Proceedings of International Conference on Mathematics
and Mathematics Education (ICMME 2018)
Turk. J. Math. Comput. Sci.
10(2018) 56–60

http://dergipark.gov.tr/tjmcs

Local T_0 Filter Convergence Spaces

Mehmet Baran

Department of Mathematics, Faculty of Sciences, Erciyes University, 38039, TURKEY.

Received: 01-08-2018 • Accepted: 10-12-2018

Abstract. In this paper, we characterize each of local T_0 filter convergence spaces and investigate the relationships between these local T_0 filter convergence spaces as well as give some invariance properties of them.

2010 AMS Classification: 54B30, 54D10, 54A05, 54A20, 18B99, 18D15.

Keywords: Topological category, filters, local filter convergence spaces, local T_0 objects.

1. Introduction

The notion of a filter which can be viewed as a generalization of sequences was introduced by Henri Cartan [10, 11] in 1937 and has been used as a valuable tool in the development of topology and its applications [9, 12, 13, 16, 22]. Concepts such as points of closure and compactness that are extremely important in general topology theory cannot be described using sequences, but can be described using general filter theory. In 1954, Kowalsky [17] gave a filter description of convergence. More details on convergence theory can, for instance, be found in [13, 19].
Baran, in [2], introduced local separation properties in set-based topological categories and then, they are generalized to point free definitions by using the generic element method of topos theory [15] or [20]. One of the use of local separation properties is to define the notion of (strong) closedness [2] in set-based topological categories which are used in the notions of Hausdorffness ([3]) , regular, completely regular, and normal objects in ([5, 6]). One of the other uses of local T_0 separation property is to define local T_2 separation property.
In this paper, we characterize each of local T_0 filter convergence spaces and investigate the relationships between these local T_0 filter convergence spaces as well as give some invariance properties of them.

2. Preliminaries

Let A be a set and K be a function which assigns to each point x of A a set of filters (proper or not, where a filter α is proper iff α does not contain the empty set, $\emptyset$,i.e, $\alpha \neq [\emptyset]$ (the filters converging to x) is called a convergence structure on A ((A, K) a local filter convergence space [24] (in [23] it is called generalized convergence space)) iff it satisfies the following two conditions.
(1) $[x] = [\{x\}] \in K(x)$ for each $x \in A$, (where $[F] = \{B \subset A : F \subset B\}$ and $[x]$ is called the principal ultra filter.
(2) $\beta \supset \alpha \in K(x)$ implies $\beta \in K(x)$ for any filter β on A.
(3) $\alpha \in K(x)$ implies $\alpha \cap [x] \in K(x)$.
A map $f : (A, K) \to (B, L)$ between local filter convergence spaces is called continuous iff $\alpha \in K(x)$ implies

Email address: baran@erciyes.edu.tr
This research was supported by the Erciyes University Scientific Research Center (BAP) under Grant No: 7174.

$f(\alpha) \in L(f(x))$ (where $f(\alpha)$ denotes the filter generated by $\{f(D) : D \in \alpha\}$. The category of local filter convergence spaces and continuous maps is denoted by ***LFCO***, [24].
Recall, [1, 14, 23], that a functor $U : \mathcal{E} \to \mathcal{B}$ is said to be topological or that $\mathcal{E}$ is a topological category over $\mathcal{B}$ if U is concrete (i.e., faithful and amnestic (i.e., if $U(f) = id$ and f is an isomorphism, then $f = id$)), has small (i.e., sets) fibers, and for which every U-source has an initial lift or, equivalently, for which each U-sink has a final lift. Note that a topological functor $U : \mathcal{E} \to \mathcal{B}$ is said to be normalized if constant objects, i.e., subterminals, have a unique structure. Note also that U has a left adjoint called the discrete functor D. Recall, in [1, 23] that an object $X \in \mathcal{E}$ is discrete iff every map $U(X) \to U(Y)$ lift to map $X \to Y$ for each object $Y \in \mathcal{E}$.
Note that the category ***LFCO*** is normalized topological category.
For filters α and β, we denote by $\alpha \cup \beta$ the smallest filter (proper or not) containing both α and β, i.e., $\alpha \cup \beta = \{M \subset A : U \cap V \subset M$ for some $U \in \alpha$ and $V \in \beta\}$.
2.1. A source $\{f_i : (A, K) \to (A_i, K_i), i \in I\}$ in ***LFCO*** is initial iff $\alpha \in K(a)$, for $a \in A$, precisely when $f_i(\alpha) \in K_i(f_i(a))$ for all $a \in A$ [19, 22].
2.2. The discrete structure (A, K) on A in ***LFCO*** is given by $K(a) = \{[a], P(A) = [\emptyset]\}$ for all $a \in A$.
2.3. An epimorphism $f : (A, S) \to (B, L)$ in ***LFCO*** is final iff for each $b \in B$, $\alpha \in L(b)$ implies that $f(\beta) \subset \alpha$ for some point $a \in A$ and filter $\beta \in S(a)$ with $f(a) = b$ [22, 24].

