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Tutte Polynomial for Graphs of Twist Knots

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Abstract. The Tutte polynomial is a two-variable polynomial that is connected by a graph, a matroid or a matrix. The Tutte polynomial has a lot of exciting applications in different areas for example combinatorics, probability, knot theory, algebra, statistical mechanics, computer sciences, chemistry and biology. It was indicated by W. T. Tutte. We transport the Tutte polynomial to knot theory. Because each knot have a corresponding graph. We study the Tutte polynomial for graphs of twist knots. We find some general forms for the Tutte polynomial of graphs belonging to twist knots and the Tutte polynomial of signed graphs belonging to twist knots. Twist knots are significant class of knots to take into account especially in contact geometry.

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1. Introduction

William Tutte described a new polynomial about graphs at the year of 1954. The Tutte polynomial is a two-variable polynomial that is connected by a graph, a matroid or a matrix. The Tutte polynomial has a lot of exciting applications in different areas for example combinatorics, probability, knot theory, algebra, statistical mechanics, computer sciences, chemistry and biology. Let G be a finite graph, its Tutte polynomial supplies a basic global characteristic concerning the deletion-contraction reduction of the graph. Many combinatorial, enumerative and algebraic characteristic of the graph can be reckoned via this polynomial like the number of spanning trees, the number of spanning forest, the number of spanning connected subgraphs and the number of acyclic orientations of the graph [1]. In 1989, Kauffman described the Tutte polynomial which is expressed for signed graphs [4].

Definition 1.1 ([2]). To understand the definition of the Tutte polynomial we need to know the two operations defined on the graphs. The ones are: edge deletion indicated by $G - e$ and edge contraction indicated by G/e.

The Tutte polynomial of a graph $G = (V, E)$ is a two-variable polynomial explained as follows:

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$$T(G;x,y)=\begin{cases}1, & E(G)=\varnothing\\ xT(G/e), & e\in E \quad \text{and} \quad e \quad \text{is a bridge}\\ yT(G-e), & e\in E \quad \text{and} \quad e \quad \text{is a loop}\\ T(G-e;x,y)+T(G/e;x,y), & e\in E \text{ and } e \text{ is a neither a loop nor a bridge}\end{cases}$$

Definition 1.2 ([4]). Let's take G as a signed graph. Suppose that e is an edge of G and the sign of e is indicated by $sign(e)$ (+ or -). The edge e can be a bridge or a loop. Assume that $b_+ = b_+(G)$ is the number of positive bridges, $b_- = b_-(G)$ is the number of negative bridges, $l_+ = l_+(G)$ is the number of positive loops and $l_- = l_-(G)$ is the number of negative loops in G. The Tutte polynomial $Q[G] = Q[G](a, b, d)$ is defined for signed graphs as follows:

(1) The edge e is neither a bridge nor a loop, where $G^{'}$ and $G^{''}$ are induced graphs on the graph G via deletion-contraction operations, so

$$Q[G] = aQ[G^{'}] + bQ[G^{''}] \text{ if } sign(e) < 0,$$
$$Q[G] = bQ[G^{'}] + aQ[G^{''}] \text{ if } sign(e) > 0$$

(2) Provided that G is connected and all of edge of G is either a loop or a bridge, so

$$Q[G] = x^{b_+ + l_-} y^{b_- + l_+}$$

where $x = a + bd$, $y = ad + b$.

(3) Let G be separated combination of graphs G_1 and G_2 in that case,

$$Q[G] = dQ[G_1]Q[G_2]$$

Definition 1.3 ([3]). A twist knot which is denoted T_n is gotten by twisting two parallel strands n times and subsequently hooking the ends together to be alternating knot, as seen in Figure 1.

