



INVERSE NODAL ANALYSIS FOR VECTOR-VALUED SINGULAR DIFFUSION OPERATORS

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Abstract: We examine a vector-valued diffusion problem given by the differential equation $-y'' + [\lambda p(x) + q(x)]y = \lambda^2 y$ with boundary conditions $\begin{cases} Ay(0) = -I_d y'(0) \\ By(\pi) = -I_d y'(\pi) \end{cases}$ where $p(x)$ and $q(x)$ denotes a continuous symmetric matrix-valued function on the interval

$[0, \pi]$ and k and K are real symmetric matrices of size $d \times d$. An eigenfunction $y(x)$ is referred to as being of type (CZ) if all isolated zeros in its components coincide with nodal points of the function itself. We demonstrate that in the two-dimensional case $d = 2$, the existence of infinitely many (CZ)-type eigenfunctions is both necessary and sufficient for the simultaneous diagonalizability of $p(x)_k$ and A, B . Consequently, if all but finitely many eigenfunctions possess the (CZ) property, then the coefficient matrix and the associated boundary data can be uniquely reconstructed up to a constant unitary transformation.

Keywords: Inverse nodal problem, Vector-valued Sturm–Liouville systems, Singular diffusion operators, Common zero property

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1. Introduction

The determination of differential operators from incomplete or indirect observations constitutes one of the central themes of inverse problem theory. Among the various types of inverse data, the nodal distribution of eigenfunctions has attracted considerable attention due to its close relationship with the qualitative behavior of solutions. Since nodal points encode information about both the oscillatory nature of eigenfunctions and the coefficients governing the underlying operator, they provide an effective framework for recovering structural characteristics of the problem under consideration (McLaughlin, 1988; Hald, 1989).

The scalar Sturm–Liouville framework has provided the foundation for much of the existing research on inverse nodal problems. A substantial body of work has demonstrated that nodal information alone is sufficient to determine the underlying potential and boundary coefficients, modulo certain unavoidable normalization constants. Moreover, a variety of reconstruction procedures based exclusively on nodal observations have been established for different classes of boundary value problems (McLaughlin, 1988;; Hald, 1989; Yang, 1997a, 1997b; Shen, 1999; Chen, 2002). Subsequent investigations have further explored how eigenvalue asymptotics, smoothness conditions, and spectral characteristics influence the accuracy and uniqueness of

the reconstruction process (Levitan and Sargsjan, 1991; Gesztesy and Simon, 1997; Hochstadt and Lieberman, 1978; Zettl, 2005; Pöschel and Trubowitz, 1987).

The transition from scalar inverse nodal problems to vector-valued and matrix Sturm–Liouville systems introduces a considerably richer mathematical structure. Unlike the scalar case, matrix coefficients generate interactions among the components of eigenfunctions, resulting in nodal patterns that are substantially more complex and difficult to analyze. While a variety of inverse spectral questions for matrix and vectorial operators have been addressed in the literature (Shen and Shieh, 1999; Carlson, 1999; Yurko, 2000; Brown and Weikard, 2000; Bondarenko, 2015), corresponding results based on nodal information remain relatively limited. Additional complications arise when the governing model contains transmission conditions, internal discontinuities, or singular terms. Such phenomena frequently occur in diffusion processes, layered materials, and composite structures, where standard techniques developed for scalar Sturm–Liouville equations are generally insufficient and must be replaced by more sophisticated analytical approaches (Agranovich and Marchenko, 1963; Kurasov, 1996; Carlson, 1998; Pivovarchik, 2000; Hryniv and Mykytyuk, 2003; Buterin, 2015).



One of the distinctive features of vector-valued Sturm-Liouville systems is the possibility that all components of an eigenfunction share the same nodal locations. This phenomenon, known as the common zero (CZ) property, establishes a strong relationship between the distribution of nodal points and the spectral structure of the underlying matrix coefficients. From an inverse perspective, the CZ property is particularly significant because it creates a bridge between vectorial models and collections of decoupled scalar problems, thereby allowing information contained in nodal data to be interpreted through a simpler spectral framework (Shen and Shieh, 1999; Brown and Weikard, 2000).

In the classical scalar case, the problem is framed through the differential equation:

$$-y'' + [\lambda p(x) + q(x)]y = \lambda^2 y$$

subject to the Robin-type boundary conditions:

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0$$

$$y(1)\cos\gamma + y'(1)\sin\gamma = 0$$

where $q \in L_1(0, \pi)$ and $\alpha, \gamma \in (0, \pi)$. For each eigenvalue λ_n , let $y_n(x)$ denote the corresponding eigenfunction, and define the nodal set $\left[x_k^{(n)} \right]_{k=1}^{n-1} X_n$ as the collection of zeros of $y_n(x)$ in the open interval $(0, \pi)$.

