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Steinhaus' Problem on Partition of A Hyperbolic Triangle

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Abstract. Hugo Steinhaus [5, 6] has asked whether inside each acute triangle there is a point from which perpendiculars to the sides divide the triangle into three parts of equal areas. In this paper, we present a solution of this problem in the Poincaré disc model of hyperbolic geometry.

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1. Introduction

Let ABC be an Euclidean acute angled triangle in the Euclidean plane. Steinhaus' problem asks a point P in its interior with pedals P_a, P_b, P_c on BC, CA, AB such that the quadrilaterals AP_bPP_c, BP_cPP_a, and CP_aPP_b have equal areas. There are some solutions of the problem in literature. To see these, we refer [3, 8] and [4]. In addition to this, A. Tyszka [7] has also shown that, for some acute triangles with rational coordinates of vertices the point solving Steinhaus' problem is not constructible with ruler and compass. Naturally, one may wonder whether the solution of the Steinhaus' problem exist in hyperbolic geometry? In this paper we try to give an affirmative answer of this question.

Hyperbolic geometry was created in the first half of the nineteenth century in order to prove the dependence of Euclids fifth postulate on the first four ones. Carl F. Gauss, Janos Bolyai, and N.I. Lobachevsky are considered the fathers of hyperbolic geometry. It is well known that there are many principal hyperbolic geometry models, for instance Poincaré upper-half plane model, Poincaré disc model, Beltrami-Klein model, Weierstrass model, etc. In this paper we choose the Poincaré disc model of hyperbolic geometry for our results. The points of this model are the points in the complex unit disc $\mathbb{D} = \{z \in C : |z| < 1\}$. The hyperbolic lines are the diameters of $\mathbb{D}$ and circular arcs orthogonal to the boundary circle of the disc. Two arcs which do not meet correspond to parallel rays, arcs which meet orthogonally correspond to perpendicular lines, and arcs which meet on the boundary are a pair of limits rays. The angles between two hyperbolic lines are the usual Euclidean angles between Euclidean tangents to the circular arcs. The advantage of the Poincaré disk model is that it is conformal, namely circles and angles are not distorted.

The hyperbolic metric density in $\mathbb{D}$ with the Gaussian curvature -1 is given by $w_{\mathbb{D}} = \frac{2}{1-|z|^2}$. The hyperbolic distance between the points $z_1, z_2 \in \mathbb{D}$ defined as

$$d_{\mathbb{D}}(z_1, z_2) = \frac{|z_1 - z_2|}{\sqrt{|z_1 - z_2|^2 + (1 - |z_1|^2)((1 - |z_2|^2)}}.$$

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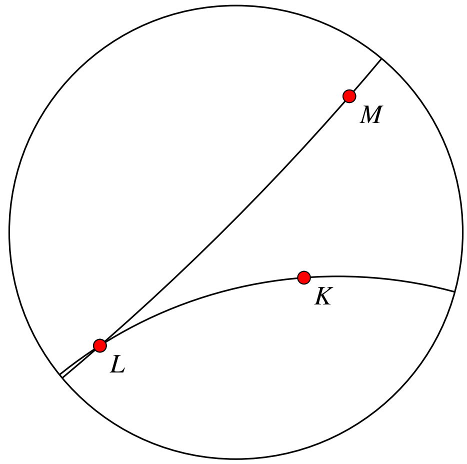

Figure 1. A hyperbolic line passing through the points K and L is a circular arc that intersect the disc $\mathbb{D}$ orthogonally. The hyperbolic lines passing through the center of disc are also correspond to chords of the disc.

In particular, for $z \in \mathbb{D}$,

$$d_{\mathbb{D}}(z_1, 0) = log\frac{1+|z|}{1-|z|} = 2arctanh|z|.$$

In full analogy with Euclidean geometry, a hyperbolic triangle consists of three line segments called sides or edges and three points called vertices. One of the important result for the hyperbolic triangle is the sum of the measures of the angles of a hyperbolic triangle must be less than π.