3. Local T_0 Filter Convergence Spaces

In this section, we give the characterization of each of T_0 local filter convergence spaces at a point p.

Let B be set and $p \in B$. Let $B \vee_p B$ be the wedge at p [2], i.e., two disjoint copies of B identified at p, or in other words, the pushout of $p : 1 \to B$ along itself (where 1 is the terminal object in **Set**, the category of sets) [21]. More precisely, if i_1 and $i_2 : B \to B \vee_p B$ denote the inclusion of B as the first and second factor, respectively, then $i_1 p = i_2 p$ is the pushout diagram. A point x in $B \vee_p B$ will be denoted by $x_1(x_2)$ if x is in the first (resp. second) component of $B \vee_p B$. Note that $p_1 = p_2$.

The principal $p-$axis map, $A_p : B \vee_p B \to B^2$ is defined by $A_p(x_1) = (x, p)$ and $A_p(x_2) = (p, x)$ and the fold map at p, $\triangledown_p : B \vee_p B \to B$ is given by $\triangledown_p(x_i) = x$ for $i = 1, 2$ [2].

Definition 3.1. Let (X, τ) be a topological space and $p \in X$. For each point x distinct from p, there exists a neighborhood of p missing x or there exists a neighborhood of x missing p, then (X, τ) is said to be T_0 at p [2, 4].

Theorem 3.2. *Let (X, τ) be a topological space and $p \in X$. Then the followings are equivalent.*
(1) (X, τ) is T_0 at p
(2) The initial topology induced by $\{A_p : X \vee_p X \to (X^2, \tau_*)$ and $\nabla_p : X \vee_p X \to (X, P(X))\}$ is discrete, where τ_* is the product topology on X^2.
(3) The initial topology induced by $\{id : X \vee_p X \to (X \vee_p X, \tau^*)$ and $\nabla_p : X \vee_p X \to (X, P(X))\}$ is discrete, where τ^* is the final topology on $X \vee_p X$ induced by the canonical injections $\{i_1, i_2 : (X, \tau) \to X \bigvee_p X\}$ and $id : X \vee_p X \to X \vee_p X$ is the identity map.

Proof. The proof is given in [4]. □

Let $\mathcal{U} : \mathcal{E} \to$ ***Set*** be a topological functor, X an object in $\mathcal{E}$ with $\mathcal{U}(X) = B$ and p is a point in B.

Definition 3.3. (1) If the initial lift of the $\mathcal{U}$-source $\{A_p : B \bigvee_p B \to \mathcal{U}(X^2) = B^2$ and $\nabla_p : B \bigvee_p B \to \mathcal{U}\mathcal{D}(B) = B\}$ is discrete, then X is called $\overline{T}_0$ at p [2].
(2) If the initial lift of the $\mathcal{U}$-source $\{id : B \bigvee_p B \to \mathcal{U}(B \bigvee_p B)' = B \bigvee_p B$ and $\nabla_p : B \bigvee_p B \to \mathcal{U}\mathcal{D}(B) = B\}$ is discrete, then X is called T_0' at p, where $(B \bigvee_p B)'$ is the final lift of the $\mathcal{U}$-sink $\{i_1, i_2 : \mathcal{U}(X) = B \to B \bigvee_p B\}$, i_1 and i_2 are the canonical injections and $id : B \vee_p B \to B \vee_p B$ is the identity map [2].

Theorem 3.4. *A local filter convergence space (B, L) is $\overline{T}_0$ at p if and only if for $x \neq p$, $[x] \notin L(p)$ or $[p] \notin L(x)$.*

Proof. Suppose (B, L) is T_0 at p. If $[x] \in L(p)$ and $[p] \in L(x)$ for some $x \neq p$, then, let $\sigma = [x_1]$. Note that σ is a filter on the wedge and

$$\pi_1 A_p \sigma = [x] \in L(\pi_1 A_p(x_2) = p)$$
$$\pi_2 A_p \sigma = [p] \in L(\pi_2 A_p(x_2) = x)$$

and

$$\nabla_p \sigma = [x] \in N(\nabla_p(x_2) = x)$$

where $\pi_i : B^2 \to B$, $i = 1, 2$, are the projection maps and N is the discrete structure on B. Since (B, L) is T_1 at p, it follows from Definition 3.3 and 2.2 that $\sigma = [x_2]$. This is a contradiction since $x \neq p$ and $\sigma = [x_1]$. Hence, we must have for all $x \neq p$, $[x] \notin L(p)$ or $[p] \notin L(x)$.