Classical results in inverse nodal analysis show that the data $(p(x), q(x))$ can be determined uniquely from the nodal distribution of eigenfunctions, modulo an additive constant in q (McLaughlin, 1988; Yang, 1997a). Moreover, nodal data have been employed not only to establish uniqueness but also to derive practical reconstruction schemes for recovering the unknown quantities of the problem. Several such approaches have been proposed and successfully implemented in different settings (Hald, J.; Yang, 1997a; Shen, 1999).

Although inverse nodal methods are well established for scalar Sturm-Liouville equations, their extension to vector-valued systems is still at a comparatively early stage. Existing studies have mainly focused on inverse eigenvalue and spectral questions for matrix-valued operators (Shen and Shieh, 1999), whereas reconstruction procedures based directly on nodal information have received much less attention. As a result, many fundamental aspects of inverse nodal analysis for vector-valued problems remain to be explored.

In this paper, we consider the following vectorial generalization:

$$\left. \begin{aligned} -Y'' + [2\lambda P(x) + Q(x)]Y &= \lambda^2 I_d Y \\ A_1 Y(0) - A_2 Y'(0) &= 0 \\ B_1 Y(\pi) + B_2 Y'(\pi) &= 0 \end{aligned} \right\} \quad (1)$$

Consider problem (1). If we take $P(x)$ as

$$P(x) = \delta(x-a)I_d,$$

$$-Y'' + Q(x)Y = \lambda^2 I_d Y \quad (2)$$

$$\begin{aligned} A_1 Y(0) - A_2 Y'(0) &= 0 \\ B_1 Y(\pi) + B_2 Y'(\pi) &= 0 \end{aligned} \quad (3)$$

$$\begin{aligned} Y(a+0) &= Y(a-0) \\ Y'(a+0) - Y'(a-0) &= \lambda I_d \sigma Y(a) \end{aligned} \quad (4)$$

problem (equation 1) is reduced to (equations 2-4).

In this formulation, λ denotes the spectral parameter, and $y(x) = [y_1(x), y_2(x), \dots, y_m(x)]^T$ is an m -dimensional vector-valued function.

The coefficient matrix $q(x) = [q_{ij}(x)]$ is assumed to be symmetric and continuous on the interval $[0, \pi]$.

The boundary matrices A_i and B_i ($i = 1, 2$) are real $d \times d$ matrices, and the identity matrix I_d appears explicitly on the right-hand side of the differential equation.

This general setting covers several classical boundary conditions as special cases. If A_2 and B_2 are both zero while A_1 and B_1 are invertible, the boundary conditions reduce to the standard Dirichlet type. Alternatively, if $A_1 = 0$ and $B_1 = 0$ and the matrices A_2 and B_2 are invertible, the system corresponds to Neumann boundary conditions.

To ensure that the system is self-adjoint, the boundary matrices must satisfy certain structural constraints, namely:

$$A_1 A_2^* = A_2 A_1^* \quad , \quad B_1 B_2^* = B_2 B_1^*$$

$$\text{rank}[A_1, A_2] = \text{rank}[B_1, B_2] = d$$

In the special case where A_2 and B_2 are both invertible, the self-adjointness conditions simplify to

$$A_1 A_2^{-1} = (A_2 A_1^{-1})^T \quad , \quad B_2^{-1} B_1 = (B_1^{-1} B_2)^T$$

which implies that the matrices $A := A_1 A_2^{-1}$ and $B := B_2^{-1} B_1$ are symmetric. Consequently, the boundary conditions can be rewritten in a normalized form:

$$A y(0) + I_d y'(0) = 0$$

$$B y(\pi) + I_d y'(\pi) = 0$$

with A, B symmetric. This general formulation encompasses various classical cases, including Neumann conditions, and forms the basis for our analysis.

2. Preliminaries

Let $Y(x)$ denote a continuous vector-valued function defined on the interval $[0, \pi]$ and taking values in $\mathbb{R}^d$. A point $x_0 \in [0, \pi]$ is said to be a nodal point of $Y(x_0)$ if $Y(x_0) = 0$, meaning that all components of the vector simultaneously vanish at x_0 .

We say that $y(x)$ is of type (CZ), short for *common zero type* if every isolated zero occurring in any single component of $y(x)$ corresponds to a nodal point, i.e., all components vanish at that location. In other words, the isolated zeros of individual components must coincide with full vector zeros.

A matrix-valued function $P(x)$ is regarded as diagonalizable in a simultaneous sense when a single unitary matrix Q exists that transforms $P(x)$ into a diagonal form for every x in the interval $[0, \pi]$. That is, under the conjugation by Q , the matrix $P(x)$ becomes diagonal uniformly with respect to x .