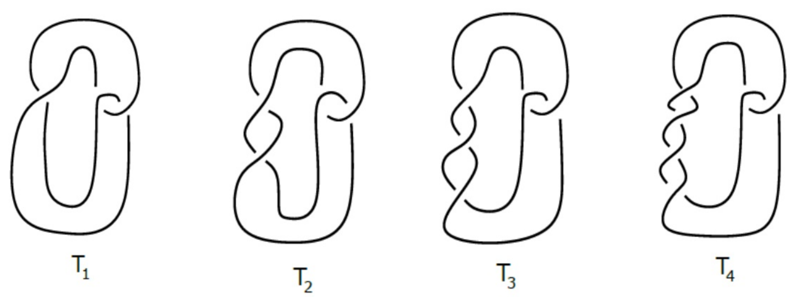


Figure 1. Some twist knots

Now we proceed that: At first, we will obtain regular projections of twist knots from their regular diagrams (see Figure 2). Then, we will shadow these projections in a checkered pattern such that the sides of an edge get different colors (see Figure 3). And then, we will get a point in the centers of each dark region. We obtain the graphs of twist knots by combining these points with the paths passing through the crossing points of the dark regions (see Figure 4 and Figure 5).

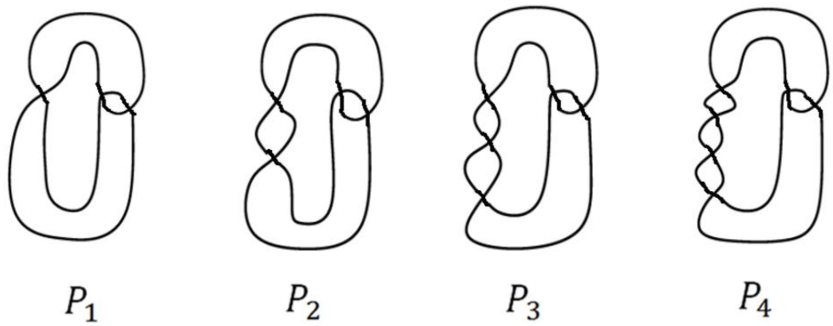


FIGURE 2. Some projections of twist knots

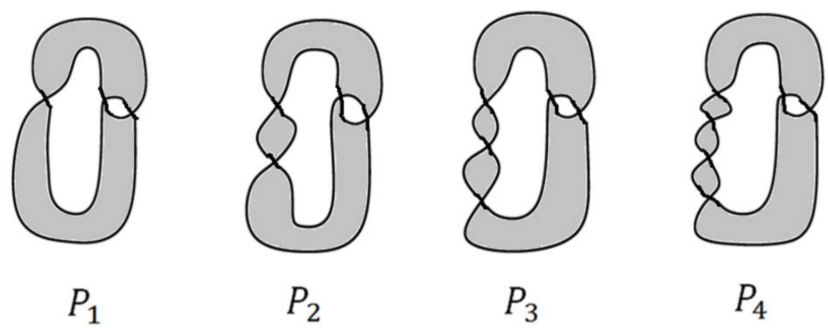


FIGURE 3. Some projections in a checkered pattern of twist knots

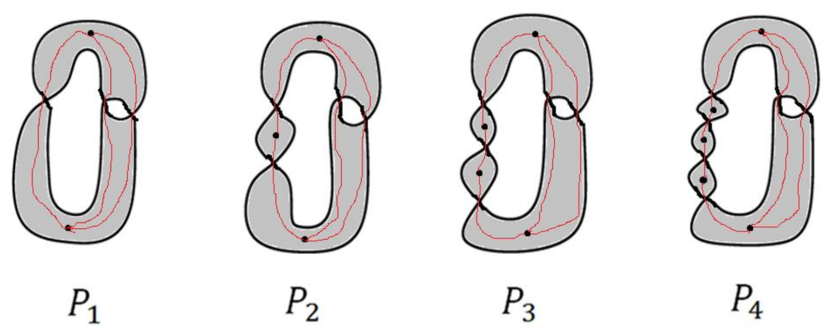


FIGURE 4. Obtaining some graphs of twist knots

Similarly we can obtain graphs of twist knots T_n for all $n \in \mathbb{N}^+$. Each path into P_n corresponds to a crossing of T_n. In G_n, every paths of P_n are signed with (+) or (−) according to the rule shown in Figure 6. We will determine the signed graphs of twist knots accordingly the rule as seen in Figure 7.