Conversely, suppose that for $x \neq p, [x] \notin L(p)$ and $[p] \notin L(x)$. We show that (B, L) is T_1 at p. Suppose also σ is a filter on the wedge which satisfies

$$\pi_1 A_p \sigma \in L(\pi_1 A_p(x_1) = x)$$
$$\pi_2 A_p \sigma \in L(\pi_2 A_p(x_1) = p)$$

and

$$\nabla_p \sigma = [x] \in N(\nabla_p(x_1) = x)$$

where N is the discrete structure on B, i.e., $\nabla_p \sigma = [x]$ or $[\emptyset]$. It follows that $\sigma = [x_1], [x_2], [\emptyset]$ or $\sigma \supset [\{x_1, x_2\}]$. We must show that $\sigma = [x_1]$ or $[\emptyset]$.

If $\sigma = [x_2]$, then $\pi_1 A_p \sigma = [p] \in L(x)$ and $\pi_2 A_p \sigma = [x] \in L(p)$, a contradiction.

If $\sigma = [\{x_1, x_2\}]$, then $\pi_1 A_p \sigma = [\{x, p\}] \subset [p]$ and $\pi_2 A_p \sigma = [\{x, p\}] \subset [x]$ which implies $[p] \in L(x)$ and $[x] \in L(p)$, a contradiction.

If $\sigma \supset [\{x_1, x_2\}]$ with $[\emptyset] \neq \sigma \neq [\{x_1, x_2\}]$, then there exists $U \in \sigma$ such that $U \neq \emptyset$ and $U \neq \{x_1, x_2\}$. Since $\{x_1, x_2\} \in \sigma$ and σ is a filter, $U \cap \{x_1, x_2\} = \{x_1\}$ or $\{x_2\}$ is in σ, i.e., $\sigma = [x_1]$ or $\sigma = [x_2]$. We showed, as above, that we can not have $\sigma = [x_2]$ and so we must have $\sigma = [x_1]$ or $[\emptyset]$.

If σ is a filter on the wedge which satisfies

$$\pi_1 A_p \sigma \in L(\pi_1 A_p(x_2) = p)$$
$$\pi_2 A_p \sigma \in L(\pi_2 A_p(x_2) = x)$$

and

$$\nabla_p \sigma = [x] \in N(\nabla_p(x_2) = x)$$

then it follows that $\sigma = [x_1], [x_2], [\emptyset]$ or $\sigma \supset [\{x_1, x_2\}]$.

If $\sigma = [x_1]$, then $\pi_1 A_p \sigma = [x] \in L(p)$ and $\pi_2 A_p \sigma = [p] \in L(x)$, a contradiction, since $x \neq p$.

If $\sigma = [\{x_1, x_2\}]$, then $\pi_1 A_p \sigma = [\{x, p\}] \subset [p]$ and $\pi_2 A_p \sigma = [\{x, p\}] \subset [x]$ which implies $[p] \in L(x)$ and $[x] \in L(p)$, a contradiction.

If $\sigma \supset [\{x_1, x_2\}]$ with $[\emptyset] \neq \sigma \neq [\{x_1, x_2\}]$, then by the same argument shown above, we must have $\sigma = [x_2]$ or $[\emptyset]$.

If σ is a filter on the wedge which satisfies

$$\pi_1 A_p \sigma \in L(\pi_1 A_p(p_2 = p_1) = p)$$
$$\pi_2 A_p \sigma \in L(\pi_2 A_p(p_1 = p_2) = p)$$

and

$$\nabla_p \sigma = [p] \in N(\nabla_p(p_1 = p_2) = p)$$

then it follows that $\sigma = [p_1]$ or $[\emptyset]$ (since $\nabla_p^{-1}(p) = \{p_1\}$). Hence, the initial lift of A_p and ∇_p is discrete, i.e., by Definition 3.3, (B, L) is T_0 at p. □

Theorem 3.5. *Every local filter convergence space is T_0' at p.*

Proof. Let (B, L) be any local filter convergence space and $p \in B$. By Definition 3.3 and 2.1-2.3, we will show that for any filter σ on the wedge $B \vee_p B$ and $z \in B \vee_p B$ which satisfies $\sigma \supset i_k(\beta)$ for some $\beta \in L(x)$ with $i_k(x) = z$ for $k = 1, 2$ and $\nabla_p \sigma = [x] \in N(\nabla_p(z) = x)$, where N is the discrete structure on B, then $\sigma = [z]$ or $[\emptyset]$.