Using this notion as a basis, we say that a vectorial Sturm–Liouville problem is simultaneously diagonalizable if the same unitary matrix Q diagonalizes not only the coefficient matrix $P(x)$ but also the associated boundary matrices A_i and B_i ($i = 1, 2$). Equivalently, the conjugated matrices $Q^*P(x)Q$, Q^*A_iQ and Q^*B_iQ all take diagonal form.

If there exists a unitary matrix Q such that the conjugations $Q^*P(x)Q$, Q^*A_iQ , Q^*B_iQ ($for i=1,2$) yield diagonal matrices, then the system described by equations (1)-(4) can be decoupled into d independent scalar Sturm–Liouville problems. Each resulting scalar problem admits a complete set of eigenfunctions, and their combination via the inverse transformation Q yields vector-valued eigenfunctions of the original system.

Crucially, the eigenfunctions of the decoupled system are naturally of type (CZ), as each scalar component possesses isolated zeros, which correspond to nodal points in the original vector setting. This leads to the following lemma characterizing the nodal structure in the diagonalizable case:

Lemma 2.1. Assume that the vectorial Sturm–Liouville problem considered in this paper is simultaneously diagonalizable. That is, there exists a unitary matrix Q such that, under the transformation $Y(x) = QZ(x)$, the coefficient matrix $Q(x)$ and the boundary matrices A_i and B_i ($i = 1, 2$) are transformed into diagonal matrices. Then, the problem admits infinitely many eigenfunctions whose nodal sets are of common zero type; namely, for each such eigenfunction, all components vanish simultaneously at every nodal point.

Lemma 2.1 shows that simultaneous diagonalizability of the vectorial Sturm–Liouville problem guarantees the existence of infinitely many eigenfunctions whose nodal sets exhibit the common zero property. This observation highlights a strong link between the algebraic structure of the coefficient matrices and nodal points.

Lemma 2.2 (Structural Consequence of the CZ Property) Let $d = 2$, and suppose that the vector-valued Sturm–Liouville problem admits infinitely many eigenfunctions of common zero type (CZ). Then the nodal sets of the two components of each such eigenfunction coincide.

Consequently, for every CZ-type eigenfunction

$$y_n(x) = (y_{n,1}(x), y_{n,2}(x))^T,$$

the ratio

$$r_n(x) = y_{n,1}(x)/y_{n,2}(x)$$

is well defined on each nodal interval and can be continuously extended across common nodal points.

Since infinitely many eigenfunctions possess the CZ property, the corresponding nodal structures become asymptotically identical. Consequently, infinitely many eigenfunctions share a common asymptotic direction in $\mathbb{R}^2$. In particular, there exists a nonzero vector $v \in \mathbb{R}^2$ such that infinitely many eigenfunctions are asymptotically aligned with v .

This implies that the coefficient matrix $Q(x)$ preserves a common one-dimensional invariant subspace for every $x \in [0, \pi]$.

Since $Q(x)$ is symmetric, the orthogonal complement of this invariant subspace is also invariant. Therefore, there exists a constant unitary matrix U such that

$$U^{-1}QU = \text{diag}(q_1(x), q_2(x))$$

for every $x \in [0, \pi]$.

Hence the coefficient matrix $Q(x)$ is simultaneously diagonalizable.

Theorem 2.3. Let $d = 2$ and consider the vectorial Sturm–Liouville problem studied in this paper, together with the associated boundary and transmission conditions. Let

$\{(\lambda_n, y_n(x))\}_{n=1}^\infty$ be its sequence of eigenpairs, where each

eigenfunction $y_n(x) = (y_{n,1}(x), y_{n,2}(x))^T$ is a two-component vector-valued function.

Proof. The existence of infinitely many CZ-type eigenfunctions implies that the nodal sets of the components coincide for infinitely many eigenfunctions. By Lemma 2.2, this forces the corresponding eigenfunctions to share a common asymptotic direction in $\mathbb{R}^2$.

Consequently, the coefficient matrix $Q(x)$ possesses a common invariant subspace for all $x \in [0, \pi]$. Since $Q(x)$ is symmetric, its orthogonal complement is invariant as well. Therefore, there exists a constant unitary matrix U such that

$$U^{-1}QU = \text{diag}(q_1(x), q_2(x))$$

for every $x \in [0, \pi]$.

Applying the same transformation to the boundary and transmission matrices yields a decoupled system consisting of two independent scalar Sturm–Liouville problems.

Hence the original vector-valued Sturm–Liouville problem is simultaneously diagonalizable.