Theorem 1.1 (The Gauss-Bonnet Theorem)**.** *Let ABC be a hyperbolic triangle with internal angles α,β, and γ. Then the hyperbolic area of ABC is $\Delta(ABC) = \pi - (\alpha + \beta + \gamma)$.*

Theorem 1.2. *The area of a hyperbolic disc of radius r is $4\pi sinh^2(\frac{r}{2})$, see [2].*

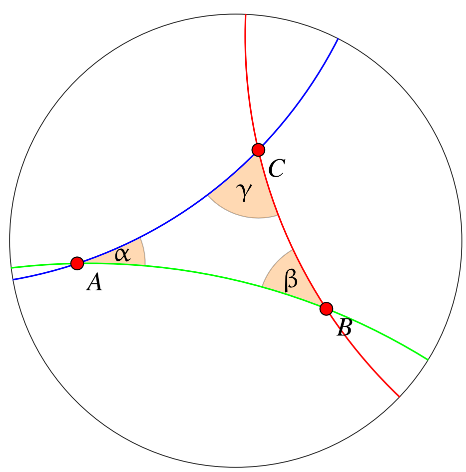

Figure 2. A hyperbolic triangle in the unit disc $\mathbb{D}$.

A Möbius transformation $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ (also called a fractional linear transformation) is a mapping of the form $w = (az + b)/(cz + d)$ satisfying $ad - bc \neq 0$, where $a, b, c, d \in \mathbb{C}$. Möbius transformations have many beautiful properties. For example, a map is Möbius if, and only if it preserves cross ratios. As for geometric aspect, circle-preserving is another important characterization of Möbius transformations. There are well-known elementary proofs that if f is a continuous injective map of the extended complex plane $\mathbb{C} \cup \{\infty\}$ that maps circles into circles, then f is Möbius. Möbius transformations are conformal bijective mappings, that is they preserve the measures of the angles with orientation. Euclidean transformations, Euclidean rotations, inversions ($z \mapsto \frac{1}{z}$) and similarities ($z \mapsto az+b,\ a \neq 0$) are the well known Möbius transformations. The most general Möbius transformation defined from $\mathbb{D}$ to itself is given by

$$z \to e^{i\theta} \frac{z_0 + z}{1 + \overline{z_0} z}$$

see [1].

2. Steinhaus' Problem on Partition of a Hyperbolic Triangle

The following lemma is well known and fundamental in Euclidean geometry.

Lemma 2.1. *The circles with the equations*

$$x^2 + y^2 + 2Ax + 2By + C = 0$$

and

$$x^2 + y^2 + 2A'x + 2B'y + C' = 0$$

are orthogonal if and only if $2(AA' + BB') = (C + C')$.

Lemma 2.2. *Let* $\theta_1, \theta_2, \cdots, \theta_n$ *be any ordered n-tuple with* $0 < \theta_n < (n-2)\pi,\ j = 1, ..., n$. *Then there exists a hyperbolic polygon P with interior angles* $\theta_1, \theta_2, \cdots, \theta_n$*, occurring in this order around* ∂P *(the boundary of P) if and only if* $\theta_1 + \theta_2 + \cdots + \theta_n < (n-2)\pi$*, see see [2].*

Lemma 2.3. *Let* l_1 *and* l_2 *be two hyperbolic lines in the complex unit disc* $\mathbb{D}$ *with*

$$l_1 = \{x + iy \in \mathbb{D}\ :\ y = 0\}$$

and

$$l_2 = \{x + iy \in \mathbb{D} :\ y = mx\}$$

for a fixed m satisfying $m > 0$. *If* $P = x_0 + iy_0$ *is a point in* $\mathbb{D}$ *satisfying* $x_0 > 0, y_0 > 0$ *and* $y_0 < mx_0$*, then the measure of the angle between the hyperbolic lines* k_1 *and* k_2 *(directed from* k_2 *to* k_1 *with respect to counterclockwise direction) is*

$$\angle(k_2, k_1) = arctan\frac{-x^2 + y^2 + 1}{2xy} - arctan\frac{x^2 + 2mxy - y^2 - 1}{mx^2 - 2xy - my^2 + m},$$

where k_1 *and* k_2 *are the hyperbolic lines passing through P and perpendicular to* l_1 *and* l_2*, respectively.*