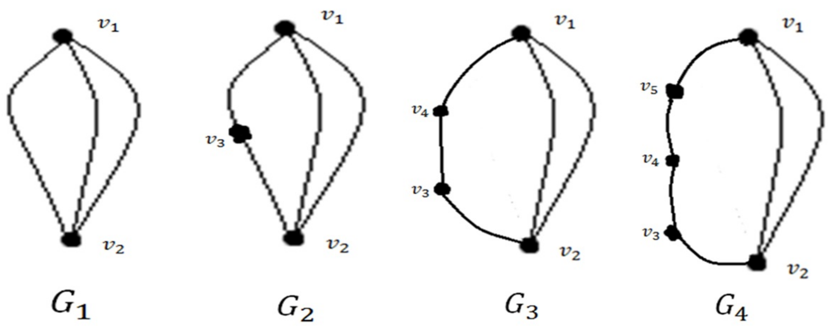


Figure 5. Some graphs of twist knots

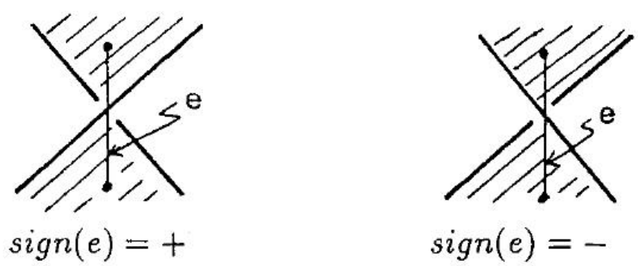


Figure 6. The rule of how to get signed graphs

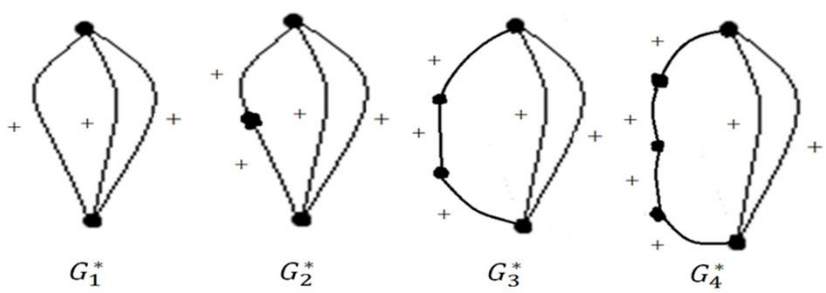


Figure 7. Some signed graphs of twist knots

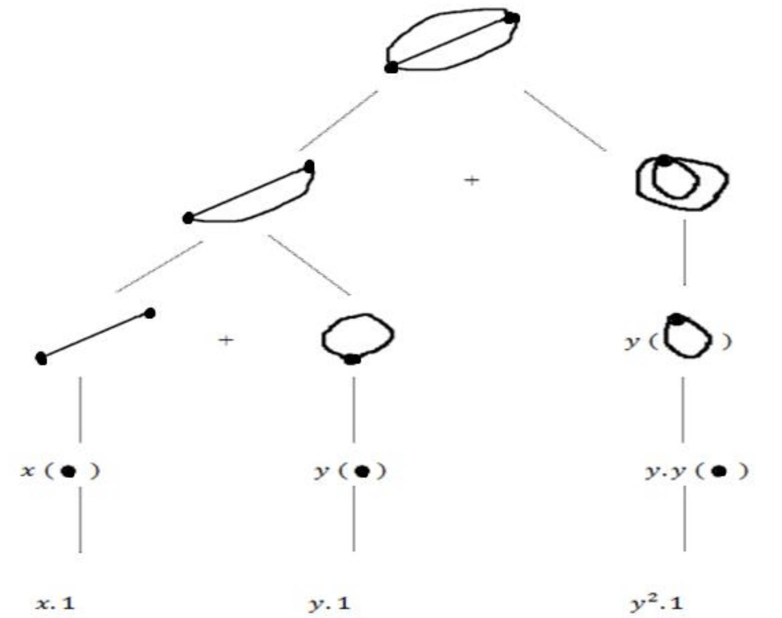
2. Results

2.1. **The Tutte polynomial for graphs of twist knots.** Let us reckon the Tutte polynomials for some graphs of twist knots by using its definition as expressed in Definition 1.1.

