambda\in\Lambda}E(R/\mathfrak{p}_\lambda)) = 0$. □

Theorem 3.9. *Let R be Gorenstein of dimension n and let*

$$0 \to R \to E^0 \to \cdots \to E^i \xrightarrow{d^i} E^{i+1} \cdots \to E^{n-1} \to E^n \to 0$$

be an injective resolution of R. Then the following statements are true:

(i) $\operatorname{Ext}_R^j(H_d^i(M), R) = 0$, *for all R-module M, $0 \leq j < n-d$ and $i \geq 0$.*

(ii) $\operatorname{Ext}_R^j(E^i, R) = 0$ *for all $i \geq 1$ and $j < i$.*

Proof. (i) It follows by Lemma 3.8.

(ii) For $i \geq 1$ we have $H^i_{n-i}(R) \cong \ker d^i$. Thus

$$0 \to H^i_{n-i}(R) \to E^i \to H^{i+1}_{n-i-1}(R) \to 0$$

is an exact sequence that induces the long exact sequence

$$\cdots \to \mathrm{Ext}^j_R(H^{i+1}_{n-i-1}(R), R) \to \mathrm{Ext}^j_R(E^i, R) \to \mathrm{Ext}^j_R(H^i_{n-i}(R), R) \to \cdots .$$

By (i), $\mathrm{Ext}^j_R(H^{i+1}_{n-i-1}(R), R) = 0 = \mathrm{Ext}^j_R(H^i_{n-i}(R), R)$, for all $j < i$. Hence $\mathrm{Ext}^j_R(E^i, R) = 0$ for all $j < i$ □

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