Proof. Let C_j be the circle defined by k_j with radius r_j and $M_j = a_j + ib_j$ be its center for $j = 1, 2$. Since the center of the circle C_j must be lie on l_j, one can easily get $M_1 = a_1$ and $M_2 = a_2 + ima_2$. By *Lemma 2.1*, we get $r_1 = \sqrt{a_1^2 - 1}$ and $r_2 = \sqrt{m^2 a_2^2 + a_2^2 - 1}$. The Euclidean tangents of the circles C_1 and C_2 at P are the Euclidean lines with the equations

$$(x_0 - a_1)x + y_0 y - y_0^2 - x_0(x_0 - a_1) = 0$$

and

$$(x_0 - a_2)x + (y_0 - ma_2)y - x_0(x_0 - a_2) - y_0(y_0 - ma_2) = 0,$$

respectively. Then we get

$$\angle(k_2, k_1) = arctan\frac{a_1 - x_0}{y_0} - arctan\frac{a_2 - x_0}{y_0 - ma_2}.$$

Moreover, since P lies on C_j for $j = 1, 2$, this yields $a_1 = \frac{x_0^2+y_0^2+1}{2x_0}$ and $a_2 = \frac{x_0^2+y_0^2+1}{2x_0+2my_0}$ which implies

$$\angle(k_2, k_1) = arctan\frac{-x^2 + y^2 + 1}{2xy} - arctan\frac{x^2 + 2mxy - y^2 - 1}{mx^2 - 2xy - my^2 + m}.$$

□

Lemma 2.4. *Let l_1 and l_2 be two hyperbolic lines in the complex unit disc $\mathbb{D}$ with*

$$l_1 = \{x + iy \in \mathbb{D} \ : \ y = 0\}$$

and

$$l_2 = \{x + iy \in \mathbb{D} \ : \ y = mx\}$$

for a fixed m satisfying $m > 0$. Then for any $\rho \in \mathbb{R}^+$ satisfying $0 < \rho < \pi sinh^2(\frac{1}{2})$, the locus of the points in

$$D_{l_1 l_2} = \{P = x + iy \in \mathbb{C} \ : \ 0 < \frac{y}{x} < m\}$$

satisfying $\Delta(PP_{l_1}0P_{l_2}) = \rho$ is a curve with the equation

$$\pi - \left(arctanm + arctan\frac{-x^2 + y^2 + 1}{2xy} - arctan\frac{x^2 + 2mxy - y^2 - 1}{mx^2 - 2xy - my^2 + m} \right) = \rho$$

where P_{l_1}, P_{l_2} are the (hyperbolic) perpendicular projections of P onto the lines l_1 and l_2, respectively.

Proof. Let k_j be a hyperbolic line passing through P and perpendicular to l_j for $j = 1, 2$. Then by *Lemma 2.3*, we get

$$\angle(k_2, k_1) = arctan\frac{-x^2 + y^2 + 1}{2xy} - arctan\frac{x^2 + 2mxy - y^2 - 1}{mx^2 - 2xy - my^2 + m}.$$

Since $\angle(l_1, l_2) = arctanm$ and by *Gauss-Bonnet theorem*, we get

$$\Delta(PP_{l_1}0P_{l_2}) = 2\pi - \left(\frac{\pi}{2} + \frac{\pi}{2} + arctanm + arctan\frac{-x^2 + y^2 + 1}{2xy} - arctan\frac{x^2 + 2mxy - y^2 - 1}{mx^2 - 2xy - my^2 + m} \right)$$

i.e.,

$$\rho = \pi - \left(arctanm + arctan\frac{-x^2 + y^2 + 1}{2xy} - arctan\frac{x^2 + 2mxy - y^2 - 1}{mx^2 - 2xy - my^2 + m} \right).$$ □

In our main result below, we denote the curve defined by the hyperbolic lines l_1 and l_2 in *Lemma 2.4* by C_α^ρ where $\alpha := \angle(l_1, l_2) = arctanm$, the triangular domain defined by the triangle ABC by $\mathcal{D}(ABC)$, the boundary of $\mathcal{D}(ABC)$ by $\partial(ABC)$, the (hyperbolic) perpendicular projection of P onto the line passing through the points A and B by P_{AB} and the area of the triangle ABC by $\Delta(ABC)$.