Assume that there exists an infinite subsequence of eigenfunctions whose nodal sets are of common zero type, that is, for each such eigenfunction, all components vanish simultaneously at every nodal point. Then the matrix-valued coefficient function of the problem is simultaneously diagonalizable by a constant unitary transformation. More precisely, there exists a unitary matrix Q such that, under the transformation $Y_n(x) = QZ_n(x)$, the original vectorial Sturm–Liouville

problem decomposes into two independent scalar Sturm–Liouville problems, each governing one component of $z_n(x)$. In fact, the result can be extended even further, as stated in our main theorem:

Theorem 2.4. Let $d=2$, equations 2-4 together with the transmission and boundary conditions. Suppose that this problem possesses infinitely many eigenfunctions $\{y_n(x)\}_{n=1}^{\infty}$, whose nodal sets are of common zero type (that is, all components vanish simultaneously at each nodal point).

Then the corresponding matrix-valued Sturm–Liouville problem is simultaneously diagonalizable, in the sense that there exists a constant invertible matrix Q such that the transformed problem obtained via $\tilde{Y}(x)=Q^{-1}Y(x)$ decouples into two independent scalar Sturm–Liouville problems.

Furthermore, when additional structural properties of the eigenfunctions are available, it becomes possible to uniquely recover the defining data of the system. This leads to the following reconstruction result:

Corollary 2.5. Let $d=2$ and consider the vectorial (2)-(4) problem defined by the differential equation and boundary conditions. Assume that all but finitely many eigenfunctions of this problem possess the common zero property.

Then, the underlying vectorial (2)-(4) problem is uniquely determined up to a constant unitary transformation. Moreover, the coefficient matrices of the problem can be explicitly reconstructed from the corresponding nodal data.

Proof. In view of Theorem 2.4, it follows that, under the stated assumptions, the coefficient matrices $Q(x)$, A , and B can be reduced simultaneously to diagonal form by a unitary transformation. More precisely, there exists a unitary matrix U such that the matrices $UQ(x)U^*$, UAU^* and UBU^* are all diagonal. In this representation, the transformed coefficient matrix $UQ(x)U^*$ has the scalar entries $u_1(x)$ and $u_2(x)$ on its diagonal, while the transformed boundary matrices UAU^* and UBU^* have constant diagonal entries a_1, a_2 and b_1, b_2 respectively.

By introducing the change of variables $Q^*y(x)=[v(x), w(x)]^T$, the original vectorial Sturm–Liouville equation decouples into two independent scalar equations 5-6.

The first component satisfies:

$$\begin{cases} -v''(x) + u(x)v(x) = \mu v(x) \\ a_1 v(0) + v'(0) = 0 \\ a_1 v(\pi) + v'(\pi) = 0 \\ v'(a+0) - v'(a-0) = \mu \beta v(a) \end{cases} \quad (5)$$

The second component satisfies:

$$\begin{cases} -w''(x) + u(x)w(x) = \eta w(x) \\ a_1 w(0) + w'(0) = 0 \\ a_1 w(\pi) + w'(\pi) = 0 \\ w'(a+0) - w'(a-0) = \eta \beta w(a) \end{cases} \quad (6)$$

These two scalar Sturm–Liouville problems can be analyzed independently. Since the nodal information is preserved under the unitary transformation Q , and because the scalar problems fall within the well-studied class of inverse nodal problems, the functions $(u_1(x), a_1, b_1)$ and can each be uniquely determined up to an additive constant using known reconstruction results. The classical inverse nodal reconstruction results of McLaughlin (1988), Hald, J.; Yang, 1997a; 1997b) can therefore be applied componentwise to the decoupled scalar problems. Consequently, the diagonal entries of the transformed coefficient matrix are uniquely recoverable from the corresponding nodal data. This completes the justification.

As a consequence, the potential functions $u_1(x)$ and $u_2(x)$, along with the associated boundary parameters (a_1, b_1) and (a_2, b_2) can all be uniquely reconstructed from their respective nodal data up to an additive constant in the potential.

Durak (2024) previously obtained eigenvalue estimates for related matrix differential operators under various boundary conditions. In contrast, the present approach yields a simpler asymptotic formulation.

We will present the eigenvalue asymptotic estimates in the next section. For the sake of completeness, certain auxiliary lemmas including Lemma 3.1 will also be provided with detailed proofs in subsequent sections.

3. Eigenvalue Asymptotics

In this section, we investigate the asymptotic behavior of the eigenvalues and corresponding nodal points for the vectorial Sturm–Liouville problem introduced earlier. The results established herein are valid for general dimension $d \geq 2$ and form a central component in the proof of our main theorem.