$\nabla_p \sigma = [\emptyset]$, then $\sigma = [\emptyset]$.

If $\nabla_p \sigma = [p]$, then $\sigma = [p]$.

If $\nabla_p \sigma = [x]$ for some $x \in B$ with $x \neq p$. It follows that $\sigma = [x_1], [x_2]$ or $\sigma \supset [\{x_1, x_2\}]$. Since $\sigma \supset i_1(\beta)$ for some

$\beta \in L(x)$ and $i_1(x) = x_1$, it follows that $\sigma = [x_1]$. If $\sigma \supset i_2(\beta)$ for some $\beta \in L(x)$ and $i_2(x) = x_2$, then $\sigma = [x_2]$. Hence, by Definition 3.3, (B, L) is T_0' at p. □

Let $\boldsymbol{T_0'LFCO}$ (resp. $\boldsymbol{\overline{T_0}LFCO}$) be the subcategory of $\boldsymbol{LFCO}$ whose objects are local T_0' (resp. $\overline{T_0}$) local filter convergence spaces.

Theorem 3.6. *$\boldsymbol{T_0'LFCO}$ and $\boldsymbol{LFCO}$ are isomorphic categories.*

Proof. It follows from Theorem 3.5. □

Theorem 3.7. *The category $\boldsymbol{\overline{T_0}LFCO}$ is closed under subspaces, products, and coproducts.*

Proof. Let (B, L) be a $\overline{T_0}$ local filter convergence space at p, $M \subset B$ with $p \in M$ and let L_M be the initial structure on M induced by the inclusion map $i : M \subset B$. Suppose for $x \in M$, $[x] \in L_M(p)$ and $[p] \in L_M(x)$. By 2.1, $i([x]) = [x] \in L(i(p) = p)$ and $i([p]) = [p] \in L(i(x) = x)$. Since (B, L) is $\overline{T_0}$ at p, by Theorem 3.4, $x = p$ and consequently, (M, L_M) is $\overline{T_0}$ at p.
Suppose that (B_i, L_i) $\overline{T_0}$ at p_i for all $i \in I$, $p_i \in B_i$ and $(B = \prod_{i\in I}, L)$, where L is the product structure on B with $p = (p_1, p_2, ...)$. We show that $(B = \prod_{i\in I}, L)$ is $\overline{T_0}$ at p.
Suppose there exist $x \in B$ with $x \neq p$ such that $[x] \in L(p)$ and $[p] \in L(x)$. It follows that there exists $m \in I$ such that $x_m \neq p_m$ in B_m. Since $[x] \in L(p)$ and $[p] \in L(x)$, we get

$$\pi_m([x]) = [x_m] \in L_m(\pi_m(p) = p_m)$$

and

$$\pi_m([p]) = [p_m] \in L_m(\pi_m(x) = x_m)$$

which contradicts to (B_m, L_m) being $\overline{T_0}$ at p_m. Hence, for any $x \neq p$ in B, $[x] \notin L(p)$ or $[p] \notin L(x)$ and by Theorem 3.4, $(B = \prod_{i\in I}, L)$ is $\overline{T_0}$ at p.
Suppose that (B_i, L_i) is $\overline{T_0}$ at p_i for all $i \in I$,$p_i \in B_i$ and $(B = \coprod_{i\in I}, L)$, where L is the coproduct structure on B and $(i, p) \in B$. It follows easily from 2.3 and Theorem 3.4 that $(B = \coprod_{i\in I}, L)$ is $\overline{T_0}$ at $(i, p) \in B$.