Theorem 2.5. *Let ABC be a hyperbolic triangle (directed counterclockwise) in $\mathbb{D}$ with three acute angles $\angle CAB = \alpha$, $\angle ABC = \beta$ and $\angle BCA = \gamma$. Then there exist a point $P \in \mathcal{D}(ABC)$ such that*

$$\Delta(PP_{AB}AP_{AC}) = \Delta(PP_{AB}BP_{BC}) = \Delta(PP_{BC}CP_{AC}) = \frac{\Delta(ABC)}{3}.$$

Proof. Assume $\sigma := \Delta(ABC)$. Since the Möbius transformations preserve the measures of the angles, the hyperbolic angle $\angle CAB$ can be moved to the center of $\mathbb{D}$ by an appropriate Möbius transformation f_1 such that the images of the segments AB and AC lie on the hyperbolic line $l_1 = \{x + iy \in \mathbb{D} \ : \ y = 0\}$ and on the hyperbolic line $l_2 = \{x + iy \in \mathbb{D} \ : \ y = (tan\alpha)x\}$, respectively. Then by *Lemma 2.4*, there exist a curve $C_\alpha^{\sigma/3}$ defined by the hyperbolic lines l_1 and l_2. Hence the image of this curve under f_1^{-1}, which is also Möbius transformation, we get the curve $f_1^{-1}(C_\alpha^{\sigma/3})$ in $\mathbb{D}$. Similarly, the angles $\angle ABC$ and $\angle BCA$ can be moved to the center of $\mathbb{D}$ under two appropriate Möbius transformations, say f_2 and f_3, such that the segments $f_2(B)f_2(C)$ and $f_2(B)f_2(A)$ lie on the lines $l_1 = \{x+iy \in \mathbb{D} : y = 0\}$ and $l_3 = \{x + iy \in \mathbb{D} \ : \ y = (tan\beta)x\}$, respectively and the segments $f_3(A)f_3(C)$ and $f_3(C)f_3(B)$ lie on the lines $l_1 = \{x + iy \in \mathbb{D} : y = 0\}$ and $l_4 = \{x + iy \in \mathbb{D} : y = (tan\gamma)x\}$, respectively. Hence we get the curves $f_2^{-1}(C_\beta^{\sigma/3})$ and $f_3^{-1}(C_\gamma^{\sigma/3})$ in $\mathbb{D}$. We claim that any of two curves must have a common point in $\mathcal{D}(ABC)$. Indeed, if such a point does not exist, then we get

$$\sigma = \Delta(ABC) > \Delta(EE_{AB}AE_{AC}) + \Delta(FF_{AB}BF_{BC}) + \Delta(GG_{BC}CG_{AC}) = 3\frac{\sigma}{3} = \sigma$$

where $E, F, G \in \mathbb{D}$ are arbitrary points on the curves $f_1^{-1}(C_\alpha^{\sigma/3})$, $f_2^{-1}(C_\beta^{\sigma/3})$ and $f_3^{-1}(C_\gamma^{\sigma/3})$, respectively. This is a contradiction. If any of two curves intersects at a point of $\partial(ABC)$, for example $f_1^{-1}(C_\alpha^{\sigma/3})$ and $f_2^{-1}(C_\beta^{\sigma/3})$ have a