Consider again the system:

$$-Y''(x) + P(x)Y(x) = \lambda I_d Y(x)$$

with initial conditions specified at the left end point as

Let $Y(x, \lambda)$ denote the matrix-valued solution to this initial value problem. Our aim is to derive an asymptotic estimate for the eigenvalues λ_n which are defined implicitly through the boundary condition at $x=1$:

$$BY(\pi, \lambda) + Y'(\pi, \lambda) = 0$$

We begin with the following key lemma adapted from McLaughlin (1988)

Lemma 3.1. Let $Y(x, \lambda)$ be the fundamental matrix solution of the differential equation $-Y'' + Q(x)Y = \lambda I_d Y$ with initial conditions $Y(0, \lambda) = I_d, Y'(0, \lambda) = -A$.

Then, it satisfies the integral representation:

$$Y(x, \lambda) = \cos(\sqrt{\lambda}x)I_d + \frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}}A + \int_0^x \frac{\sin(\sqrt{\lambda}(x-t))}{\sqrt{\lambda}}P(t)Y(t, \lambda)dt$$

This expression provides the foundation for analyzing the spectral properties of the system. In particular, we can derive asymptotic formulas for the eigenvalues λ_n by examining the behavior of the determinant of the associated boundary operator.

Theorem 3.2. Let λ_n denote the n -th eigenvalue of (2)-

(4). Define $m = \left\lfloor \frac{n}{d} \right\rfloor$ where d is the dimension of the

system. Then, for each $j=1,2,\dots,d$ the eigenvalues satisfy the following asymptotic relation for sufficiently large m :

$$\lambda_{dm+j} = m + \frac{a_{0j}}{m\pi^2} + O\left(\frac{1}{m^2}\right). \quad (7)$$

Here, the ordering is in the form of $a_{i1} \leq a_{i2} \leq \dots \leq a_{id}$, $i=0,1,2,3$. a_{ij} are the eigenvalues of the following expressions, respectively;

$$\begin{aligned} & \left[A - \frac{1}{4}\beta \int_0^a Q(t)dt + \frac{1}{2}\int_0^\pi Q(t)dt \right], \\ & \left[-\frac{1}{2}\beta \left(A - \frac{1}{2}\int_0^a Q(t)dt + \int_0^\pi Q(t)dt \right) \right], \\ & \left[\frac{1}{2}\beta \left(A - \frac{3}{2}\int_0^a Q(t)dt + \int_0^\pi Q(t)dt \right) \right], \left[\frac{1}{4}(1-\beta^2) \int_0^a Q(t)dt \right]. \end{aligned}$$

Proof. Using the asymptotic formula established by Amirov and Durak (2024), we obtain (equation 8),

$$\lambda_{nj} = \lambda_{nj}^0 + \frac{d_{nj}}{\lambda_{nj}^0} + o\left(\frac{1}{\lambda_{nj}^0}\right). \quad (8)$$

where,

$$\begin{aligned} d_{nj} = & \frac{1}{\Delta_0(\lambda_{nj}^0)} \left[(w_0(\pi) + A) \sin(\lambda_{nj}^0 \pi) I_d \right. \\ & + \left(\frac{\beta}{2} A - w_1(\pi) \right) \cos(\lambda_{nj}^0 \pi) I_d \\ & \left. + \left(w_2(\pi) - \frac{\beta}{2} A \right) \cos \lambda_{nj}^0 (2a - \pi) I_d + w_3(\pi) \sin \lambda_{nj}^0 (2a - \pi) I_d \right]. \end{aligned}$$

$$F_Q(\lambda) = B\phi(\pi, \lambda) + \phi'(\pi, \lambda)$$

$$\begin{aligned} & = B \cos \lambda \pi I_d + \frac{\beta}{2} B (\sin \lambda \pi - \sin \lambda (2a - \pi)) I_d \\ & + \frac{1}{\lambda} B I_d w_0(\pi) \sin \lambda \pi + \frac{1}{\lambda} B w_1(\pi) \cos \lambda \pi I_d \\ & + \frac{(1-2\beta^2)}{4\lambda} B \sin \lambda (2a - \pi) \int_0^a q(t) dt I_d \\ & + \frac{1}{2\lambda} B w_2(\pi) \cos \lambda (2a - \pi) I_d - \lambda \sin \lambda \pi I_d \\ & + \frac{\lambda \beta}{2} \cos \lambda \pi I_d + \frac{\beta \lambda}{2} \cos \lambda (2a - \pi) I_d \\ & + w_0(\pi) \cos \lambda \pi I_d - w_1(\pi) \sin \lambda \pi I_d \\ & - \frac{(1-2\beta^2)}{4} \cos \lambda (2a - \pi) \int_0^a q(t) dt I_d \\ & + w_2(\pi) \sin \lambda (2a - \pi) I_d + o\left(\frac{\exp|\tau|\pi}{\lambda} I_d\right) \end{aligned}$$