For G_1: The Figure at the beginning of page 65th

$$\Rightarrow T(G_1; x, y) = x + y + y^2$$

For G_2: We have $T(G_2; x, y) = x^2 + x + xy + y + y^2$.
For G_3: We have $T(G_3; x, y) = x^3 + x^2 + x + x^2y + xy + y + y^2$.
For G_4: We have $T(G_4; x, y) = x^4 + x^3 + x^2 + x + x^3y + x^2y + xy + y + y^2$.
Thus we will indicate the theorem below for the Tutte polynomials of graphs belonging to twist knots.



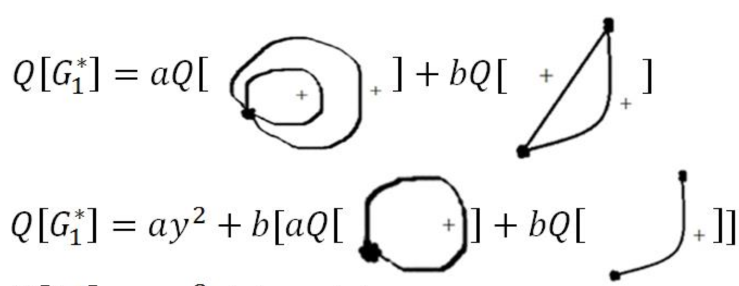
Theorem 2.1. *For all $n \in \mathbb{N}^+$*

$$T(G_n; x, y) = x^n + x^{n-1} + x^{n-2} + \cdots + x + x^{n-1}y + x^{n-2}y + \cdots + xy + y + y^2$$
$$T(G_n; x, y) = x^n + x^{n-1} + x^{n-2} + \cdots + x + (x^{n-1} + x^{n-2} + \cdots + x + 1)y + y^2$$
$$T(G_n; x, y) = \Big[\sum_{i=1}^{n} (x^i + x^{i-1}y)\Big] + y^2$$

Proof. It is obvious to prove the theorem by induction. □

2.2. **The Tutte polynomial for signed graphs of twist knots.** Let us reckon the Tutte polynomials for some signed graphs of twist knots by using its definition as expressed in Definition 1.2. We show this calculation method on a sample.

Example 2.2. We compute the Tutte polynomial for the signed graph G_1^*.



$$Q[G_1^*] = ay^2 + bay + bx$$

$$Q[G_1^*] = ay^2 + aby + b^2x$$

Similarly we perform the same calculations for a few signed graphs:
For G_2^*: We have $Q[G_2^*] = a^2y^2 + a^2by + ab^2xy + a^2bxy + ab^2x^2 + ab^2x + b^3x^2$.

For G_3^*: We have $Q[G_3^*] = a^3y^2 + a^3by + a^2b^2xy + a^3bxy + a^2b^2x^2 + a^2b^2x + ab^3x^2 + a^3b^2x^2y + a^2b^3x^3 + a^2b^3x^2y + ab^4x^3 + a^2b^3x^2y + ab^4x^3 + ab^4x^2y + b^5x^3$.

Thus we will indicate the theorem below for the Tutte polynomials of signed graphs belonging to twist knots.

Theorem 2.3. *For all $n \in \mathbb{N}^+$ we have the following recurence formula:*

$$Q[G_n^*](a,b,d) = aQ[G_{n-1}^*] + b(a+b)^{n-1}x^{n-1}[ay+bx]$$

where $Q[G_0^*] = y^2$.

Proof. It is obvious to prove the theorem by induction. □

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