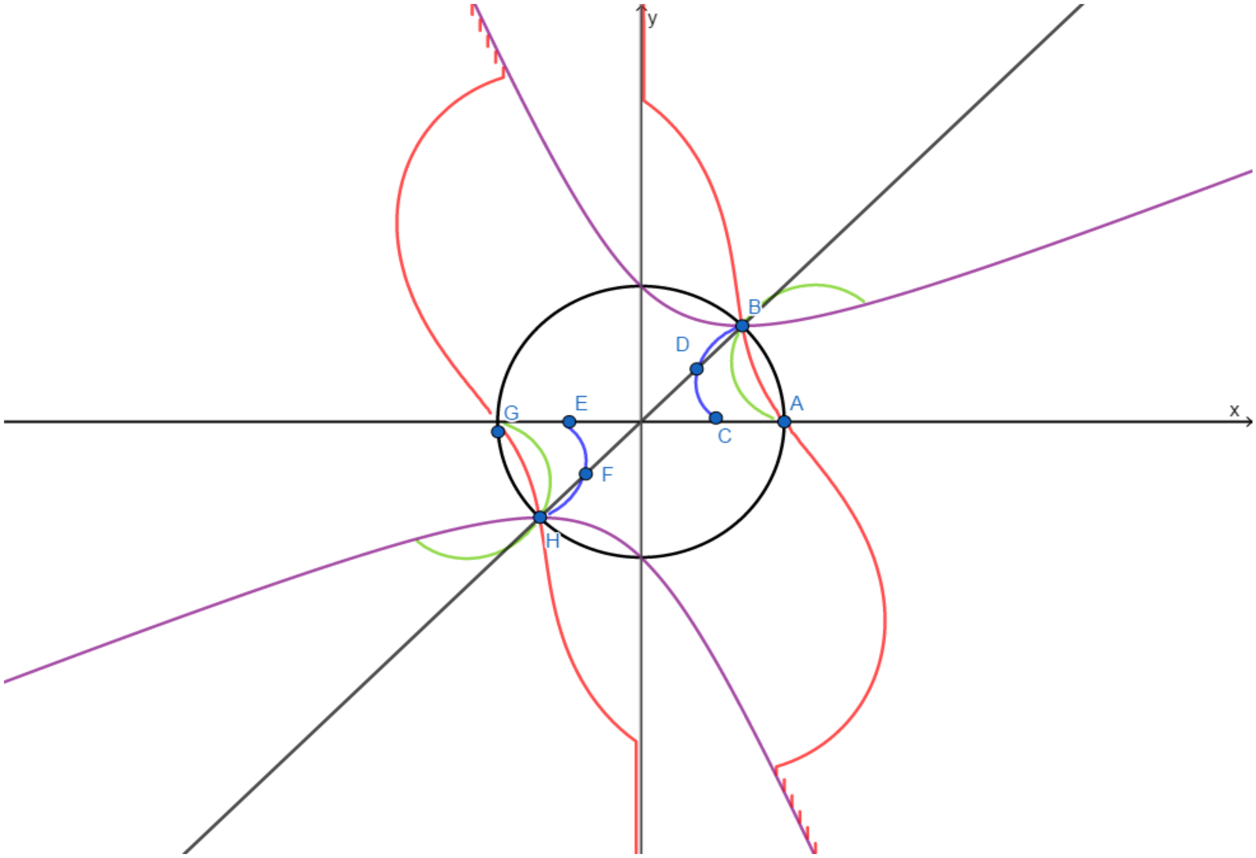

FIGURE 3. The curves $C_{\pi/4}^{13\pi/20}$, $C_{\pi/4}^{5\pi/12}$, $C_{\pi/4}^{7\pi/36}$ are shown by blue, green and red colors, respectively. The curve with purple color has the equation $x^2 - 2xy - y^2 + 1 = 0$

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common point in $\partial(ABC)$, say Q, then $f_3^{-1}(C_\gamma^{\sigma/3})$ must intersect $f_1^{-1}(C_\alpha^{\sigma/3})$ or $f_2^{-1}(C_\beta^{\sigma/3})$ at a interior point of $\mathcal{D}(ABC)$. Otherwise,

$$\sigma = \Delta(ABC) > \Delta(EE_{AB}AE_{AC}) + \Delta(FF_{AB}BF_{BC}) + \Delta(GG_{BC}CG_{AC}) = 3\frac{\sigma}{3} = \sigma$$

where $E, F, G \in \mathbb{D}$ are arbitrary points on the curves $f_1^{-1}(C_\alpha^{\sigma/3})$, $f_2^{-1}(C_\beta^{\sigma/3})$ and $f_3^{-1}(C_\gamma^{\sigma/3})$, respectively. This is again a contradiction. Therefore, there exist a point $P \in \mathcal{D}(ABC)$ such that

$$\Delta(PP_{AB}AP_{AC}) = \Delta(PP_{AB}BP_{BC}) = \Delta(PP_{BC}CP_{AC}) = \frac{\Delta(ABC)}{3}. \qquad \square$$

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