$$\begin{aligned} & = \left(B + \frac{\beta}{2} I_d \right) \cos \lambda \pi + \left(\frac{\beta}{2} \beta - \lambda I_d \right) \sin \lambda \pi - \frac{\beta}{2} B \sin \lambda (2a - \pi) I_d \\ & + \frac{\beta}{2} \cos \lambda (2a - \pi) I_d + \left(\frac{\beta}{\lambda} \sin \lambda \pi + \cos \lambda \pi I_d \right) w_0(\pi) I_d \\ & + \left(\frac{1}{\lambda} B \cos \lambda \pi - \sin \lambda \pi I_d \right) w_1(\pi) I_d \\ & + \left(\frac{1}{2\lambda} B \cos \lambda (2a - \pi) + \sin \lambda (2a - \pi) I_d \right) w_2(\pi) I_d \\ & + \left(\frac{1}{\lambda} B \sin \lambda (2a - \pi) - \cos \lambda (2a - \pi) I_d \right) w_3(\pi) I_d \\ & + o\left(\frac{\exp|\tau|\pi}{\lambda} I_d\right) \end{aligned}$$

Define the set

$$\begin{aligned} G_Q(\lambda) = & \frac{1}{\lambda} B \left[A - \frac{1}{4}\beta \int_0^a Q(t)dt + \frac{1}{2}\int_0^\pi Q(t)dt \right] \sin \lambda \pi I_d \\ & + \left[A - \frac{1}{4}\beta \int_0^a Q(t)dt + \frac{1}{2}\int_0^\pi Q(t)dt \right] \cos \lambda \pi I_d + o\left(\frac{\exp|\tau|\pi}{\lambda} I_d\right). \end{aligned}$$

If we use Rouché's theorem for $\lambda \in R \setminus \{0\}$ when $\exp|\operatorname{Im} \lambda| = 1$, we get the zeros of $\det(G_Q(\lambda))$. Here,

$\{a_{01}, a_{02}, \dots, a_{0d}\}$ is the set of eigenvalues of

$$B \left[A - \frac{1}{4}\beta \int_0^a Q(t)dt + \frac{1}{2}\int_0^\pi Q(t)dt \right].$$

$$\lambda_{dm+j} = \lambda_{dm+j}^0 + \delta_{dm}^j = m + h_{dm}^j + \delta_{dm}^j$$

$$\tan(m\pi + h_{dm}^j \pi + \delta_{dm}^j \pi) = \tan(h_{dm}^j \pi + \delta_{dm}^j \pi) = \frac{a_{0j}}{m\pi + h_{dm}^j + \delta_{dm}^j}$$

$$h_{dm}^j + \delta_{dm}^j = \frac{a_{0j}}{m\pi + h_{dm}^j + \delta_{dm}^j} + O\left(\frac{1}{m^2}\right)$$

$$= \frac{a_{0j}}{m\pi} \cdot \frac{1}{1 + \frac{h_{dm}^j + \delta_{dm}^j}{m\pi}} = \frac{a_{0j}}{m\pi} + o\left(\frac{1}{m}\right).$$

$$\lambda_{dm+j} = \lambda_{dm+j}^0 + \delta_{dm}^j = m + \frac{a_{0j}}{m\pi^2} + O\left(\frac{1}{m^2}\right)$$

The remaining assertions can be established by arguments analogous to those used above and are therefore omitted.

Theorem 3.3. Consider the boundary value problem of (2)-(4) and let λ_n be its eigenvalues with corresponding eigenfunctions $y_n(x)$ normalized in any fixed way.

Denote by

$$0 < x_1^{(n)} < x_2^{(n)} < \dots < x_v^{(n)} < \pi$$

the nodal points (zeros) of $y_n(x)$ in $(0, \pi)$, counted in increasing order.

Let λ_n^0 be the "unperturbed" spectral sequence associated with the characteristic function $\Delta_0(\lambda)$, so that

$\lambda_n^0 \rightarrow \infty$ and $\lambda_n^0 \cong n$ as $n \rightarrow \infty$. Then, for all sufficiently large n , the j -th nodal point satisfies the uniform asymptotic formula (equation 9),

$$x_j^{(n)} = \left(\frac{j-1}{2} \right) \pi + O\left(\frac{1}{(\lambda_n^0)^2}\right), \quad j=1,2,\dots,v \quad (9)$$

Proof. From the standard large-parameter asymptotics of the fundamental solution, we may write, uniformly for $x \in [0, \pi]$,

$$y_n(x) = C_n \left(\cos(\lambda_n^0 x) + R_n(x) \right), \quad \|R_n\|_{L^r(0,\pi)} = O\left(\frac{1}{\lambda_n^0}\right)$$

where $C_n \neq 0$ is a nonzero scalar depending on the normalization.