□

Remark 3.8. (1) Note that, by Theorem 3.2, for the category $\boldsymbol{Top}$ of topological spaces, $\overline{T_0}$ at p and T_0' at p are equivalent and reduce to the usual T_0 at p.
(2) Let $\mathcal{U} : \mathcal{E} \to \boldsymbol{Set}$ be a topological functor, X an object in $\mathcal{E}$ and $p \in \mathcal{U}(X)$ be a retract of X, i.e., the initial lift $h : \bar{1} \to X$ of the $\mathcal{U}$-source $p : 1 \to \mathcal{U}(X)$ is a retract, where 1 is the terminal object in $\boldsymbol{Set}$. Then if X is $\overline{T_0}$ at p, then X is T_0' at p [5] but, by Theorems 3.4 and 3.5, the reverse implication is not true.
(3) If $\mathcal{U} : \mathcal{E} \to \boldsymbol{Set}$ is a normalized topological functor, then $\overline{T_0}$ at p implies T_0' at p [5].
(4) In a topological category, T_1 at p and $\overline{T_0}$ at p objects may be equivalent [7, 18]. All T_1 at p, $\overline{T_0}$ at p, and T_0' at p objects may be equivalent [5]. Moreover, $\overline{T_0}$ at p objects could be only discrete objects and T_0' at p objects could be all objects [8].
(5) One of the use of $\overline{T_0}$ at p and T_0' at p is to define the notion of local T_2 objects in set-based topological categories [2].

Acknowledgement: This research was supported by the Erciyes University Scientific Research Center (BAP) under Grant No: 7174.

References

[1] Adámek, J., Herrlich, H., Strecker, G.E., Abstract and Concrete Categories, New York,USA: Wiley, 1990. 2
[2] Baran, M., *Separation properties*, Indian J Pure Appl Math **23**(1991), 333-341. 1, 3, 3.1, 3.3, 3.8
[3] Baran, M., Altındiş, H., *T_2-Objects in topological categories* Acta Math Hungar, **71**(1996), 41–48. 1
[4] Baran, M., *Separation properties in topological categories*, Math Balkanica **10**(1996), 39–48. 3.1, 3
[5] Baran, M., *Completely regular objects and normal objects in topological categories*, Acta Math Hungar **80**(1998), 211–224. 1, 3.8
[6] Baran, M., *T_3 and T_4 -objects in topological categories*, Indian J Pure Appl Math. **29**(1998), 59–69. 1
[7] Baran, M. Kula, S., and Erciyes, A., *T_0 and T_1 semiuniform convergence space*, Filomat, **27**(2013), 537–546. 3.8
[8] Baran, T. M., *Local T_0 pseudo-quasi-semi metric spaces* , Proceedings of 1st International Mediterranean Science and Engineering Congress (IMSEC 2016) Çukurova University, Congress Center, October 26-28, 2016, Adana / TURKEY, 286–291. 3.8

[9] Birkhoff, G., *A new definition of limit*, Bull. Amer. Math. Soc., **41**(1935), 636. 1

[10] Cartan, H., *Filtres et ultrafiltres*, CR Acad. Paris, **205**(1937), 777–779. 1

[11] Cartan, H., *Théorie des filtres*, CR Acad. Paris, **205**(1937), 595–598. 1

[12] Cook, H.C., and Fisher, H.R., *On equicontinuity and continuous convergence*, Math. Ann. **159**(1965), 94–104. 1

[13] Gõhler, W., *Convergence structures-historical remarks and the monadic approach*, Bremen Mathematik Arbeitspapiere, **48**(1997), 171–193. 1

[14] Herrlich, H., *Topological functors*, Gen Topology Appl **4**(1974), 125-142. 2

[15] Johnstone, PT., Topos Theory, L.M.S Mathematics Monograph: No. 10 New York UAS; Academic Press, 1977. 1

[16] Kent, D.C., *Convergence functions and their related topologies*, Fund. Math. **54**(1964), 125–133. 1

[17] Kowalsky, H.J., *Beitrõge zur topologischen algebra*, Math. Nachrichten **11**(1954), 143–185. 1

[18] Kula, M., *A note on Cauchy spaces*, Acta Math Hungar **133**(2011), 14–32. 3.8

[19] Lowen-Colebunders, E., Function Classes of Cauchy Continuous Maps, New York USA; Marcel Dekker Inc, 1989. 1, 2

[20] MacLane, S., Moerdijk I., Sheaves in Geometry and Logic, Springer- Verlag, 1992. 1

[21] Mielke, M V., *Separation axioms and geometric realizations*, Indian JPure Appl Math **25**(1994), 711–722. 3

[22] Nel, L.D., *Initially structured categories and cartesian closedness*, Canadian J.Math., **27**(1975), 1361–1377. 1, 2

[23] Preuss, G., Theory of Topological Structures, An Approach to Topological Categories, Dordrecht; D Reidel Publ Co, 1988. 2

[24] Schwarz, F., *Connections between convergence and nearness*, Lecture Notes in Math. Springer-Verlag **719**(1978), 345–354. 2