Fix $j \in \{1, 2, \dots, v\}$ and define two "bracketing" points:

$$x_{j,-} = \frac{\left(j - \frac{1}{2}\right)\pi - \frac{\pi}{4}}{\lambda_n^0}, \quad x_{j,+} = \frac{\left(j - \frac{1}{2}\right)\pi + \frac{\pi}{4}}{\lambda_n^0}.$$

Then

$$\cos(\lambda_n^0 x_{j,-}) = \cos\left(\left(j - \frac{1}{2}\right)\pi - \frac{\pi}{4}\right) = +\frac{\sqrt{2}}{2}$$

$$\cos(\lambda_n^0 x_{j,+}) = \cos\left(\left(j - \frac{1}{2}\right)\pi + \frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

Since $R_n(x) = O\left(\frac{1}{\lambda_n^0}\right)$ uniformly, for all sufficiently large n

we have

$$\cos(\lambda_n^0 x_{j,-}) + R_n(x_{j,-}) > 0, \quad \cos(\lambda_n^0 x_{j,+}) + R_n(x_{j,+}) < 0$$

Multiplying by $C_n \neq 0$ preserves a sign change, hence

$$y_n(x_{j,-}) \cdot y_n(x_{j,+}) < 0 \text{ for large } n.$$

By continuity of y_n , the intermediate value theorem implies the existence of at least one point

$$x_j^{(n)} \in (x_{j,-}, x_{j,+}) \text{ such that } y(x_j^{(n)}) = 0.$$

By the standard oscillation properties of Sturm-Liouville-type problems (and the fact that zeros are simple for large n , this zero corresponds to the j -th nodal point.

Consequently, there exists $\varepsilon_{j,n}$ with $|\varepsilon_{j,n}| < \frac{\pi}{4}$ such that

$$\lambda_n^0 x_j^{(n)} = \left(j - \frac{1}{2}\right)\pi + \varepsilon_{j,n}.$$

$$\text{Equivalently, } x_j^{(n)} = \frac{\left(j - \frac{1}{2}\right)\pi}{\lambda_n^0} + \frac{\varepsilon_{j,n}}{\lambda_n^0}.$$

Evaluating the eigenfunction at the nodal point and using the asymptotic representation,

$$0 = y_n(x_j^{(n)}) = C_n \left(\cos(\lambda_n^0 x_j^{(n)}) + R_n(x_j^{(n)}) \right)$$

so

$$\cos(\lambda_n^0 x_j^{(n)}) = -R_n(x_j^{(n)}) = O\left(\frac{1}{\lambda_n^0}\right)$$

Using $\lambda_n^0 x_j^{(n)} = \left(j - \frac{1}{2}\right)\pi + \varepsilon_{j,n}$, we get

$$\cos\left(\left(j - \frac{1}{2}\right)\pi + \varepsilon_{j,n}\right) = (-1)^n \sin(\varepsilon_{j,n}). \text{ Hence}$$

$$\sin(\varepsilon_{j,n}) = O\left(\frac{1}{\lambda_n^0}\right)$$

Since $|\varepsilon_{j,n}| < \frac{\pi}{4}$, we have $\sin(\varepsilon_{j,n}) \cong \varepsilon_{j,n}$ and therefore

$$\varepsilon_{j,n} = O\left(\frac{1}{\lambda_n^0}\right). \text{ Substituting back gives}$$

$$x_j^{(n)} = \frac{\left(j - \frac{1}{2}\right)\pi}{\lambda_n^0} + O\left(\frac{1}{\lambda_n^0}\right)$$

uniformly in j . This completes the proof.

Corollary 3.4. Define the nodal lengths $l_j^{(n)} = x_{j+1}^{(n)} - x_j^{(n)}$,

$$j = 1, 2, \dots, v-1.$$

Let λ_n^0 denote the unperturbed spectral sequence used in Theorem 2.3. Then, as $n \rightarrow \infty$,

$$\text{a) } l_j^{(n)} = \frac{\pi}{\lambda_n^0} + O\left(\frac{1}{\lambda_n^0}\right), \quad j = 1, 2, \dots, v-1$$

$$\text{b) } \lambda_n^0 l_j^{(n)} = \pi + O\left(\frac{1}{\lambda_n^0}\right), \quad j = 1, 2, \dots, v-1$$

$$\text{c) } \lambda_n^0 l_j^{(n)} = \left(j - \frac{1}{2}\right)\pi + O\left(\frac{1}{\lambda_n^0}\right), \quad j = 1, 2, \dots, v-1$$

4. Proof of Theorem 2.4

Lemma 4.1. Let $Y(x, \lambda_n)$ be the fundamental matrix solution of equation (8), and let $v_n \in \mathbb{R}^d$ be a unit null vector in the kernel of $TP(\lambda_n)$ such that $y_n(x) := Y(x, \lambda_n)$ is an eigenfunction of (2) and (4) associated with eigenvalue λ_n and further assume that $y_n(x)$ is of type (CZ) with nodal set $\{x_k^{(n)}\}$. Then, as $n \rightarrow \infty$

a) For every fixed $x \in (0, \pi)$

$$\lim_{n \rightarrow \infty} \lambda_n^0 \sum_{j: x_{j+1}^{(n)} \leq x} \left(l_j^{(n)} - \frac{\pi}{\lambda_n^0} \right) = -\frac{1}{2} \int_0^x q(t) dt.$$

b) For almost every $x \in (0, \pi)$, one has

$$\text{c) } \lim_{n \rightarrow \infty} \lambda_n^0 \left(\lambda_n^0 l_{j(x,n)}^{(n)} - \pi \right) = -\frac{1}{2} \int_{x_j^{(n)}}^{x_{j(x,n)+1}^{(n)}} q(t) dt.$$

where $j(x, n)$ is the unique index satisfying $x \in (x_{j(x,n)}^{(n)}, x_{j(x,n)+1}^{(n)})$.

Proof of Theorem 2.4

Let $\{x_j^{(n)}\}$ be the nodal points of the eigenfunction corresponding to the eigenvalue λ_n and let

$$l_j^{(n)} = x_{j+1}^{(n)} - x_j^{(n)}, \quad j = 1, 2, \dots, v-1. \text{ denote the nodal lengths.}$$

By Corollary 3.4(a), the nodal lengths admit the asymptotic representation (equation 10),

$$l_j^{(n)} = \frac{\pi}{\lambda_n^0} + O\left(\frac{1}{\lambda_n^0}\right) \quad (10)$$

uniformly in j . Moreover, by Lemma 3.1(b), the deviation of $l_j^{(n)}$ from $\frac{\pi}{\lambda_n^0}$ satisfies

$$\lambda_n^0 \left(\lambda_n^0 l_{j(x,n)}^{(n)} - \pi \right) = -\frac{1}{2} \int_{x_j^{(n)}}^{x_{j(x,n)+1}^{(n)}} q(t) dt + o(1), \quad n \rightarrow \infty.$$

Fix $x \in (0, \pi)$ and let $j = j(x, n)$ be the unique index such that $x \in (x_j^{(n)}, x_{j+1}^{(n)})$

Since $l_j^{(n)} \rightarrow 0$ as $n \rightarrow \infty$, the interval $(x_j^{(n)}, x_{j+1}^{(n)})$ shrinks to the point x .

Dividing the relation in (10) by $l_j^{(n)}$ and using the asymptotic formula for $l_j^{(n)}$, we obtain (equation 11),

$$\frac{1}{I_j^{(n)}} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q(t) dt = -2 \left(\lambda_n^0 \right)^2 \left(I_j^{(n)} - \frac{\pi}{\lambda_n^0} \right) o(1). \quad (11)$$

Let x be a Lebesgue point of q . Then, by the Lebesgue differentiation theorem,

$$\lim_{n \rightarrow \infty} \frac{1}{I_j^{(n)}} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q(t) dt = q(x).$$

Taking limits in the identity obtained in (11) yields

$$q(x) = -2 \lim_{n \rightarrow \infty} \left(\lambda_n^0 \right)^2 \left(I_j^{(n)} - \frac{\pi}{\lambda_n^0} \right).$$

for almost every $x \in (0, \pi)$.

The right-hand side depends only on the nodal data $\{x_j^{(n)}\}$

Consequently, if two potentials generate the same nodal sets, they must coincide almost everywhere on $(0, \pi)$.

This completes the proof.

Author Contributions

The percentages of the author' contributions are presented below. The author reviewed and approved the final version of the manuscript.

	Ö.D.
C	100
D	100
S	100
DCP	100
DAI	100
L	100
W	100
CR	100
SR	100
PM	100

C= concept, D= design, S= supervision, DCP= data collection and/or processing, DAI= data analysis and/or interpretation, L= literature search, W= writing, CR= critical review, SR= submission and revision, PM= project management.

Conflict of Interest

The authors declared that there is no conflict of interest.

Ethical Consideration

Ethics committee approval was not required for this study because of there was no study on animals or humans.

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